Review of *Euclid and His Twentieth Century Rivals* by Nathaniel Miller

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It is commonplace to view the rigor of the mathematics in Euclid's *Elements* in the way an experienced teacher views the work of an earnest beginner: respectable relative to an early stage of development, but ultimately flawed. Given the close connection in content between Euclid's *Elements* and high school geometry classes, this is understandable. Euclid, it seems, never realized what everyone who moves beyond elementary geometry into more advanced mathematics is now customarily taught: a fully rigorous proof cannot rely on geometric intuition. In his arguments he seems to call illicitly upon our understanding of how objects like triangles and circles behave rather than grounding everything rigorously in axioms.

Though widespread, the attitude is in a historical sense puzzling. For over two millenia, mathematicians of all levels studied the arguments in *Elements* and found nothing substantial missing. The book, on the contrary, represented the limit of mathematical explicitness. It served as *the* paradigm for careful and exact reasoning. How it could enjoy this reputation, for so long, is mysterious if careful and exact reasoning demands that all inferences be grounded in a modern axiomatic theory in the way Hilbert did in his famous *Foundations of Geometry.* By these standards, Euclid's work is deeply flawed. The holes in his arguments are not minor and excusable but massive and cryptic.

With his book *Euclid and His Twentieth Century Rivals*, Nathaniel Miller makes substantial progress in clearing this mystery up. The book is an explication of **FG**, a formal system of proof developed by Miller which reconstructs Euclid's deductions as essentially diagrammatic. The holes in Euclid's arguments are taken to appear precisely at those steps which are unintelligible without an accompanying geometric diagram. Interpreting the reasoning

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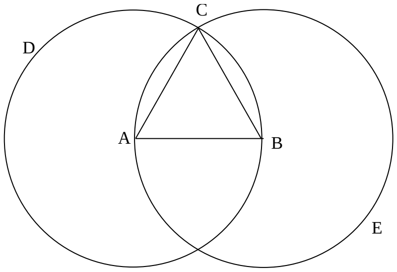
[1] The reader ought to be aware, especially when reading the critical remarks in the latter half of the review, that my 2006 dissertation provides a formalization of Euclidean diagrammatic reasoning similar to Miller's. Though Miller's work did not directly inspire my project, I learned of it early on. My formalization would not have the character it has had I not been driven to improve upon Miller's work. Indeed, I take one of the main contributions of my thesis to be its alternative account of what secures the generality of Euclid's results. One should thus keep in mind that my criticisms in what follows are closely connected to my views on how my work and Miller's compare with respect to capturing Euclid's reasoning.

in the *Elements* in terms of a modern axiomatization (as is often done), we have no resources for understanding what underlies such steps. All we can do is talk of some vague intuitive process, which is less of an explanation than an evasion. In *Euclid and His Twentieth Century Rivals* Miller confronts the question directly, and provides a sharp, well-motivated account of Euclid's diagrammatic method of proof. The book thus deserves the attention of anyone who is interested in the nature of mathematical reasoning, and believes that Euclid engaged it.

The key idea behind **FG** is that geometric diagrams justify inferences via their topology, and nothing else. Though the diagrams which appear in the *Elements* invariably have metric properties, Euclid never appeals to them in a proof. When reading properties off the diagram he restricts himself to topological relations—e.g. intersections between lines and circles, the containment of one region in another.[2] For example, the diagram for the proof of proposition 1 in book I is:



The proof of the proposition shows how to construct an equilateral triangle on any given segment. What the diagram contributes to the proof is the intersection point C of circles D and E. Though the curves D and E look circular, this is not read from the diagram. That they are circles is stipulated from the outset. And the conclusion that the triangle ABC is equilateral follows from this stipulation, the definition of a circle, and the transitivity of equality (laid out earlier as an axiom). That the triangle looks equilateral in the diagram plays no part in the argument. Thus, we can replace the diagram with a topologically equivalent one where the stipulated circles do not look circular and the triangle does not look equilateral

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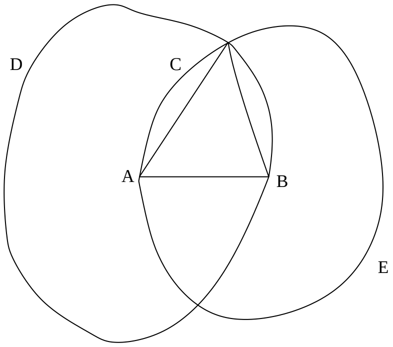
[2] The first to emphasize this fact, and build an analysis of Euclid's proofs around it, is Ken Manders, who did so in his unpublished but widely circulated 1995 piece *The Euclidean Diagram*.



without doing any damage to the proof. In an essay on topology[3], Poincaré mentions that it "has often been remarked that geometry is the art of reasoning correctly about figures which are poorly constructed." All of the *Elements* (not just proposition 1) confirm the remark Poincaré cites for the geometry Euclid practiced. The proofs in the book still go through if we replace all metrically exact diagrams with poorly drawn, topologically equivalent ones.

