Bond and Stock Market Equilibrium with Heterogeneous Agents Receiving Unspanned Income

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Abstract: We provide the first closed-form solution for the equilibrium risk-free rate and the equilibrium stock price in a continuous-time economy where agents have heterogeneous preferences and unspanned labor income risk. We show that lowering the fraction of income risk spanned by the market produces a lower equilibrium risk-free rate and a lower stock market Sharpe ratio, partly due to changes in aggregate consumption dynamics. If we fix the aggregate consumption dynamics, the Sharpe ratio will be the same as in the corresponding representative agent economy where all risks are spanned, whereas the risk-free rate (and expected stock return) will be lower in our economy. The reduction in the risk-free rate is highest when the most risk-averse individuals face the largest unspanned income uncertainty. In stylized numerical examples the risk-free rate is reduced by several percentage points. Our closed-form solution hinges on exponential utility and normally distributed dividends, but nevertheless our results show that unspanned income risk may in general play an important role in explaining the so-called risk-free rate puzzle.

Keywords: Unspanned income, heterogeneous preferences, continuous-time equilibrium, risk-free rate puzzle, equity premium, incomplete markets, Brownian motion

JEL-Classification: G12, G11, D53

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1 Introduction

Labor income is an important source of wealth for most individuals, cf. Heaton and Lucas (2000) and Campbell (2006), with a potentially large effect on their consumption and portfolio decisions and thus on equilibrium asset prices. If labor income uncertainty could be completely hedged by appropriate financial assets, it could easily be included in the modern models for individual consumption and portfolio choice and in equilibrium asset pricing models. However, a typical individual's labor income is only weakly correlated with the financial market and thus has a large unspanned and unhedgeable component, which complicates these models tremendously. This paper provides the first closed-form solution for the equilibrium risk-free rate and the equilibrium stock price in a continuous-time economy where agents have heterogeneous preferences as well as unspanned labor income risk. Our analysis is performed by conjecturing the form of the equilibrium risk-free rate as well as the stock price and then we solve for the optimal consumption and investment strategy of each individual. Aggregating over individuals, we derive an equilibrium consistent with our conjecture. In equilibrium, lowering the fraction of income risk that is spanned by the market leads to a lower covariance between the dividends of the stock and aggregate consumption and, consequently, a lower stock market Sharpe ratio. The equilibrium risk-free rate decreases due to increased precautionary savings. If we fix the aggregate consumption dynamics, the Sharpe ratio will be the same and the risk-free rate (and expected stock return) will be lower in our economy than in the corresponding representative agent economy where all risks are spanned. The reduction in the risk-free rate depends on the magnitude of all individuals' unspanned income risk and their risk aversion. The reduction is highest when the most risk-averse individuals face the largest unspanned income uncertainty. Even though our closed-form solution hinges on very specific assumptions on the preferences, the dividend process, and the income processes, our results suggest that unspanned income risk may in more general settings play an important role in explaining the risk-free rate puzzle produced by the standard representative agent models.

Our economy is formulated in a continuous-time, finite horizon model with a single consumption good, a risk-free asset and a single risky asset. The risky asset is a claim to an exogenously given dividend stream represented by a Gaussian stochastic process (an arithmetic Brownian motion). There is a finite number of consumer-investors maximizing time-additive negative exponential utility over consumption, with heterogeneity in the subjective time preference rate and the absolute risk aversion. Each individual receives an exogenously given income stream represented by another Gaussian process, which is imperfectly correlated with the stock's dividend process and therefore is also imperfectly correlated with the stock's price process itself. In other words, the individual's income process contains an unhedgeable risk component. We conjecture and later prove that in such a setting, the equilibrium risk-free rate, stock price volatility, and Sharpe ratio are all deterministic processes.

First, we solve the utility maximization problem of each individual assuming a deterministic risk-free rate, stock price volatility, and Sharpe ratio. This result makes a contribution to the literature on optimal consumption and portfolio choice. As long as labor income risk is spanned by the traded assets, the introduction of labor income risk does not complicate the solution of multi-period utility maximization problems, cf. e.g., Merton (1971) and Bodie, Merton, and Samuelson (1992). However, when the income process is partially unspanned, the investor's optimization problem becomes tremendously more complicated. To the best of our knowledge explicit solutions have only been found in two such cases, both involving negative exponential utility, a normally distributed income stream, a constant risk-free rate, and a constant drift and volatility of stock prices. Svensson and Werner (1993) solve for the optimal consumption and portfolio strategies with an infinite horizon, while Henderson (2005) assumes a finite horizon and utility of terminal wealth only.[2] Henderson conjectures [last paragraph of her concluding remarks] that "if we were to incorporate consumption, it is unlikely that the model could be solved analytically." Our results do indeed generalize her findings to the case of consumption over a finite lifetime.

We show that the optimal consumption at any time is equal to what the investor could get by spreading out evenly—annuitizing—the individual's perceived total wealth over the remaining time period. The perceived total wealth is the sum of the financial wealth, corrected upwards due to the future investment opportunities, and an appropriate measure of human wealth. Any idiosyncratic income shock, positive or negative, leads to an immediate identical change in consumption. The optimal stock investment is a combination of the standard myopic term and an income hedge term that basically "undoes" the stock-like risk inherent in the income process. The optimal investment in the risk-free asset is increasing in the magnitude of the unspanned income risk component due to precautionary savings. Our proof of the optimal strategies involves a mathematically interesting Ornstein-Uhlenbeck type bridge process, which is needed to ensure that the investor does not end up indebted at the terminal date. It seems difficult—if not impossible—to move beyond the assumptions of negative exponential utility and a Gaussian income process and

[2] Henderson (2005) also finds near-explicit solutions for more general income processes. Duffie and Jackson (1990) and Teplá (2000) derive similar solutions for investors receiving an unspanned income only at the terminal date.

still obtain closed-form solutions with unspanned income risk. Several recent papers have numerically solved for optimal consumption and portfolio strategies in various settings, see e.g., Cocco, Gomes, and Maenhout (2005), Koijen, Nijman, and Werker (2009), Lynch and Tan (2009), and Munk and Sørensen (2009).

Next, we aggregate over the individual consumer-investors and impose the market clearing conditions. We derive simple explicit solutions for the equilibrium stock price and risk-free rate confirming the conjecture we used in the individual utility maximization problems, i.e., the risk-free rate, the Sharpe ratio, and the volatility of the stock are all deterministic processes. As a result, we see that a decrease in the fraction of income risk which is spanned by the market, implies a lower market Sharpe ratio and a lower risk-free rate.

We also compare our equilibrium to the equilibrium in a similar economy with all income risk being spanned, but the same heterogeneous preferences and the same aggregate consumption. In such an economy a representative agent exists, which immediately leads to the equilibrium risk-free rate and stock market Sharpe ratio. In both economies the Sharpe ratio is the product of the volatility of aggregate consumption and the reciprocal of the aggregate absolute risk tolerance; the latter being lower relative to an economy with homogeneous risk aversions. In contrast, the risk-free rate is markedly different and is always lower in our economy with unspanned income risk, as described in the opening paragraph. In both economies the equilibrium risk-free rate consists of three terms: The first term is a weighted average of the subjective time preference rates of the individuals where the weight for a given individual is the ratio of her absolute risk tolerance to the aggregate absolute risk tolerance. This weighted average may be lower or higher than the (non-weighted) average time preference rate. This phenomenon is also discussed by Gollier and Zeckhauser (2005) who consider an economy with no uncertainty. The second term is the product of the expected growth rate of the aggregate consumption and the reciprocal of the aggregate risk tolerance. Neither the first or the second term are affected by the presence of unspanned income risk. The third term is negative and due to precautionary savings. This is the channel through which unspanned income risk affects the equilibrium risk-free rate. The precautionary savings are higher when income risk cannot be hedged which produces a lower equilibrium risk-free rate. Other things being equal, the reduction in the risk-free rate will be higher, the larger the unspanned component of the income shocks. The heterogeneity in risk aversion is crucial for the magnitude of the reduction in the risk-free rate. The reduction is highest when the most risk-averse individuals face the largest unspanned income uncertainty. In stylized numerical examples we find a reduction in the risk-free rate of several percentage points. Since the stock market Sharpe ratio is the same, the equilibrium expected return on the stock is lower than in the economy with identical aggregate consumption but spanned income risk.

Only a few existing papers on equilibrium asset pricing take labor income into account at all. Including labor income in a representative agent modeling framework is straightforward. Santos and Veronesi (2006) consider a representative agent economy where consumption is the sum of a given labor income stream and dividends from financial assets. The relative weights of income and dividend in consumption vary over time, which lead to time-varying covariances between the stock's dividend process and state prices. Consequently, the risk premia is time-varying in line with conclusions from empirical studies. However, the representative agent framework requires complete or effectively complete markets, which is not the case when individuals have non-traded income streams. Telmer (1993) considers a model with two agents who have identical preferences and initial endowments, but later their incomes may diverge due to transitory idiosyncratic shocks. The agents can only trade in a risk-free asset so they are unable to hedge the income shocks. Telmer solves numerically for the equilibrium and finds that the equilibrium state-price deflator is only weakly affected by the unhedgeable income shocks. The agents can self-insure against adverse income shocks by buffer savings and therefore the equilibrium interest rate is lower, but for reasonable parameterizations this reduction is small. This small impact may partly be due to the assumption that shocks have no persistent effect on income. Constantinides and Duffie (1996) assume that agents have identical and standard preferences but are subject to persistent non-hedgeable shocks. Their model can generate virtually generate *any* pattern of stock and bond prices, as long as the dispersion of income shocks across individuals is set appropriately. In particular, if this cross-sectional dispersion is counter-cyclical and sufficiently large, the model prices are consistent with the observed high equity premium. However, Cochrane (2005, Ch. 21) argues that cross-sectional income data do not show such large dispersion but agrees that persistent idiosyncratic income shocks may have a substantial effect on equilibrium asset prices. In our model income shocks are persistent and can produce a significant reduction of the equilibrium risk-free rate.

