Computing the stability number of a graph via linear and semidefinite programming

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Outline

1. Stable set problem
2. Stable set via copositive programming
3. Approximating the copositive cone
   - Polyhedral approximations
   - SOS approximations
   - Examples

Stable Set Problem

$G = (V, E)$ loopless undirected graph.
$S \subseteq V$ is *stable* if $\{i, j\} \notin E$ for all $i, j \in S$.

Stability number

$$\alpha(G) := \max\{|S| : S \subseteq V \text{ stable in } G\}.$$

Computing $\alpha(G)$ is NP-hard, even NP-hard to approximate.

Stable set via copositive programming

Copositive cone:

$$\mathcal{C}_n = \{M \in \mathbb{S}^n : x^{\mathrm{T}} M x \geq 0, \text{ for all } x \in \mathbb{R}^n_+\}.$$

Theorem (De Klerk & Pasechnik)

Let $n = |V|$. Then

$$\alpha(G) = \min\{\lambda : \lambda(I + A(G)) - \mathbb{1} \in \mathcal{C}_n\}.$$

Related to Motzkin & Straus' Theorem:

$$\frac{1}{\alpha(G)} = \min\{x^{\mathrm{T}}(I + A(G))x : x \in \Delta\},$$

where $\Delta = \{x \geq 0 : \sum_{i=1}^n x_i = 1\}$.

Theme

The cone $\mathcal{C}_n$ is difficult to handle.

Approximate $\mathcal{C}_n$, and consequently $\alpha(G)$, via positive polynomials.

Polyhedral approximations to $\mathcal{C}_n$

By Pólya's Theorem, if $M \in \text{int}(\mathcal{C}_n)$ then for some $r \in \mathbb{N}$

$$\left(\sum_{i=1}^{n} x_i\right)^r x^{\mathrm{T}} M x \text{ has non-negative coefficients.}$$

De Klerk & Pasechnik:

$$\mathcal{C}_n^r := \left\{ M \in \mathbb{S}^n : \left(\sum_{i=1}^{n} x_i\right)^r x^{\mathrm{T}} M x \text{ has non-negative coefficients} \right\},$$

and

$$\zeta^{(r)}(G) := \min\{\lambda : \lambda(I + A(G)) - \mathbb{1} \in \mathcal{C}_n^r\}.$$

Polyhedral approximations to $\mathcal{C}_n$

Observe

- $\mathcal{C}_n^r$ is polyhedral, so $\zeta^{(r)}(G)$ can be computed via LP.
- By Pólya's Thm $\mathcal{C}_n^r \uparrow \mathcal{C}_n$, so $\zeta^{(r)}(G) \downarrow \alpha(G)$.
- How fast does it converge?

Closed-form for $\zeta^{(r)}(G)$

Theorem (Vera & P)

Assume $r+2 = u\alpha(G) + v$ *where* $0 \le v < \alpha(G)$. *Then*

$$\zeta^{(r)}(G) = \frac{\binom{r+2}{2}}{\binom{u}{2}\alpha(G) + vu}.$$

Corollary

If $\alpha(G) > 1$ *then* $\zeta^{(r)}(G) > \alpha(G)$.

Corollary (De Klerk & Pasechnik)

- $\zeta^{(r)}(G) < \infty$ *if and only if* $r \ge \alpha(G) - 1$.
- $\lfloor \zeta^{(r)}(G) \rfloor = \alpha(G)$ *if and only if* $r \ge \alpha(G)^2 - 1$.

SOS approximations to $\mathcal{C}_n$

Parrilo, 2000:

Put $x \circ x := \begin{bmatrix} x_1^2 & \cdots & x_n^2 \end{bmatrix}^{\mathrm{T}}$.
Observe:

$$\begin{array}{rcl} M \in \mathcal{C}_n^r & \Leftrightarrow & \left(\sum_{i=1}^n x_i\right)^r x^{\mathrm{T}} M x = \sum_{|\beta|=r+2} c_\beta x^\beta, \; c_\beta \geq 0 \\ & \Rightarrow & \left(\sum_{i=1}^n x_i^2\right)^r (x \circ x)^{\mathrm{T}} M (x \circ x) \text{ is sos} \\ & \Rightarrow & M \in \mathcal{C}_n. \end{array}$$

Define

$$\mathcal{K}_n^r := \left\{ M \in \mathbb{S}^n : \left(\sum_{i=1}^n x_i^2 \right)^r (x \circ x)^{\mathrm{T}} M (x \circ x) \text{ is sos} \right\}.$$

SDP approximations to $\alpha(G)$

De Klerk & Pasechnik, 2002:

$$\vartheta^{(r)}(G) := \min\{\lambda : \lambda(I + A(G)) - \mathbb{1} \in \mathcal{K}_n^r\}$$

- $\mathcal{K}_n^r \uparrow \mathcal{C}_n$ also, and consequently $\vartheta^{(r)}(G) \downarrow \alpha(G)$.
- Each $\vartheta^{(r)}(G)$ can be computed via SDP.
- How much better than $\zeta^{(r)}(G)$ is each $\vartheta^{(r)}(G)$?