This insight about Euclid's proofs immediately suggests the strategy Miller adopts for formalizing them. As topological objects, Euclid's diagrams are discrete. What identifies them, topologically, is the way their lines and circles partition a bounded region of the plane into a finite set of regions. Thus, the discrete syntactic objects which are to function as diagrams in a formalization ought to be individuated in the same manner. The first challenge, then, is to define such syntactic objects precisely. The second is to formulate suitable rules for how objects so defined are to be used in proofs.

Miller meets the first challenge in two stages. He first defines the notion of a *nicely well-formed diagram* in terms of four primitive symbols: frames, dots, lines, and dotted lines. The frame is a rectangular box which forms the outer boundary of an **FG** diagram. Within this boundary, a finite number of dots, lines, and dotted lines appear. The lines (which need not be straight) represent segments of Euclidean straight lines, while the dotted lines (which need not be circular) represent arcs of circles. They connect the dots of the diagram to each other and to the boundary of the diagram. The 'nicely well-formed part' of the definition consists of a list of conditions ensuring that lines and dotted lines form configurations which behave in a rough topological way like Euclidean lines and circles. One condition, for instance, ensures that two circles intersect no more than twice.

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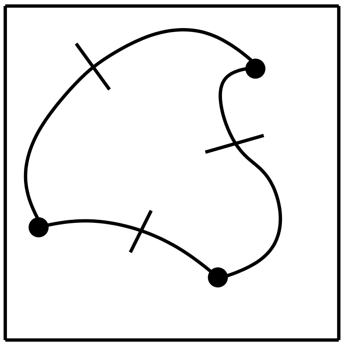
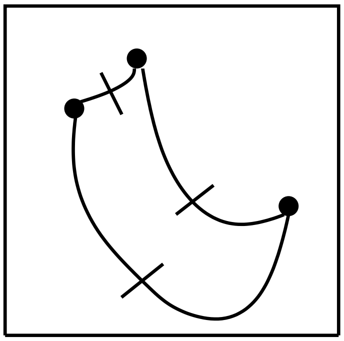
[3] 'Why Space Has Three Dimensions' which can be found in *Mathematics and Science: Last Essays* (Dover, New York, 1963).

In the second stage he defines a mapping between nicely well-formed diagrams and planar graphs. The image of an **FG** diagram under the mapping is termed the *corresponding graph structure* of the diagram. Two diagrams are defined as equivalent if they have the same corresponding graph structure. The point of this equivalence relation is to identify diagrams which induce the same partitions, and so function identically in proofs. So that an **FG** diagram can express equalities, Miller allows hash-marks to be placed on the magnitudes depicted by a diagram. In accordance with the familiar convention (which Euclid did not himself observe) two magnitudes are represented as equal when the same hash-mark appears on both. Thus, in **FG**, the two diagrams

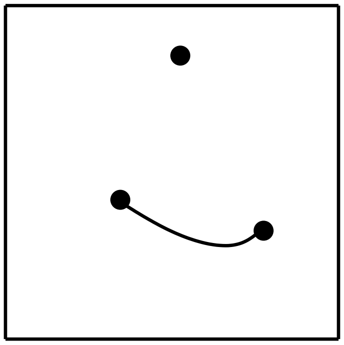




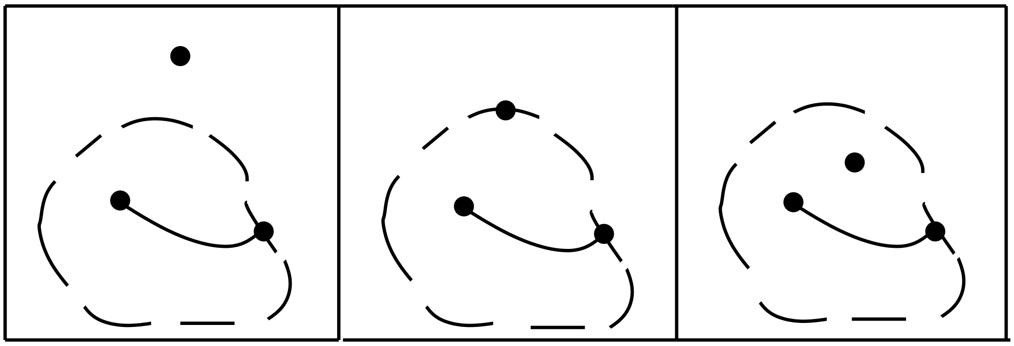
are equivalent, and represent an equilateral triangle.