The reduction in the equilibrium risk-free rate that we find with unspanned income risk can explain the so-called risk-free rate puzzle, which was first identified by Weil (1989). The risk-free rate puzzle is based on the observation that the historical risk-free rate is smaller than the risk-free rate predicted by a simple consumption-based representative agent model with the high level of risk aversion needed to explain the high observed equity premium. Only few papers have offered explanations of the risk-free rate puzzle in rational asset pricing models. In a calibrated version of a relatively simple overlapping generations model, Constantinides, Donaldson, and Mehra (2002) present numerical results showing that borrowing constraints restraining young individuals from borrowing against future labor income can significantly reduce the risk-free rate and partly explain the risk-free rate puzzle, but they do not discuss the role of unspanned income risk in the determination of the risk-free rate. Bansal and Yaron (2004) consider an economy with a representative agent displaying Epstein-Zin recursive utility. The ability to disentangle the risk aversion from the elasticity of intertemporal substitution adds important flexibility in matching the low historical risk-free rate. Bansal and Yaron (2004) do not include labor income explicitly and by virtue of their representative agent framework there is no room for unspanned income risk. The present paper seems to be the first to identify unspanned labor income risk as a potential explanation of the risk-free rate puzzle.

The paper proceeds as follows. Section 2 presents the economy, i.e., the individuals' preferences and endowments as well as the assets they can trade in. Section 3 solves for any given individual's optimal consumption and investment strategy under the assumption that the risk-free rate, the excess expected return and the volatility of the stock are deterministic functions of time. Section 4 then shows by aggregation over individuals that the conjectured form of the equilibrium is correct and pins down the precise equilibrium risk-free rate and stock price dynamics. Finally, we conclude in Section 5.

2 The economy

We consider a continuous-time economy over the time interval $[0, T]$, $T > 0$. The economy offers one consumption good, which is the numéraire. Two financial assets are available for trading throughout the time interval. The first asset is a risk-free asset with B_t denoting its time t price initialized at $B_0 = 1$. The second is a risky asset which is a claim to a continuous exogenously given dividend process $D = (D_t)$ evolving as

$$dD_t = \mu_D(t)\, dt + \sigma_D(t)\, dW_t, \quad t \in [0, T], \quad D_0 \in \mathbb{R}, \tag{1}$$

for two deterministic functions μ_D and σ_D. In (1), the process $W = (W_t)$ is a standard Brownian motion meaning that the dividend process D is an arithmetic Brownian motion. We assume that the risky asset is in unit net supply and denote its time t price by S_t. We will specify the individuals such that in equilibrium the risk-free asset provides a deterministic rate of return, i.e.,

$$dB_t = r(t) B_t\, dt, \quad t \in [0, T],$$

where r is a deterministic function of time. The equilibrium price dynamics of the risky asset will have the form

$$dS_t = \big(S_t r(t) + \mu_S(t) - D_t\big)\, dt + \sigma_S(t)\, dW_t, \quad t \in [0, T], \tag{2}$$

for some deterministic functions μ_S and σ_S. Here, μ_S denotes the total expected excess return over the risk-free rate and σ_S is the (absolute) price volatility. The ratio $\lambda_S(t) = \mu_S(t)/\sigma_S(t)$ is the market price of risk associated with the Brownian motion W and is also identical to the Sharpe ratio of the risky asset S. For later use we introduce the notation

$$\beta(t, s) = \exp\left(-\int_t^s r(u)\, du\right), \quad 0 \le t \le s \le T,$$

which denotes the time t price of a zero-coupon bond paying 1 at time s for $s \in [t, T]$. Furthermore, the process

$$A(t) = \int_t^T \beta(t,u)\, du, \quad t \in [0,T],$$

denotes the annuity factor for $[t,T]$, i.e., the time t value of a continuous payment stream at the rate 1 until time T. We note that $A(t) \to 0$ as $t \to T$ and therefore $A(t)^{-1} \to \infty$ for $t \to T$.

We assume that the economy is populated by I agents all living on the time interval $[0,T]$ and having time-additive negative exponential utility of consumption. The agents can have different degrees of risk aversion as well as different time preference rates. Agent i is thus maximizing $\mathbb{E}[\int_0^T U_i(s, c_s)\, ds]$, where

$$U_i(t,c) = -e^{-\delta_i t} e^{-a_i c}, \quad c \in \mathbb{R}, \quad t \in [0,T], \tag{3}$$

and $\delta_i \geq 0$ and $a_i \geq 0$ are the time preference rate and the absolute risk aversion coefficient of the agent. We assume that each agent i receives income according to an exogenously given income rate process $Y_i = (Y_{it})$, i.e., the cumulative income up to time $t \in [0,T]$ is given by $\int_0^t Y_{iu}\, du$. We will assume that Y_i has the dynamics

$$dY_{it} = \mu_{Yi}(t)\, dt + \sigma_{Yi}(t) \left(\rho_i\, dW_t + \sqrt{1-\rho_i^2}\, dZ_{it} \right), \quad t \in [0,T], \tag{4}$$

where μ_{Yi} and σ_{Yi} are deterministic functions, and where $Z_i = (Z_{it})$ is a standard Brownian motion so that $W, Z_1, \dots, Z_I$ are independent.[3] The constant $\rho_i \in [-1, +1]$ controls the income-stock correlation. Hence, our model allows for both common income risk that is hedgeable via trading in the risky asset and allows for idiosyncratic unspanned income risk that cannot be hedged. We note that all shocks to the income rate are persistent.

We end this section by making two assumptions on the market structure. The first assumption is made on the model's basic primitives and is therefore completely exogenous.

Assumption 1. The deterministic functions $(\sigma_{Yi}, \sigma_D, \mu_{Yi}, \mu_D)$, $i = 1, ..., I$, are continuous and finitely valued on the interval $[0,T]$. $\diamondsuit$

The second assumption is made on the price dynamics and has to be verified by the equilibrium parameter processes.

Assumption 2. The interest rate r is a deterministic and continuous function on $[0,T]$ which satisfies

$$\lim_{s \uparrow T} \int_0^s A(t)^{-1}\, dt = +\infty.$$

[3]The analysis goes through when $Z_1, \dots, Z_I$ are allowed to be correlated, but at the expense of increased notational complexity.

The deterministic functions (μ_S, σ_S) are continuous on the interval $[0, T]$ and are such that the market price of risk $\lambda_S(t) = \mu_S(t)/\sigma_S(t)$ is well-defined and belongs to $\mathcal{L}^p$ for all $p \geq 1$, i.e.,

$$\int_0^T \left(\lambda_S(t)\right)^p dt < \infty.$$

$\diamondsuit$

As we shall see, Assumption 1 implies that the equilibrium parameters r, μ_S and σ_S automatically satisfy Assumption 2. Indeed, Assumption 1 leads to a bounded deterministic interest rate and such rates satisfy the integrability requirement: if $r(t) \geq \underline{r}$ for all $t \in [0, T]$ for some constant $\underline{r}$ we have

$$\beta(t, s) = \exp\left(-\int_t^s r(u)\, du\right) \leq e^{-\underline{r}(s-t)}, \quad 0 \leq t \leq s \leq T.$$

Consequently, we see that

$$A(t)^{-1} \geq \frac{1}{\underline{r}}(1 - e^{-\underline{r}(T-t)}),$$

and the right-hand-side does not have a finite integral over $[0, T]$.

3 The individual investor's problem

In this section we solve the utility maximization problem of each agent. In order to simplify the notation we suppress the i subscripts identifying the agent throughout this section.

3.1 Admissible strategies

The agent has to choose a consumption process $c = (c_t)$ and an investment process $\theta = (\theta_t)$, where θ_t represents the number of units of the risky asset owned by the agent at time t for $t \in [0, T]$. The remaining wealth is invested in the risk-free asset. Given a consumption and investment strategy (c, θ), we let $X_t^{(c,\theta)}$ denote the financial wealth of the agent at time t and let $x \in \mathbb{R}$ be the initial wealth of the agent. For $t \in [0, T]$ we define the self-financing wealth dynamics by

$$\begin{aligned} dX_t^{(c,\theta)} &= \left(X_t^{(c,\theta)} - \theta_t S_t\right) r(t)\, dt + \theta_t (dS_t + D_t\, dt) + (Y_t - c_t)\, dt \\ &= \left(X_t^{(c,\theta)} r(t) + Y_t - c_t\right) dt + \theta_t \mu_S(t)\, dt + \theta_t \sigma_S(t)\, dW_t, \end{aligned} \tag{5}$$

with $X_0^{(c,\theta)} = x$. We note that the dividend process D does not appear in the wealth dynamics. In order to ensure that $X^{(c,\theta)}$ is well-defined, we require that $c \in \mathcal{L}^1$ and

$\theta\sigma_S \in \mathcal{L}^2$, i.e., we require the following two integrability conditions hold $\mathbb{P}$-a.s.

$$\int_0^T |c_t|\, dt < \infty, \quad \int_0^T \big(\theta_t \sigma_S(t)\big)^2\, dt < \infty. \tag{6}$$

Under Assumption 2, we see that $\theta\sigma_S \in \mathcal{L}^2$ and Cauchy-Schwartz's inequality imply that $\theta\mu \in \mathcal{L}^1$ and as a consequence the wealth dynamics given by (5) are well-defined.

