Homework 9

1. Let $P : \textbf{Set} \to \textbf{Set}$ be the functor which sends each set $A$ to its powerset $P(A)$, and each function $f : A \to B$ to the function $P(f) : P(A) \to P(B)$ defined by $P(f)(U) = \{ f(a) | a \in U \}$ for $U \subseteq A$. Show that the functions $\eta_A : A \to P(A), \ a \mapsto \{a\}$ define a natural transformation $1_{\textbf{Set}} \to P$.

2. Let $A$ be an object in a cartesian closed category $\mathbb{C}$. Let $P_A : \mathbb{C} \to \mathbb{C}$ be the ‘product with $A$’ functor with object part $B \mapsto B \times A$, and let $E_A : \mathbb{C} \to \mathbb{C}$ be the ‘exponentiation by $A$’ functor, with object part $B \mapsto B^A$.

   (a) Show that the evaluation maps $\varepsilon_B : B^A \times A \to B$ define a natural transformation $\varepsilon : P_A \circ E_A \to 1_{\mathbb{C}}$.

   (b) Define a natural transformation $\eta : 1_{\mathbb{C}} \to E_A \circ P_A$.

3. Given three objects $A, B, C$ of a cartesian closed category $\mathbb{C}$, we can define an ‘internal composition’ map

   $$m : C^B \times B^A \to C^A$$

   as exponential transpose of the composite map

   $$C^B \times B^A \times A \xrightarrow{\text{CB} \times \varepsilon} C^B \times B \xrightarrow{\varepsilon} C.$$ 

   In the case where $C$ is $\textbf{Cat}$ and $A, B, C$ are small categories, we obtain a functor of type

   $$\text{Fun}(B, C) \times \text{Fun}(A, B) \to \text{Fun}(A, C),$$

   with object part $(G, F) \mapsto G \circ F$. Give an explicit description of the morphism part of this functor.
4. (*) Show that for any category $C$, the natural transformations
\[ \eta : \text{id}_C \to \text{id}_C \]
from the identity functor to itself form a *commutative* monoid under composition.
What is this monoid in the case of the categories $\text{Sets}$, $\text{Rel}$, and $\text{CMon}$? In the last case you can just guess, no proof required.

5. Given an object $C$ of a locally small category $C$ we we have seen the functors
\[ \text{hom}(C, -) : C \to \text{Set} \quad \text{and} \quad \text{hom}(-, C) : C^{\text{op}} \to \text{Set} \]
called *hom-functors* or *representable functors*. More generally, a functor $F : C \to \text{Set}$ or $G : C^{\text{op}} \to \text{Set}$ is called representable, if it is *isomorphic* to one of the form $\text{hom}(C, -)$ or $\text{hom}(-, C)$, respectively, for some $C$.

(a) Show that the functor $U : \text{Cat} \to \text{Set}$ which sends each small category to its set of objects is representable.
(b) Show that the functor $U : \text{Mon} \to \text{Set}$ which sends each monoid to its underlying set is representable.
(c) Show that the functor $P$ from exercise 1 is *not* representable.