So that these symbols can be used to establish Euclid's results in a formalization, the stipulated rules of proof must do two things. First, the rules must regulate the construction of diagrams. And second, they must restrict diagrammatic inferences to positional relations which are general within the context of the proof. A typical proof in the *Elements* first lays out the construction of a geometric figure, and then draws conclusions about the constructed figure, which are based, in part, on how things appear in a particular, representative diagram. The classic worry this method immediately raises is: how can we be sure that the particular diagram is representative in the way required by the proof? Each of Euclid's proofs have a general scope extending beyond any individual diagram. What justifies reading off some features from an individual diagram as general? Though Euclid never mistakes a property particular to an individual diagram as general, he does not provide any explicit criteria for how the separation of the general from the particular is to be made. What a formalization such as Miller's must do, if it is to count as a formalization, is furnish such criteria via its rules of proof.

The rules of **FG** do this via disjunctive diagram arrays. A Euclidean construction in **FG** is not carried out via a single, representative diagram, but via an array of representative diagrams. In applying a construction step (such as joining two points in a segment, or drawing a circle on a radius) to a diagram D, one must produce the array representing all topological cases which could possibly result from applying the step to a figure represented by D. For instance, if D is the diagram



then constructing a circle on the segment of D produces in **FG** the array



As only topological features can be read off from a diagram, D contains no information about the distance of the left-endpoint of the segment to the point off the segment. And so, it is consistent with D that the latter point sit outside, on, or inside the constructed circle.

Applying a construction step to an array A produces the array of all diagrams obtained by applying the step to each diagram of A. Thus, the array produced after n construction steps contains all topological cases which could possibly result from those n steps. Logically, the array is similar to a propositional statement in disjunctive normal form. It asserts that the geometric figure of the proof has the properties of one of the diagrams in the array. Once all contradictory diagrams (diagrams whose markings equate the part of a whole to the whole) are thrown out, one is then in a position to discern what holds in general. This consists in those properties manifest in all diagrams of the array.

The central technical achievement of Miller's work is the specification of a mechanical procedure which given any initial diagram and any geometric construction as input outputs the appropriate array. Though it is clear what all the cases are when a circle is added to a diagram consisting only of a segment and a point, it is not clear if there is a general method for enumerating cases given any construction step and any diagram. The diagram does not have to become that much more complex for the range of possible cases consistent with a construction step to become obscure. Even if some cases can be seen, one usually lacks a guarantee that these constitute *all* cases to be considered.

The purely topological character of his diagrams, however, allows Miller to specify a method which has such a guarantee built in. The reason that the range of cases which come with a construction step is obscure is that it is not immediate how the metric symmetries of lines and circles restrict what is and isn't possible topologically. Yet once we allow line and circles to bend any which way, the obscurity vanishes. The range of cases emerges as the range of all topological possibilities consistent with the conditions for a nicely well formed diagram. These possibilities can be generated by straightforward, if tedious, procedure, which is implemented in a computer program named **CDEG** (for Computerized Diagrammatic Euclidean Geometry). Though Miller does not provide all the details of the procedure, he discusses **CDEG**'s treatment of two diagrammatic constructions. From this the reader can get a general sense of how it works.

CDEG constitutes the sum of Miller's efforts to realize the aim, stated in the introduction, to formalize Euclid's proofs "in a way that preserves their inherently diagrammatic nature." On a first appraisal, his efforts are successful. The nicely well-formed diagrams of his proof system are so structured to express the geometric information Euclid relies on diagrams to express. In building around these symbols a system of proof which gives precise formal versions of Euclid's arguments, Miller develops an inherently diagrammatic formalization and so challenges the standard presumption that diagrammatic reasoning cannot be rigorous.

One may still ask, however, if the diagrammatic method of proof embodied by **CDEG** is Euclid's. Comparing the diagram arrays of Miller's formalizations to the single diagram proofs in the *Elements* raises doubts. Requiring that a proof branch into cases with each construction step can lead very quickly to a case explosion. Miller regards this as a inevitable consequence of rigorizing Euclid's use of diagrams. Yet when a case-heavy **FG** formalization is laid side-by-side Euclid's original version, the original does not appear deficient. Rather, the multitude of cases generated by the rules of **FG** appear excessive. The geometric differences recorded by a case-branching often do not seem material to the issue the proof decides.