We place two additional requirements on the investor's possible choices. The first is the natural economic requirement that at the end of the time horizon the investor has no remaining debt obligations and so we require the processes (c, θ) to ensure

$$\mathbb{P}\left(X_T^{(c,\theta)} \geq 0\right) = 1. \tag{7}$$

As we shall see, the optimal strategies $(\hat{c}, \hat{\theta})$ have the property that $X_T^{(\hat{c},\hat{\theta})} = 0$. The second requirement is purely technical and is an artifact of our continuous time setting: to provide rigorous proofs we need a certain degree of regularity of the investor's possible choices. To state this regularity condition we define the auxiliary deterministic function

$$g(t) = \delta - r(t) + \frac{1}{2}\lambda_S(t)^2 + a\left(\mu_Y(t) - \rho\sigma_Y(t)\lambda_S(t)\right) - \frac{1}{2}(1-\rho^2)a^2\sigma_Y(t)^2, \tag{8}$$

for $t \in [0, T]$. Given this definition, we can then define the function

$$V(t, x, y) = -A(t) \exp\left\{-aA(t)^{-1}x - ay - A(t)^{-1} \int_t^T \beta(t, s) \int_t^s g(u)\, du\, ds\right\}, \tag{9}$$

for $t \in [0, T]$ and $x, y \in \mathbb{R}$. We denote by V_t, V_x and V_y the respective partial derivatives of V. We can then finally define the admissibility concept we adopt in this paper as follows:

Definition 1. A pair (c, θ) of progressively measurable processes satisfying (6) and (7) as well as ensuring that the following stochastic integrals for $t \in [0, T]$

$$\int_0^t e^{-\delta u} V_x\left(u, X_u^{(c,\theta)}, Y_u\right) \theta_u \sigma_S(u)\, dW_u \quad \text{and} \quad \int_0^t e^{-\delta u} V_y\left(u, X_u^{(c,\theta)}, Y_u\right) \sigma_Y(u)\, dZ_u, \tag{10}$$

are genuine martingales, are deemed *admissible*. In this case we will write $(c, \theta) \in \mathcal{A}$. ◇

It is a consequence of (6) and Itô's lemma that the stochastic integrals (10) are always well-defined local martingales. In order to rigorously prove our main existence result stated in the next section, we insist on only allowing the investor to use those strategies that produce genuine martingales in (10). For square integrable integrands the stochastic integrals are indeed martingales, see e.g., Protter (2004, Ch. IV.2).

3.2 The optimal strategies

The investor is assumed to maximize expected utility of running consumption and therefore seeks a pair $(\hat{c},\hat{\theta}) \in \mathcal{A}$ attaining the following maximum

$$\sup_{(c,\theta)\in\mathcal{A}} \mathbb{E}\left[\int_0^T -e^{-\delta s} e^{-ac_s}\, ds\right], \quad X_0^{(c,\theta)} = x.$$

We do not require the investor's strategies to produce finite expectation, i.e., we allow for strategies such that $\mathbb{E}\left[\int_0^T e^{-ac_s} ds\right] = +\infty$. Of course, the optimal strategies $(\hat{c},\hat{\theta})$ produce a finite expectation.

As usual, in order to apply the dynamic programming principle, the problem is embedded into a family of problems and since $\left(X^{(c,\theta)}, Y\right)$ form a Markovian system, we can define

$$J(t,x,y) = \sup_{(c,\theta)\in\mathcal{A}} \mathbb{E}_t\left[\int_t^T -e^{-\delta(s-t)} e^{-ac_s}\, ds\right], \quad t\in[0,T], \quad X_t^{(c,\theta)} = x, \quad Y_t = y, \quad (11)$$

and we refer to J as the value function or the indirect utility function.

The explicit solution to the investor's problem is stated in the following theorem and the proof can be found in the Appendix.

Theorem 1. *When Assumptions 1 and 2 are satisfied, the value function J defined in* (11) *is identical to the function V defined in* (9)*, i.e., $J = V$. Furthermore, the optimal consumption and investment strategies are given by*

$$\hat{c}(t,x,y) = A(t)^{-1}x + y + \frac{1}{a}A(t)^{-1}\int_t^T \beta(t,s)\int_t^s g(u)\, du\, ds, \quad (12)$$

$$\hat{\theta}(t,x,y) = A(t)\left(\frac{\lambda_S(t)}{a\sigma_S(t)} - \frac{\rho\sigma_Y(t)}{\sigma_S(t)}\right), \quad (13)$$

for $x,y\in\mathbb{R}$ and $t\in[0,T]$, where the deterministic function g is defined by (8)*.*

As shown in the Appendix, the optimal strategies are such that the dynamics of the optimal financial wealth $\hat{X}_t = X_t^{(\hat{c},\hat{\theta})}$ is given by

$$d\hat{X}_t = \left[m(t) - \left(A(t)^{-1} - r(t)\right)\hat{X}_t\right]\, dt + A(t)\left[a^{-1}\lambda_S(t) - \rho\sigma_Y(t)\right] dW_t, \quad (14)$$

for the mean reversion level m defined by

$$m(t) = A(t)\lambda_S(t)[a^{-1}\lambda_S(t) - \rho\sigma_Y(t)] - a^{-1}A(t)^{-1}\int_t^T \beta(t,s)\int_t^s g(u) du ds. \quad (15)$$

In other words, the financial wealth is an Ornstein-Uhlenbeck type process with time-dependent coefficients. As $t \to T$, the speed of mean reversion $A(t)^{-1} - r(t) \to \infty$, while both $m(t)$ and the volatility converge to zero. As we prove in the Appendix, this forces the optimal terminal wealth to become zero at time T, i.e., we have $\hat{X}_T = 0$ almost surely and in particular (7) is satisfied.

3.3 Discussion

We can measure how the agent values the income stream by the extra initial wealth that is needed compensate the agent for the loss of the entire income stream. For the case of no income at all, the value function is given by

$$\bar{V}(t,x) = -A(t)^{-1} \exp\left\{-aA(t)^{-1}x - A(t)^{-1}\int_t^T \beta(t,s)\int_t^s \bar{g}(u)\,du\,ds\right\},$$

where

$$\bar{g}(t) = \delta - r(t) + \frac{1}{2}\lambda_S(t)^2 = g(t) - a\left(\mu_Y(t) - \rho\sigma_Y(t)\lambda_S(t) - \frac{a}{2}(1-\rho^2)\sigma_Y(t)^2\right).$$

The wealth equivalent of the income stream, $L(t,y)$, is then defined by $\bar{V}(t, x + L(t,y)) = V(t,x,y)$, which implies that

$$L(t,y) = A(t)y + \int_t^T \beta(t,s)\int_t^s \left(\mu_Y(u) - \rho\sigma_Y(u)\lambda_S(u) - \frac{a}{2}(1-\rho^2)\sigma_Y(u)^2\right) du\,ds. \quad (16)$$

The first term is the present value of a constant payment stream at the rate y whereas the second term corrects for income growth, covariance with the stock price, and unspanned income risk. This term equals the expectation of the discounted future income under the risk-adjusted measure $\mathbb{Q}$. The measure $\mathbb{Q}$ is defined by applying the market-given price of risk $\lambda_S(t)$ for the risk represented by W and the agent-specific price of risk $\lambda_Y(t) = \frac{a}{2}\sqrt{1-\rho^2}\sigma_Y(t)$ associated with the unspanned income risk represented by Z. The income dynamics can be written as

$$dY_t = \left(\mu_Y(t) - \rho\sigma_Y(t)\lambda_S(t) - \frac{a}{2}(1-\rho^2)\sigma_Y(t)^2\right) dt + \sigma_Y(t)\left(\rho\,dW_t^{\mathbb{Q}} + \sqrt{1-\rho^2}\,dZ_t^{\mathbb{Q}}\right),$$

where $W^{\mathbb{Q}}$ and $Z^{\mathbb{Q}}$ are independent $\mathbb{Q}$-Brownian motions. This produces the expectation

$$\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \beta(t,s)Y_s\,ds\right] = L(t, Y_t).$$

The value function (9) can be rewritten as

$$V(t,x,y) = -A(t)\exp\left\{-aA(t)^{-1}\left[x + H(t) + L(t,y)\right]\right\},$$

where

$$H(t) = a^{-1} \int_t^T \beta(t,s) \int_t^s \left(\delta - r(u) + \frac{1}{2} \lambda_S(u)^2 \right) du\, ds,$$

can be interpreted as a correction of current wealth due to the future investment opportunities offered by the stock. The optimal consumption (12) can be expressed as

$$\hat{c}(t,x,y) = A(t)^{-1} \left[x + H(t) + L(t,y) \right],$$

so the optimal consumption at time t is equal to the constant consumption we get by "annuitizing" the perceived total wealth $X_t + H(t) + L(t, Y_t)$ over the remaining lifetime. As perceived total wealth changes and the remaining time period shortens, the optimal consumption changes. We also note that the agent with initial wealth x and the given income stream consumes exactly as an agent with initial wealth $x + L(t,y)$ and no income.

