Homework 10

1. Given categories \( C, D \) and an adjunction between \( C \) and \( D \), consisting of functors \( F : C \to D \) and \( U : D \to C \) and a family of bijections

\[
\varphi_{A,B} : \text{hom}(FA, B) \cong \text{hom}(A, UB)
\]

natural in \( A \in \text{obj}(C) \) and \( B \in \text{obj}(D) \), show that the morphisms

\[
\varepsilon_B = \varphi_{U_B,B}^{-1}(1_{UB}) : FUB \to B \quad \text{for } B \in \text{obj}(D)
\]

constitute a natural transformation \( \varepsilon : F \circ U \to 1_D \).

2. To every small category \( C \) we can associate a preorder \( P(C) \) whose elements are the objects of \( C \), with \( A \leq B \) iff there exists a morphism from \( A \) to \( B \) in \( C \). Show that the inclusion functor \( J : \text{Preord} \to \text{Cat} \) has a left adjoint which maps every category \( C \) to \( P(C) \). Describe the unit and counit of this adjunction.

3. Let \( U : \text{SGph} \to \text{Set} \) be the forgetful functor from simple graphs.

   (a) Does \( U \) have a left adjoint? (Hint: look at the old homeworks.)
   (b) Does \( U \) have a right adjoint?

4. The category of elements \( \int F \) of a presheaf \( F : C^{\text{op}} \to \text{Sets} \) on a locally small category \( C \) has as objects pairs \((C, x)\) with \( C \in C \) and \( x \in F(C) \), and morphisms \( f : (C, x) \to (D, y) \) are morphisms \( f : C \to D \) in \( C \) satisfying \( F(f)(y) = x \). Composition and identities are inherited from \( C \). Show that \( F \) is representable if and only if \( \int F \) has a terminal object.

5. (*) Given a functor \( F : C \to D \) where \( C \) is small and \( D \) is locally small, the nerve functor \( N_F : D \to \hat{C} \) maps each object \( D \in D \) to the presheaf

\[
\text{hom}_D(F(-), D) : C^{\text{op}} \to \text{Sets}
\]

   (a) Complete the definition of \( N_F \) by giving its morphism part.
A functor \( F : \mathbb{C} \to \mathbb{D} \) is called \emph{dense}, if its nerve \( N_F \) is full and faithful. A subcategory \( \mathbb{C} \subseteq \mathbb{D} \) is called dense, if the nerve \( N_I \) of the canonical inclusion functor \( I : \mathbb{C} \hookrightarrow \mathbb{D} \) is full and faithful.

(b) Let \( \Delta_0 \) be the full subcategory of \emph{Preord} containing only the terminal preorder \( \{0\} \), and let \( \Delta_1 \) be the full subcategory of \emph{Preord} containing the terminal preorder \( \{0\} \) and the two-element preorder \( \{0 \leq 1\} \). Show that \( \Delta_0 \) is not dense in \emph{Preord}, but \( \Delta_1 \) is.

(c) Since every preorder is a category, \( \Delta_1 \) can also be regarded as a full subcategory of \emph{Cat}. Show that \( \Delta_1 \) is not dense in \emph{Cat}.

(d) (Optional) \( \Delta_2 \) is the full subcategory of \emph{Cat} containing besides the objects of \( \Delta_1 \) also the preorder \( \{0 \leq 1 \leq 2\} \). Show that \( \Delta_2 \) is dense in \emph{Cat}.