The capacity of **FG** for case-analysis overkill is not apparent in the formalization of proposition 1 of book I, which Miller exhibits in the book. But it shows itself, with a vengeance, in the formalization of proposition 2, which Miller does not discuss. The rules of **FG** demand that many, many cases be considered for the seven step construction. My efforts to work out all of them yielded 57 at the fourth step. Pushing further with the whole construction on three of these yielded 50 more. (At this point my will to continue gave out.) What distinguishes the cases are positional relations which are irrelevant to the inferences Euclid makes later, in that the relations need not be attended to for the soundness of the inferences to be confirmed.[4] In verifying the generality of the result, we can focus in on certain relations in a single representative diagram and ignore others. We do not have to check that the result holds in a long list of cases. That the proof allows us to do this does not seem to be an accidental feature. It seems, rather, to be a key mathematical insight of the proof. With proposition 2 and others throughout the *Elements*, Euclid seems deliberately to frame his arguments so that it suffices, or almost suffices, to consider a single diagram. Miller's formal account fails to bring this out. The one and only way to secure a general result with diagrams is by a brute enumeration of cases.

It would be wrong, of course, to deny any place for the examination of various cases in Euclid's mathematics. The insight that certain positional relations hold generally was no doubt helped along by viewing various cases. Yet it is unlikely that the case analysis method prescribed by **FG** *justified* a conclusion for Euclid. The method of **FG** justifies because it gives a sharp

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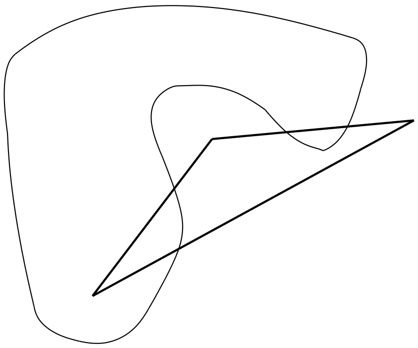
[4] For the details, see my *Proofs, Pictures, and Euclid*

characterization of the range of *all* cases. But this characterization, recall, depends on modern topological concepts. Stretching lines and circles into their topological equivalents seems safely beyond the Euclidean tradition. That Euclid only looked to the topological features of lines and circles in diagrams does not mean he had a purely topological conception of them. And the abstractness of the conception makes it improbable. We have little reason to believe, for instance, that Euclid ever considered



as a possible way for a triangle and a circle to partition the plane. But if he justified his results as **FG** prescribes, he would have had to. The conditions for a nicely well-formed diagram do not rule it out.

The foreignness of arbitrarily bending curves to Euclid's geometry diminishes the force of some conclusions Miller draws from **FG**. He shows how the complexity of an **FG** proof dramatically decreases after a lemma has been incorporated. This, he claims, explains the ubiquity of lemmas in classical geometry. But it is not clear how much the decrease in complexity is bound up with features particular to **FG**. Proving a result in **FG** with a lemma usually requires less cases than proving it without. Yet a great number of the additional cases are configurations from combinatorial topology, not Euclidean geometry. It is unlikely that the prospect of them all loomed before Euclid as he sought to prove the same result. The point is especially pertinent to the way Miller handles superposition in **FG**. The proof system has a technique for translating a figure within a diagram. When used to formalize proposition 4, the first place Euclid invokes superposition, the result is a diagram array detailing a staggering number of cases. Miller remarks that we should thus not be surprised that Euclid is reluctant to use superposition throughout *Elements*. Yet to consider these cases as geometric possibilities, one has to relax one's conception of a rigid angle and allow lines to snake through other lines in unusual, non-linear ways. It is hard to believe that such configurations had any influence on Euclid's theoretical attitude towards superposition.

And so, Miller's work does not provide a wholly convincing account of the *Elements*' longstanding reputation (in his words) as "the gold standard for careful reasoning and mathematical rigor." **FG** presupposes a mathematical conception of curves which did not emerge fully until the end of the nineteenth century. These shortcomings, however, do not diminish Miller's great achievement in leading us to a more convincing account. We cannot understand Euclid's method without understanding his use of geometric diagrams. Miller's first steps towards formalizing this use are giant, innovative ones along a path that is standardly thought to be completely blocked off. The received view portrays Euclid's arguments as hopelessly informal *because* they rely on diagrams. Miller fully developed formal account explodes this view, and provides a basis from which further investigation and discussion can fruitfully proceed.