Given the dynamics of financial wealth in (14), the dynamics of the sum of financial wealth and the present value of future income is of the form

$$d\Big(\hat{X}_t + L(t,Y_t)\Big) = \ldots\, dt + A(t) a^{-1} \lambda_S(t)\, dW_t + A(t) \sqrt{1-\rho^2} \sigma_Y(t)\, dZ_t.$$

Therefore, the optimal investment strategy with income is such that the total wealth has the same sensitivity towards the hedgeable shocks represented by dW_t as the financial wealth has in the case without income. The presence of the unhedgeable shock represented by dZ_t does not affect the desired exposure to the hedgeable shock. The dynamics of the optimal consumption $\hat{c}_t = \hat{c}(t, \hat{X}_t, Y_t)$ becomes

$$d\hat{c}_t = \mu_c(t)\, dt + a^{-1} \lambda_S(t)\, dW_t + \sqrt{1-\rho^2} \sigma_Y(t)\, dZ_t,$$

for some deterministic function μ_c. We note that any idiosyncratic income shock (positive or negative) leads to an immediate identical change in consumption.

The above theorem shows that the optimal number of stocks held by the agent is deterministic and, in particular, independent of wealth and income. The amount optimally invested in the stock at time t is

$$\hat{\theta}(t,x,y) S_t = A(t) \left(\frac{\lambda_S(t)}{a \sigma_S(t)/S_t} - \frac{\rho \sigma_Y(t)}{\sigma_S(t)/S_t} \right),$$

where $\sigma_S(t)/S_t$ is the relative volatility of the stock. The optimal stock position is a combination of the standard myopic or speculative term and a term correcting for the stock-like component of the income process (an "income hedge" term), in line with the standard results for dynamic portfolio problems, cf., e.g., Merton (1969, 1971) and Bodie, Merton, and Samuelson (1992).

Henderson (2005) maximizes negative exponential utility of terminal wealth assuming a constant interest rate r (so that $A(t) = (1 - e^{-r(T-t)})/r$), a constant Sharpe ratio λ_S of the stock, and a Gaussian income like (4) but with constant drift and volatility. Henderson finds that the amount optimally invested in the stock at time t is

$$\theta^{\text{Hend}}(t, x, y)S_t = e^{-r(T-t)}\frac{\lambda_S}{a\sigma_S/S_t} - A(t)\frac{\rho\sigma_Y}{\sigma_S/S_t}.$$

Henderson assumes that the relative volatility of the stock, σ_S/S_t, is constant, but this has no effect on the structure of the optimal portfolio.[4] The appropriate weight on the speculative investment is $e^{-r(T-t)}$ with utility of terminal wealth, but $A(t)$ in our case with utility of intermediate consumption. In Henderson's case, the weight is the present value of getting a payment of 1 at the terminal date. In our case, the weight is the present value of getting a continuous payment at the rate of 1 over the remaining life. Furthermore, we note that if we let $T \to \infty$ our results become similar to those found in the infinite-horizon setting of Svensson and Werner (1993).

Finally, we consider the optimal amount invested in the risk-free asset: $\hat{X}_t - \hat{\theta}_t S_t$. The unspanned component of the income dynamics affect the wealth dynamics in (14) via the mean reversion level $m(t)$ in (15). More precisely, we see that the variance of the unspanned income shock decreases the function $g(t)$ in (8) and thus increases the mean reversion level of wealth via the last term in (15). The effect of unspanned income risk on $m(t)$ increases proportionally with the absolute risk aversion a. The unspanned income risk thus leads to an increase in financial wealth and, since stock holdings are unaffected, the wealth increase comes via an increase in the position in the risk-free asset. This is the precautionary savings caused by risk aversion and unhedgeable, persistent income shocks.

4 Equilibrium

4.1 Derivation of the equilibrium

We derive the equilibrium risk-free rate and stock price in the economy by aggregation over the agents. We let $\tau_i = 1/a_i$ denote the absolute risk tolerance of agent i and $\tau_\Sigma = \sum_{i=1}^{I} \tau_i$ the aggregate absolute risk tolerance in the economy.

As shown in Theorem 1, agent i optimally invests in

$$\hat{\theta}_i(t) = A(t)\left(\tau_i \frac{\lambda_S(t)}{\sigma_S(t)} - \frac{\rho_i \sigma_{Yi}(t)}{\sigma_S(t)}\right), \quad t \in [0, T], \tag{17}$$

units of the stock. To ensure that the stock market clears, we need $\sum_i \hat{\theta}_i(t) = 1$, which

[4] Our expression for the optimal portfolio generalizes to the case with any stochastic volatility as long as the risk-free rate and the Sharpe ratio of the stock are deterministic, see e.g., Detemple, Garcia, and Rindisbacher (2003) and Nielsen and Vassalou (2006).

implies that the equilibrium Sharpe ratio of the stock (or the market price of risk) is

$$\lambda_S(t) = \frac{1}{\tau_\Sigma}\left[A(t)^{-1}\sigma_S(t) + \sum_i \rho_i \sigma_{Yi}(t)\right], \tag{18}$$

where the volatility $\sigma_S(t)$ is still to be identified. From (18), we see that the Sharpe ratio increases with the income-stock correlations ρ_i. The larger the income-stock correlations are, the more negative is the income hedge term of the agents. Therefore, to maintain market clearing, the Sharpe ratio $\lambda_S(t)$ of the stock has to increase as well.

The stock price S is given as the appropriately risk-adjusted expectation of the discounted future dividends. The adjustment for the risk affecting the dividend stream (1) is given by the market price of risk $\lambda_S(t)$. Under our conjecture that the risk-free rate is deterministic, the value of the discounted dividends is not affected by the idiosyncratic income shocks. We wish to show that the stock price defined as

$$S_t = \mathbb{E}_t^{\mathbb{Q}^{\min}}\left[\int_t^T \beta(t,u) D_u \, du\right], \quad t \in [0,T],$$

is the equilibrium stock price. Here $\mathbb{Q}^{\min}$ is the minimal martingale measure defined by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}^{\min}}{d\mathbb{P}} = \exp\left\{-\frac{1}{2}\int_0^T \lambda_S(t)^2 \, dt - \int_0^T \lambda_S(t) \, dW_t\right\}.$$

The dividend dynamics (1) can be rewritten as

$$dD_t = \big[\mu_D(t) - \lambda_S(t)\sigma_D(t)\big] \, dt + \sigma_D(t) \, dW_t^{\min},$$

where $W_t^{\min} = W_t + \int_0^t \lambda_S(u) \, du$ defines a standard Brownian motion under $\mathbb{Q}^{\min}$. By means of Fubini's theorem we now find that

$$\begin{aligned} S_t &= \int_t^T \beta(t,u) \mathbb{E}_t^{\mathbb{Q}^{\min}}[D_u] \, du \\ &= \int_t^T \beta(t,u)\Big(D_t + \int_t^u \big(\mu_D(s) - \lambda_S(s)\sigma_D(s)\big) \, ds\Big) \, du \\ &= A(t) D_t + \int_t^T \beta(t,u) \int_t^u \big(\mu_D(s) - \lambda_S(s)\sigma_D(s)\big) \, ds \, du. \end{aligned} \tag{19}$$

In particular, the (absolute) volatility of the stock price is $\sigma_S(t) = A(t)\sigma_D(t)$. By substituting this expression for σ_S into (18) we obtain the following expression for the Sharpe ratio $\lambda_S(t)$ in terms of only exogenous quantities:

$$\lambda_S(t) = \frac{1}{\tau_\Sigma}\left[\sigma_D(t) + \sum_i \rho_i \sigma_{Yi}(t)\right], \quad t \in [0,T]. \tag{20}$$

Therefore we see that Assumption 1 ensures that λ_S given (20) satisfies the second part of Assumption 2.

The bond market clearing condition is equivalent to the condition that the total financial wealth is invested in the stock, i.e., $S_t = \sum_i \hat{X}_{it}$ for all $t \in [0, T]$ where $\hat{X}_i$ denotes investor i's optimal wealth process given by (14). We next show that S given by (19) satisfies this relationship. Since $\hat{X}_{iT} = 0$ for all i, it is immediate that the relation holds at $t = T$ and so to show the validity at time t, $t < T$, it suffices to verify that $dS_t = \sum_i d\hat{X}_{it}$.

