Homework 1 – Structured Sets and Structure Preserving Functions

1. Posets and monotone maps. Given posets \((A, \leq)\) and \((B, \leq)\), a (not necessarily monotone) function \(f : A \to B\) is called order-reflecting, if the implication

\[
f(x) \leq f(y) \Rightarrow x \leq y
\]

holds for all \(x, y \in A\).

(a) Show that order-reflecting functions between posets are injective.

(b) Show that isomorphisms in the category \(\textbf{Pos}\) are order-reflecting.

(c) Given an example of a bijective monotone map which is not an isomorphism.

The poset \((A, \leq)\) is called a total order if any two elements are comparable, i.e. we have \(x \leq y\) or \(y \leq x\) for all \(x, y \in A\).

(d) Show that if \((A, \leq)\) is a total order and \(f : (A, \leq) \to (B, \leq)\) is a surjective monotone map, then \((B, \leq)\) is also a total order.

Given posets \((A, \leq)\) and \((B, \leq)\), a function \(f : A \to B\) is called antimonotone, if we have

\[
x \leq y \Rightarrow f(y) \leq f(x)
\]

for all \(x, y \in A\).

(e) We have seen that posets and monotone functions together form a category \(\textbf{Pos}\).

Do posets and antimonotone functions also form a category? What about posets together with order-reflecting functions?

Hint: To verify that a class of structures together with a class of functions forms a category, we have to show that the functions are closed under composition and contain identities.
2. **Simple graphs.** A *simple graph* is a pair \((V, E)\) of a set \(V\) and a binary relation \(E \subseteq V \times V\) which is reflexive and symmetric, i.e. we have 
\[(x, x) \in E \quad \text{and} \quad (x, y) \in E \Rightarrow (y, x) \in E\]
for all \(x, y \in V\). We call \(V\) the set of *vertices*, and \(E\) the set of *edges*.

Given simple graphs \((V, E)\) and \((W, F)\), a *morphism of simple graphs* \(f : (V, E) \to (W, F)\) is a function \(f : V \to W\) which ‘preserves edges’, i.e. we have \((f(v), f(v')) \in F\) whenever \(v, v' \in V\) and \((v, v') \in E\).

(a) Show that simple graphs and their morphisms form a category \(\text{SGph}\).

(b) A *path* in a simple graph \((V, E)\) is a sequence \((v_0, \ldots, v_n)\) \((n \geq 0)\) of vertices such that we have \((v_j, v_{j+1}) \in E\) for all \(0 \leq j < n\). The graph is called *connected* if for all \(v, w \in V\) there exists a path \((v_0, \ldots, v_n)\) with \(v_0 = v\) and \(v_n = w\).

Show that if \(f : (V, E) \to (W, F)\) is a morphism of simple graphs such that \((V, E)\) is connected, and \(f\) is surjective as a function, then \((W, F)\) is connected.

(c) Up to isomorphism, how many connected simple graphs with exactly three vertices are there? How many with four vertices? Draw pictures.

3. Suppose that \(f : A \to B\), \(g : B \to C\) and \(h : A \to C\) are morphisms in a category \(\mathbf{C}\) such that \(g \circ f = h\).

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{h} & & \downarrow{g} \\
B & \xrightarrow{g} & C
\end{array}
\]

Show that if any two of \(f, g, h\) are isomorphisms, then the third must be as well (this is known as the 2-out-of-3 property).

4. \(\ast\) Show that in the category \(\text{Mon}\) of monoids and homomorphisms, the isomorphisms are exactly the bijective homomorphisms.