From (5) and (12), the optimal wealth dynamics of agent i can be written as

$$\begin{aligned} d\hat{X}_{it} &= \left(\hat{X}_{it} r(t) + Y_{it} - \hat{c}_{it}\right) dt + \hat{\theta}_{it}\mu_S(t)\, dt + \hat{\theta}_{it}\sigma_S(t)\, dW_t \\ &= \left[\left(r(t) - A(t)^{-1}\right) \hat{X}_{it} - A(t)^{-1} \int_t^T \beta(t,s) \int_t^s \tau_i g_i(u)\, du\, ds\right] dt \\ &\quad + \hat{\theta}_{it}\mu_S(t)\, dt + \hat{\theta}_{it}\sigma_S(t)\, dW_t. \end{aligned}$$

By summing up and using $\sum_i \hat{\theta}_{it} = 1$, we obtain

$$\begin{aligned} \sum_i d\hat{X}_{it} = \Bigg[&\left(r(t) - A(t)^{-1}\right) \sum_i \hat{X}_{it} + \mu_S(t) \\ &- A(t)^{-1} \int_t^T \beta(t,s) \int_t^s \Big(\sum_i \tau_i g_i(u)\Big)\, du\, ds\Bigg] dt + \sigma_S(t)\, dW_t. \end{aligned}$$

We rewrite the stock price dynamics (2) as

$$dS_t = \left[\left(r(t) - A(t)^{-1}\right) S_t + \mu_S(t) + A(t)^{-1} S_t - D_t\right] dt + \sigma_S(t)\, dW_t,$$

to see that $dS_t = \sum_i d\hat{X}_{it}$ if and only if

$$S_t = A(t) D_t - \int_t^T \beta(t,s) \int_t^s \Big(\sum_i \tau_i g_i(u)\Big)\, du\, ds. \tag{21}$$

By comparing (19) and (21), we conclude that (21) is equivalent to

$$\mu_D(t) - \lambda_S(t)\sigma_D(t) = -\sum_i \tau_i g_i(t), \quad t \in [0, T].$$

Therefore, we can substituting in $g_i(t)$ from (8) and rearrange to obtain

$$\begin{aligned} r(t)\tau_\Sigma = &\sum_i \tau_i \delta_i + \mu_D(t) + \sum_i \mu_{Yi}(t) \\ &- \lambda_S(t) \left[\sigma_D(t) + \sum_i \rho_i \sigma_{Yi}(t) - \frac{1}{2}\tau_\Sigma \lambda_S(t)\right] - \frac{1}{2}\sum_i a_i (1 - \rho_i^2)\sigma_{Yi}(t)^2. \end{aligned}$$

Finally, the expression (20) for $\lambda_S(t)$ can be used. We summarize our findings in the following theorem.

Theorem 2. *Given Assumption 1, the equilibrium risk-free rate is*

$$\begin{aligned} r(t) = &\sum_i \frac{\tau_i}{\tau_\Sigma}\delta_i + \frac{1}{\tau_\Sigma}\Big[\mu_D(t) + \sum_i \mu_{Yi}(t)\Big] \\ &- \frac{1}{2\tau_\Sigma^2}\Big[\sigma_D(t) + \sum_i \rho_i\sigma_{Yi}(t)\Big]^2 - \frac{1}{2\tau_\Sigma}\sum_i a_i(1-\rho_i^2)\sigma_{Yi}(t)^2, \end{aligned} \tag{22}$$

and the equilibrium stock price S_t is

$$S_t = A(t)D_t + \int_t^T \beta(t,u)\int_t^u \Big(\mu_D(s) - \tau_\Sigma\sigma_D(s)\Big[\sigma_D(s) + \sum_i \rho_i\sigma_{Yi}(s)\Big]\Big)\,ds\,du, \tag{23}$$

which leads to a stock price volatility of $\sigma_S(t) = A(t)\sigma_D(t)$ and the Sharpe ratio

$$\lambda_S(t) = \frac{1}{\tau_\Sigma}\Big[\sigma_D(t) + \sum_i \rho_i\sigma_{Yi}(t)\Big], \quad t \in [0,T]. \tag{24}$$

In particular, $r(t)$ satisfies the first part and $\lambda_S(t)$ the second part of Assumption 2.

We note that the Sharpe ratio increases when more of the income risk is spanned. Higher $\rho_i\sigma_{Yi}$ leads to a larger (negative) income hedge demand for the stock, cf. (17), and thus a lower total stock demand. To maintain market clearing, a higher Sharpe ratio is necessary in order to increase the speculative demand. The investors' risk attitudes determine by how much the Sharpe ratio has to increase. An increase in $\rho_i\sigma_{Yi}$ does not affect the absolute volatility of the stock, but it lowers the stock price and thus increases the relative volatility $\sigma_S(t)/S_t$. The expected excess rate of return on the stock, $\mu_S(t)/S_t = \lambda_S(t)\sigma_S(t)/S_t$, is therefore also increasing in $\rho_i\sigma_{Yi}$. The equilibrium risk-free rate depends on the degree of spanning, i.e., the income-stock correlations, via the last two terms in (22). For simplicity, we assume for a moment that $a_i = a = 1/\tau$, $\rho_i = \rho$, and $\sigma_{Yi}(t) = \sigma_Y(t)$ for all I individuals, and let $\bar\sigma_D(t) = \sigma_D(t)/I$ be the absolute volatility of per capita dividends. Then the last two terms in (22) add up to $-\frac{1}{2\tau^2}\sigma_Y(t)^2 - \frac{1}{2\tau^2}[\bar\sigma_D(t)^2 + 2\rho\bar\sigma_D(t)\sigma_Y(t)]$, which is decreasing in the income-dividend correlation. Intuitively, a higher income-dividend correlation induces investors to shift funds from the risky to the risk-free asset and therefore, to maintain market clearing, the equilibrium risk-free rate have to decrease.

4.2 Comparison with a representative agent equilibrium

Aggregate consumption in our economy is $C_{\Sigma t} = \sum_i \hat c_{it} = D_t + \sum_i Y_{it}$ with dynamics

$$dC_{\Sigma t} = \Big[\mu_D(t) + \sum_i \mu_{Yi}(t)\Big]\,dt + \Big[\sigma_D(t) + \sum_i \rho_i\sigma_{Yi}(t)\Big]\,dW_t + \sum_i \sqrt{1-\rho_i^2}\,\sigma_{Yi}(t)\,dZ_{it}. \tag{25}$$

If we vary the income-stock correlations, the aggregate consumption dynamics change which in itself causes changes in asset pricing moments. To get a clearer understanding of the impact on unspanned income risk, we will next compare the equilibrium in our setting to the equilibrium in an economy with the same aggregate consumption dynamics, same dividend process, and same preferences, but where all income risk is spanned. In that complete-market economy, a representative agent exists and maximizes time-additive expected utility $\mathbb{E}[\int_0^T \exp\{-\delta t\} U(C_{\Sigma t})\, dt]$. If we write the aggregate consumption dynamics as $dC_{\Sigma t} = \mu_{Ct}\, dt + \sigma_{Ct}^\top\, dB_t$, where B is a possibly multi-dimensional standard Brownian motion, it is well-known (see, e.g., Breeden 1986) that the equilibrium risk-free rate is

$$r_t^{\text{REP}} = \delta + \left(-\frac{U''(C_{\Sigma t})}{U'(C_{\Sigma t})} \right) \mu_{Ct} - \frac{1}{2} \frac{U'''(C_{\Sigma t})}{U'(C_{\Sigma t})} \|\sigma_{Ct}\|^2, \tag{26}$$

where $||\cdot||$ denotes the standard Euclidian norm, and the Sharpe ratio of the risky asset is

$$\lambda_t^{\text{REP}} = \left(-\frac{U''(C_{\Sigma t})}{U'(C_{\Sigma t})} \right) \rho_{CD} \sigma_D(t) \|\sigma_{Ct}\|, \tag{27}$$

where ρ_{CD} is the dividend-consumption correlation. It is also well-known that when a complete (or effectively complete) market economy is populated by I agents having the exponential utility (3) with different time preference rates δ_i and different absolute risk tolerances $\tau_i = 1/a_i$, then a representative agent exists and has negative exponential utility with time preference rate $\sum_i \tau_i \delta_i / \tau_\Sigma$ and absolute risk tolerance τ_Σ; see Huang and Litzenberger (1988, Sec. 5.26). The risk-free rate (26) is then

$$r_t^{\text{REP}} = \sum_i \frac{\tau_i}{\tau_\Sigma} \delta_i + \frac{1}{\tau_\Sigma} \mu_{Ct} - \frac{1}{2\tau_\Sigma^2} \|\sigma_{Ct}\|^2$$

and the Sharpe ratio is

$$\lambda_t^{\text{REP}} = \frac{1}{\tau_\Sigma} \rho_{CD} \sigma_D(t) \|\sigma_{Ct}\|.$$

We first consider the Sharpe ratio. If the aggregate consumption dynamics in the corresponding representative agent economy is given by (25), the Sharpe ratio in that economy is identical to the Sharpe ratio in our setting. By Jensen's inequality we have $1/\tau_\Sigma \leq \sum_i a_i$ which renders the Sharpe ratio lower in our heterogeneous economy than in a homogeneous economy where all agents are equipped with an absolute risk aversion equal to the average risk aversion in the heterogeneous economy.

Next we focus on the risk-free rate. If the aggregate consumption dynamics are given by (25), the equilibrium risk-free rate in the corresponding representative agent economy becomes deterministic and equal to

$$r^{\mathrm{REP}}(t) = \sum_i \frac{\tau_i}{\tau_\Sigma}\delta_i + \frac{1}{\tau_\Sigma}\Big[\mu_D(t) + \sum_i \mu_{Yi}(t)\Big] - \frac{1}{2\tau_\Sigma^2}\Big(\Big[\sigma_D(t) + \sum_i \rho_i\sigma_{Yi}(t)\Big]^2 + \sum_i (1-\rho_i^2)\sigma_{Yi}(t)^2\Big). \tag{28}$$

In absence of idiosyncratic income risk, i.e., when $\rho_i^2 = 1$ for all agents i, our economy does allow for a representative agent and, consequently, the two expressions (22) and (28) are identical. In general, a representative agent does not exist in our framework and we cannot hope for an expression of the type (26).

The two first terms in the expressions (22) and (28) are identical and thus not affected by unspanned income risk. The first term is a weighted average of the individual time preference rates, where the weight for agent i equals her share of the aggregate absolute risk tolerance. The time preference rates of the most risk tolerant agents have the highest weights in determining the "common" time preference rate. This is consistent with Gollier and Zeckhauser (2005) who consider how a group of heterogeneous agents share an exogenous and deterministic consumption process. They show for general time-additive utility functions that the collective decisions correspond to the decisions of a representative agent equipped with a time preference rate equal to a risk tolerance weighted average of the individual time preference rates. In general the weights—and hence the representative time preference rate—vary over time as the individual consumption rates vary, but in our case with exponential utility the weights are constant over time. If the most risk tolerant agents tend to have the highest [lowest] time preference rates, the "common" time preference rate is higher [lower] than the average time preference rate with the obvious consequence for the equilibrium risk-free rate. The averaging of the time preference rates is independent of the risk structure in the economy and thus the same in our model and the corresponding representative agent model. The second term in the risk-free rate is the product of the expected growth rate of aggregate consumption and the reciprocal of the aggregate risk tolerance. As for the Sharpe ratio, this term is smaller in our heterogeneous economy compared to a homogeneous economy where all agents are equipped with an absolute risk aversion equal to the average risk aversion in the heterogeneous economy.

The unspanned income risk affects the equilibrium risk-free rate via the precautionary savings motive as reflected by the difference in the last term on the right-hand sides of (22) and (28). Risk-averse individuals exposed to unhedgeable income shocks increase their demand for the risk-free asset leading to a higher equilibrium price and a lower equilibrium risk-free rate. Other things being equal, more risk-averse individuals increase their bond demand by more. Therefore, it is natural that the unspanned income risk of each individual is scaled by the individual's risk aversion. Indeed, this is reflected by our solution for the optimal consumption and investment strategies as discussed in the final paragraph of Section 3. With identical aggregate consumption, the difference between the risk-free rate in our economy and in the representative agent economy is

$$r(t) - r^{\text{REP}}(t) = -\frac{1}{2\tau_\Sigma}\sum_i \Big(a_i - \frac{1}{\tau_\Sigma}\Big)(1-\rho_i^2)\sigma_{Yi}(t)^2. \tag{29}$$

This difference is negative whenever $\rho_i^2 < 1$ and $\sigma_{Yi}(t) > 0$ for some i since $a_i = 1/\tau_i > 1/\sum_i \tau_i = 1/\tau_\Sigma$. We therefore have the following corollary:

Corollary 1. *When Assumption 1 is satisfied, the equilibrium risk-free rate in our economy with unspanned income shocks is smaller than the equilibrium risk-free rate in the corresponding complete market economy with identical aggregate consumption.*

As explained in the introduction, the present paper seems to be the first to suggest and quantify the role of unspanned income risk in the determination of the equilibrium risk-free rate and as a potential explanation of the risk-free rate puzzle. With a lower risk-free rate and the same Sharpe ratio, it is clear that the expected return on the stock—keeping the volatility fixed—is also lower in our setting than in the corresponding representative agent setting with identical aggregate consumption dynamics.

To assess the quantitative impact of unspanned income shocks we suppose that all I individuals have the same absolute risk aversion $a_i = a$, the same constant income volatility $\sigma_{Yi}(t) = \sigma_Y$, and the same income-stock correlation $\rho_i = \rho$. Then $\tau_\Sigma = I/a$ and we have

$$r(t) - r^{\text{REP}}(t) = -\frac{1}{2}a^2\left(1 - \frac{1}{I}\right)(1-\rho^2)\sigma_Y^2 \to -\frac{1}{2}a^2(1-\rho^2)\sigma_Y^2, \quad \text{for } I \to \infty.$$

Various studies report that individuals typically seem to have a relative risk aversion around 2 (see e.g., Szpiro 1986; Guo and Whitelaw 2006; Paiella and Attanasio 2007) and labor income with a percentage volatility of around 10% and a correlation with the stock market near zero (see e.g., Gourinchas and Parker 2002; Cocco, Gomes, and Maenhout 2005; Davis, Kubler, and Willen 2006). If we think of wealth and income being fairly evenly distributed and initial wealth having the same magnitude, the above limiting interest rate difference becomes[5] $-0.5\cdot 2^2(1-0^2)0.1^2 = -0.02$, i.e., the model with idiosyncratic income risk produces a risk-free rate which is 2 percentage points lower than in the corresponding representative agent model. This difference is increasing in and highly dependent on the risk aversion level and the income volatility level, but less dependent on the income-stock correlation (as long as it is near zero). For a risk aversion of 3, the risk-free rate reduction is 4.5 percentage points.

[5] Then $a^2\sigma_Y^2 = (\gamma/W)^2(v_Y Y)^2 = \gamma^2 v_Y^2$, where γ is the relative risk aversion and v_Y is the percentage income volatility.

To get a feeling of the importance of heterogeneity in risk aversion and income uncertainty, we next suppose that one third of all individuals has a risk aversion of 1 and an income volatility of 0.05, another third has a risk aversion of 2 and an income volatility of 0.1, whereas the last third has a risk aversion of 3 and an income volatility of 0.15. Then we get $\tau_\Sigma = (11/18)I$ and letting $I \to \infty$, the interest differential $r(t) - r^{\text{REP}}(t)$ approaches approximately -0.0245. Conversely, if we suppose that the individuals with a risk aversion of 1 have an income volatility of 0.15 and the individuals with a risk aversion of 3 have an income volatility of 0.05, whereas the individuals with risk aversion 2 still have an income volatility of 0.1, we see that the interest rate differential becomes approximately -0.0136. These examples illustrate that the quantitative effect of idiosyncratic income risk on the equilibrium risk-free rate is depending on the cross-sectional relation between agents' risk aversion and income uncertainty.

5 Conclusion

This paper has offered the first closed-form solution for the equilibrium risk-free rate and the equilibrium stock price in a dynamic economy where agents have heterogeneous preferences and unspanned labor income risk. The degree of spanning of the individual labor income risk has clear effects on the equilibrium consumption and asset prices. A smaller fraction of income risk being spanned produces a lower equilibrium risk-free rate and a lower stock market Sharpe ratio, partly due to changes in aggregate consumption dynamics.

On the other hand, if we hold the aggregate consumption dynamics fixed, the Sharpe ratio is the same and the risk-free rate and expected stock return is lower in our economy than in the corresponding representative agent economy with all income risk spanned. The reduction in the risk-free rate depends on the magnitude of all individuals' unspanned income risk as well as their risk aversion coefficients, and the reduction is highest when the most risk-averse individuals face the largest unspanned income uncertainty. In a small stylized numerical example we have illustrated that the reduction in the risk-free rate—and thus the expected stock return for a fixed volatility—can be several percentage points. Our model therefore suggests that unspanned income risk may play an important role in explaining the risk-free rate puzzle.

Obviously our assumptions of exponential utility and normally distributed incomes and dividends are unrealistic. It seems impossible to obtain closed-form solutions under more appropriate assumptions and even the fundamental theoretical question of equilibrium existence is currently unclear. Therefore, a natural next step is to perform a numerical analysis of such more realistic cases which may shed further light on the quantitative effects of unspanned income risk on equilibrium prices of financial assets.

A Proofs

Proof of Theorem 1

The Hamilton-Jacobi-Bellman (HJB) equation associated with the problem (11) is

$$\delta V = \sup_{c,\theta\in\mathbb{R}} \Big\{ -e^{-ac} + V_t + V_x\big\{r(t)x - c + y + \theta\mu_S(t)\big\} + \frac{1}{2}V_{xx}\theta^2\sigma_S(t)^2 + V_y\mu_Y(t) + \frac{1}{2}V_{yy}\sigma_Y(t)^2 + V_{xy}\theta\rho\sigma_S(t)\sigma_Y(t)\Big\}, \quad t\in[0,T), \quad x,y\in\mathbb{R},$$

with subscripts on V indicating partial derivatives. Since our investor only receives utility of running consumption, the terminal condition is given by

$$V(T,x,y) = 0, \quad x \geq 0, \quad y\in\mathbb{R}, \tag{30}$$

where the requirement $x \geq 0$ is due to our admissibility requirement (7). The proof proceeds through the usual steps required in order to apply the dynamic programming principle.

Step I: Explicit solution to the HJB equation. The first-order conditions in the HJB equation lead to the optimal control candidates

$$\hat{c}_t = -a^{-1}\log\left(a^{-1}V_x\right), \qquad \hat{\theta}_t = -\frac{V_x}{V_{xx}}\frac{\mu_S(t)}{\sigma_S(t)^2} - \frac{V_{xy}}{V_{xx}}\frac{\rho\sigma_Y(t)}{\sigma_S(t)}. \tag{31}$$

By substituting these expressions into the HJB equation, we obtain the PDE

$$\delta V = V_t + V_x\left(r(t)x + y - a^{-1}\right) + a^{-1}V_x\log\left(a^{-1}V_x\right) - \frac{1}{2}\frac{V_x^2}{V_{xx}}\lambda_S(t)^2 - \frac{1}{2}\frac{V_{xy}^2}{V_{xx}}\rho^2\sigma_Y(t)^2 + V_y\mu_Y(t) + \frac{1}{2}V_{yy}\sigma_Y(t)^2 - \frac{V_xV_{xy}}{V_{xx}}\rho\sigma_Y(t)\lambda_S(t),\ t\in[0,T),\ x,y\in\mathbb{R}.$$

We try a solution of the form $V(t,x,y) = -\exp\{-H(t)x - G(t)y - F(t)\}$ for smooth deterministic functions F, G and H. The optimal control candidates can the be expressed as

$$\hat{c}_t = -a^{-1}\Big(\log\left(a^{-1}H(t)\right) - H(t)x - G(t)y - F(t)\Big),$$
$$\hat{\theta}_t = \frac{1}{H(t)\sigma_S(t)}\Big(\rho\sigma_Y(t)G(t) - \lambda_S(t)\Big).$$

Formally inserting these expressions in to the PDE gives us the wealth dynamics (see (5))

$$dX_t^{(\hat{c},\hat{\theta})} = \Big(X_t^{(\hat{c},\hat{\theta})}\big\{r(t) - a^{-1}H(t)\big\} + Y_t\big\{1 - a^{-1}G(t)\big\} + a^{-1}\Big(\log\left(a^{-1}H(t)\right) - F(t)\Big) + \frac{\lambda_S(t)}{H(t)}\Big(\lambda_S(t) - G(t)\rho\sigma_Y(t)\Big)dt + \frac{1}{H(t)}\Big(\lambda_S(t) - \rho\sigma_Y(t)G(t)\Big)dW_t.$$

Based on the below lemma, we see that to ensure admissibility of the optimal control candidates, we wish to have

$$\lim_{t\to T} H(t) = +\infty, \quad \lim_{t\to T} G(t) = a, \quad \lim_{t\to T} m(t) = \lim_{t\to T} \frac{1}{H(t)}\Big(\rho\sigma_Y(t)G(t) - \lambda_S(t)\Big) = 0,$$

where we have defined the mean reversion function m by

$$m(t) = a^{-1}\Big(\log\big(a^{-1}H(t)\big) - F(t)\Big) + \frac{\lambda_S(t)}{H(t)}\Big(\lambda_S(t) - G(t)\rho\sigma_Y(t)\Big), \quad t \in [0,T].$$

To compute F, G and H explicitly, we compute the involved derivatives and insert them into the above PDE. After dividing through with the exponential term, we collect the x and y-terms and thereby obtain the following coupled system of ODEs

$$\begin{aligned} x-\text{terms}: &\quad H'(t) = a^{-1}H(t)^2 - H(t)r(t), \quad H(T) = +\infty,\\ y-\text{terms}: &\quad G'(t) = a^{-1}H(t)G(t) - H(t), \quad G(T) = a. \end{aligned}$$

Direct calculations show that $G(t) = a$ for $t \in [0,T]$ whereas the Ricatti equation for H has the solution $H(t) = aA(t)^{-1}$, $t \in [0,T]$, where A is the annuity factor. To get the terminal condition for F, we insert these expressions into the mean reversion function m

$$m(t) = -a^{-1}\Big(\log\big(A(t)\big) + F(t)\Big) + A(t)\Big(a^{-1}\lambda_S(t) - \rho\sigma_Y(t)\Big). \quad t \in [0,T],$$

From this expression it follows that F's terminal condition is

$$\lim_{t\to T}\Big(\log\big(A(t)\big) + F(t)\Big) = 0.$$

Since $A(t) \to 0$ for $t \to T$, it must be the case that $F(t) \to \infty$ for $t \to T$ and this implies that the boundary condition (30) is satisfied for $x \geq 0$ and $y \in \mathbb{R}$. To get the explicit form for F, we collect the remaining terms of the PDE and find the linear ODE

$$F'(t) = A(t)^{-1}\Big[\log\big(A(t)\big) + 1 + F(t)\Big] - g(t) - r(t),$$

where we have defined the deterministic function

$$g(t) = \delta - r(t) + \frac{1}{2}\lambda_S(t)^2 + a\left(\mu_Y(t) - \rho\sigma_Y(t)\lambda_S(t)\right) - \frac{1}{2}a^2(1-\rho^2)\sigma_Y(t)^2, \quad t \in [0,T].$$

By using Leibnitz's rule, we can verify that F is given by

$$F(t) = \int_t^T e^{-\int_t^s A(u)^{-1}\,du} g(s)\,ds - \log\big(A(t)\big) = \tilde{F}(t) - \log\big(A(t)\big).$$

Furthermore, since $A'(t)/A(t) - r(t) = -A(t)^{-1}$, we have

$$-\int_t^s A(u)^{-1}\,du = \int_t^s \frac{A'(u)}{A(u)}\,du - \int_t^s r(u)\,du = \ln A(s) - \ln A(t) - \int_t^s r(u)\,du,$$

and thus $\exp\left(-\int_t^s A(u)^{-1}\,du\right) = A(t)^{-1}A(s)\beta(t,s)$. By substituting this into $\tilde{F}(t)$ and integrating by parts, we get

$$\tilde{F}(t) = A(t)^{-1}\int_t^T A(s)\beta(t,s)g(s)\,ds = A(t)^{-1}\int_t^T \beta(t,s)\int_t^s g(u)\,du\,ds.$$

This proves that V defined by (9) solves the HJB equation.

Step II: Admissibility of the optimal control candidates. We then turn to show that the optimal candidate policies indeed are admissible in the sense of Definition 1. The candidate for the optimal investment strategy is given by

$$\hat{\theta}_t = A(t)\left(\frac{\lambda_S(t)}{a\sigma_S(t)} - \frac{\rho\sigma_Y(t)}{\sigma_S(t)}\right).$$

Since A converges to zero, Assumptions 1+2 ensure the needed integrability of $\hat{\theta}$. The candidate process for optimal consumption is given by

$$\hat{c}(t,x,y) = A(t)^{-1}x + y + \frac{1}{a}A(t)^{-1}\int_t^T \beta(t,s)\int_t^s g(u)\,du\,ds.$$

Here the dynamics of $X^{(\hat{c},\hat{\theta})}$ are given by

$$dX_t^{(\hat{c},\hat{\theta})} = [m(t) + \left(r(t) - A(t)^{-1}\right)X_t^{(\hat{c},\hat{\theta})}]dt + A(t)[a^{-1}\lambda_S(t) - \rho\sigma_Y(t)]dW_t,$$

where m is the the above deterministic function, i.e.,

$$m(t) = A(t)\lambda_S(t)[a^{-1}\lambda_S(t) - \rho\sigma_Y(t)] - a^{-1}\tilde{F}(t).$$

The last term in $\hat{c}$ is equal to $a\tilde{F}(t)$ which converges to zero as $t \to T$ and is therefore integrable. The second term is Y_t which is an OU process with nicely behaved mean and variance functions, cf. Assumption 1. The first term, $A(t)^{-1}X_t^{(\hat{c},\hat{\theta})}$, is more complicated because $A(u) \to 0$ for $u \to T$. By means of Ito's lemma we find the dynamics

$$\begin{aligned} d\left[A(t)^{-1}X_t^{(\hat{c},\hat{\theta})}\right] &= X_t^{(\hat{c},\hat{\theta})}\left[\left(A(t)^{-1}\right)^2 - r(t)A(t)^{-1}\right]dt + A(t)^{-1}dX_t^{(\hat{c},\hat{\theta})} \\ &= A(t)^{-1}m(t)dt + [a^{-1}\lambda_S(t) - \rho\sigma_Y(t)]dW_t. \end{aligned}$$

From this and Assumptions 1 and 2 it follows that the variance function is finite

$$\int_0^T \Big(a^{-1}\lambda_S(t) - \rho\sigma_Y(t)\Big)^2 dt < \infty.$$

However, the mean function is more evolved. By integrating, we find the mean function to be

$$\begin{aligned}\psi(t) &= \mathbb{E}\left[A(t)^{-1}X_t^{(\hat{c},\hat{\theta})}\right] \\ &= A(0)^{-1}X_0^{(\hat{c},\hat{\theta})} + \int_0^t \lambda_S(u)[a^{-1}\lambda_S(u) - \rho\sigma_Y(u)]du - a^{-1}\int_0^t \frac{\tilde{F}(u)}{A(u)}du.\end{aligned}$$

We wish to show that ψ is a continuous function on $[0,T]$ and that ψ does not blow up at T. This follows if we show that the ratio $\tilde{F}(u)/A(u)$ has a finite limit when $u \to T$ because in that case $\tilde{F}/A$ is a continuous function on the compact interval $[0,T]$. By the rules of Leibnitz and L'Hopital as well as the definition of $\tilde{F}$ we find

$$\begin{aligned}\lim_{u\uparrow T}\frac{\tilde{F}(u)}{A(u)} &= \lim_{u\uparrow T}\frac{\tilde{F}(u)A(u)}{A(u)^2} \\ &= \lim_{u\uparrow T}\frac{r(u)\tilde{F}(u)A(u) - g(u)A(u)}{2A(u)[r(u)A(u)-1]} \\ &= \lim_{u\uparrow T}\frac{r(u)\tilde{F}(u) - g(u)}{2[r(u)A(u)-1]} = \frac{1}{2}g(T),\end{aligned}$$

which is finite by Assumptions 1 and 2. Therefore, $\tilde{F}/A$ is a continuous function on $[0,T]$ and as a consequence the mean function ψ is also continuous on $[0,T]$. In conclusion the needed integrability condition (6) is satisfied.

Lemma 1 stated and proved below shows that $\hat{c}$ and $\hat{\theta}$ satisfy $X_T^{(\hat{c},\hat{\theta})} = 0$ and therefore, we only we need to check the martingality of the two stochastic integrals in (10):

$$\int_0^t e^{-\delta u}V_x\hat{\theta}_u\sigma_S(u)\,dW_u \quad \text{and} \quad \int_0^t e^{-\delta u}V_y\sigma_Y(u)\,dZ_u.$$

To do so, it suffices to show square integrability of the integrands, i.e., we need

$$\mathbb{E}\int_0^T \Big(e^{-\delta u}V_x\hat{\theta}_u\sigma_S(u)\Big)^2 du < \infty \quad \text{and} \quad \mathbb{E}\int_0^T \Big(e^{-\delta u}V_y\sigma_Y(u)\Big)^2 du < \infty.$$

By using the explicit form of V and the optimal control candidates $(\hat{c},\hat{\theta})$ we have

$$\begin{aligned}V_x\hat{\theta}_u\sigma_S(u) &= a\exp\Big(-aA(u)^{-1}X_u^{(\hat{c},\hat{\theta})} - aY_u - \tilde{F}(u)\Big)A(u)\left(\frac{\lambda_S(u)}{a} - \rho\sigma_Y(u)\right), \\ V_y\sigma_Y(u) &= aA(u)\exp\Big(-aA(u)^{-1}X_u^{(\hat{c},\hat{\theta})} - aY_u - \tilde{F}(u)\Big)\sigma_Y(u).\end{aligned}$$

Under Assumptions 1 and 2, we can use Cauchy-Schwartz's inequality repeatedly to see that it suffices to show that

$$\exp\left(A(u)^{-1}X_u^{(\hat{c},\hat{\theta})}\right) \quad \text{and} \quad \exp(Y_u),$$

have all moments. To argue that this indeed is true, we recall that for a normally distributed random variable U, $U \sim \mathcal{N}(\mu, \sigma^2)$, we have for $p \in \mathbb{R}$ the identity

$$\mathbb{E}[\exp(pU)] = \exp\left(p\mu + \frac{1}{2}p^2\sigma^2\right).$$

Since both $A(u)^{-1}X_u^{(\hat{c},\hat{\theta})}$ and Y_u are normally distributed, this formula shows that it suffices to argue that the mean function and the variance functions are continuous on $[0,T]$ and do not blow up at T. Again, because of Assumption 1, this holds for Y's mean and variance function and we just proved that it holds for the mean and variance function for $A(t)^{-1}X_t^{(\hat{c},\hat{\theta})}$ too.

Step III: Verifying that V indeed equals the value function J. We first show that we have $V \geq J$ and so we let $(c,\theta) \in \mathcal{A}$ be arbitrary. We can assume that $\mathbb{E}[\int_0^T e^{-ac_s}ds] < \infty$ - otherwise there is nothing to prove. Itô's lemma renders the following relation for $t \in [0,T]$:

$$e^{-\delta T}V(s, X_T^{(c,\theta)}, Y_T) = e^{-\delta t}V(t, X_t^{(c,\theta)}, Y_t) + \int_t^T e^{-\delta u}\text{drift}_u^{(c,\theta)}du$$
$$+ \int_t^T e^{-\delta u}V_x\theta_u\sigma_S(u)dW_u + \int_t^T e^{-\delta u}V_y\sigma_Y(u)\Big(\rho dW_u + \sqrt{1-\rho^2}dZ_u\Big),$$

where $e^{-\delta t}\text{drift}_t^{(c,\theta)}$ is the dt-part of $e^{-\delta t}V(t, X_t^{(c,\theta)}, Y_t)$. By the HJB equation we have

$$\text{drift}_t^{(c,\theta)} - e^{-ac_t} \leq 0,$$

and therefore by taking conditional expectations through and using the martingale property (10), we find

$$\mathbb{E}_t\left[e^{-\delta T}V\left(T, X_T^{(c,\theta)}, Y_T\right)\right] \leq e^{-\delta t}V(t, X_t^{(c,\theta)}, Y_t) + \mathbb{E}_t\left[\int_t^T e^{-\delta u}e^{-ac_u}du\right].$$

However, by the boundary condition (30) and by the admissibility requirement (7), the above left-hand-side vanishes. All in all, we have the estimate

$$V(t, X_t^{(c,\theta)}, Y_t) \geq \mathbb{E}_t\left[\int_t^T -e^{-\delta(u-t)}e^{-ac_u}du\right],$$

from which the inequality $V \geq J$ follows. To justify that we have $V = J$, we note that given the admissibility of the optimal control candidates $(\hat{c}, \hat{\theta})$, we can re-do the above calculations but replacing the inequalities with equalities and the claim follows.

$\diamondsuit$

Lemma 1 (Ornstein-Uhlenbeck Bridge)**.** *Given the assumptions stated in Theorem 1. Let T be a positive constant and let m and c be two continuous finitely valued deterministic functions defined on $[0,T]$. We assume that*

$$\lim_{s \uparrow T} m(s) = 0.$$

Then there exists a unique stochastic process $X = (X_t)$ solving the affine SDE

$$dX_t = \big(m(t) + \big(r(t) - A(t)^{-1}\big)X_t\big)\,dt + c(t)A(t)\,dW_t, \quad t \in (0,T), \quad X_0 \in \mathbb{R}, \tag{32}$$

and $X_t \to 0$, $\mathbb{P}$-a.s., as $t \to T$.

Proof. We define the deterministic function $b(t) = r(t) - A(t)^{-1}$ and we note that $b(t) \to -\infty$ as $t \to T$. A direct application of Itô's product rule gives us that the stochastic process

$$X_s = e^{\int_0^s b(u)\,du}\left(X_0 + \int_0^s e^{-\int_0^t b(u)\,du}\Big(m(t)\,dt + c(t)A(t)\,dW_t\Big)\right), \quad s \in [0,T),$$

satisfies (32). Furthermore, L'Hopital's rule gives us

$$\lim_{s \uparrow T} \frac{\int_0^s e^{-\int_0^t b(u)\,du} m(t)\,dt}{e^{\int_0^s -b(u)\,du}} = \lim_{s \uparrow T} \frac{m(s)}{-b(s)} = 0.$$

The proof can therefore be concluded by showing

$$e^{\int_0^s b(u)\,du} M_s = e^{\int_0^s b(u)\,du} \int_0^s e^{-\int_0^t b(u)\,du} c(t)A(t)\,dW_t \to 0, \quad \mathbb{P}\text{-a.s.},$$

as $s \uparrow T$. The quadratic variation of M is given by

$$\langle M \rangle_s = \int_0^s e^{-2\int_0^t b(u)\,du} \frac{c(t)^2}{A(t)^{-2}}\,dt, \quad s \in [0,T).$$

If $\langle M \rangle_T < \infty$ we trivially have that M is a continuous martingale on the interval $[0,T]$ and in particular M_T is a real valued random variable and the claim follows. If $\langle M \rangle_T = \infty$, we can use Exercise II.15 in Protter (2004) to see that

$$0 = \lim_{t \uparrow T} \frac{M_t}{\langle M \rangle_t}, \quad \mathbb{P}\text{-almost surely.}$$

L'Hopital's rule gives us

$$\lim_{s\uparrow T}\langle M\rangle_s e^{\int_0^s b(u)\,du} = \lim_{s\uparrow T}\frac{\int_0^s e^{-2\int_0^t b(u)\,du}\frac{c(t)^2}{A(t)^{-2}}\,dt}{e^{-\int_0^s b(u)\,du}} = \lim_{s\uparrow T}\frac{e^{-\int_0^s b(u)\,du}c(s)^2}{-b(s)A(s)^{-2}}. \tag{33}$$

We want to show that this limit is zero and to do so it suffices to show that

$$\lim_{s\uparrow T}\frac{e^{\int_0^s A(u)^{-1}\,du}}{A(s)^{-2}} = 0. \tag{34}$$

Since $\frac{\partial}{\partial t}A(t)^{-2} = 2A(t)^{-2}\Big(A(t)^{-1} - r(t)\Big)$, we have the representation

$$A(s)^{-2} = A(0)^{-2}e^{2\int_0^s (A(u)^{-1}-r(u))\,du},$$

from which (34) follows with the use of Assumption 2. $\diamondsuit$

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