SDP approximations to $\alpha(G)$

For $v \in V$ define

$$v^{\perp} = \{v\} \cup \Gamma(v),$$

where $\Gamma(v) = \{u \in V : \{u, v\} \in E\}$.
Observe

$$\alpha(G) = 1 + \max_{v \in V} \alpha(G \setminus v^{\perp}).$$

Theorem (De Klerk & Pasechnik)

$$\vartheta^{(1)}(G) \leq 1 + \max_{v \in V} \vartheta^{(0)}(G \setminus v^{\perp}).$$

Thus $\vartheta^{(1)}(G) = \alpha(G)$ for certain graphs.
In particular, $\vartheta^{(1)}(G) = \alpha(G)$ if $\alpha(G) \leq 2$.

SDP approximations to $\alpha(G)$

Conjecture (De Klerk & Pasechnik)

If $r \geq \alpha(G) - 1$,

$$\vartheta^{(r)}(G) = \alpha(G).$$

Gvozdenović & Laurent 2004/2005, Vera & P 2004/2005:
Partial solutions to the conjecture.

Advertisement: M. Laurent's talk tomorrow.

A weaker SOS approximation to $\mathcal{C}_n$

Proposition (Zuluaga, Vera, P., 2003)

$M \in \mathcal{K}_n^r$ if and only if

$$\left(\sum_{i=1}^n x_i\right)^r x^{\mathrm{T}} M x = \sum_{|\beta| \le r+2} g_\beta(x) x^\beta$$

where each g_β is sos.

Define

$$\mathcal{Q}_n^r := \left\{ M \in \mathbb{S}^n : \left(\sum_{i=1}^n x_i\right)^r x^{\mathrm{T}} M x = \sum_{|\beta|=r} q_\beta(x) x^\beta \right\},$$

Each $q_\beta(x)$ of the form $x^{\mathrm{T}}(P+N)x$ with $P \succeq 0$, $N \ge 0$.

A weaker SOS approximation to $\mathcal{C}_n$

By construction, $\mathcal{C}_n^r \subseteq \mathcal{Q}_n^r \subseteq \mathcal{K}_n^r$.

Proposition

$\mathcal{Q}_n^0 = \mathcal{K}_n^0, \; \mathcal{Q}_n^1 = \mathcal{K}_n^1.$

In general, $\mathcal{Q}_n^r \subsetneq \mathcal{K}_n^r,$ for $r \geq 2$.

Define

$$\nu^{(r)}(G) := \min\{\lambda : \lambda(I + A(G)) - \mathbb{1} \in \mathcal{Q}_n^r\}.$$

A weaker SOS approximation to $\mathcal{C}_n$

Observe:

- $\nu^{(r)}(G) \downarrow \alpha(G)$ because $\mathcal{C}_n^r \subseteq \mathcal{Q}_n^r$
- $\nu^{(r)}(G) \geq \vartheta^{(r)}(G)$ because $\mathcal{Q}_n^r \subseteq \mathcal{K}_n^r$.
- $\nu^{(r)}(G)$ can be computed via SDP.
- The SDP for $\nu^{(r)}(G)$ is simpler than that for $\vartheta^{(r)}(G)$.

Theorem (Vera & P)

For $r = 1, 2, 3$

$$\nu^{(r)}(G) \le r + \max_{S \subseteq V \text{ stable}, \, |S| = r} \nu^{(0)}(G \setminus S^{\perp}).$$

Corollary

For $\alpha(G) \le 5$

$$\nu^{(\alpha(G)-1)}(G) = \alpha(G).$$

Partial result for a **stronger** version of the conjecture.

Gvozdenović & Laurent 2004/2005: show a stronger version of the above.

Examples

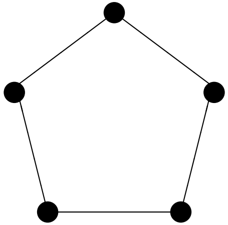
- If $\alpha(G) = \chi(\bar{G})$ then $\vartheta^{(0)}(G) = \alpha(G)$

  Direct verification.
  Let $V_1, \dots, V_\chi$ be a vertex coloring of $\bar{G}$. Then

$$x^{\mathrm{T}}(\chi \cdot (I + A(G)) - \mathbb{1})x = \frac{1}{2}\sum_{1 \le j < k \le \chi}\left(\sum_{i \in V_j} x_i - \sum_{i \in V_k} x_i\right)^2 + \chi \cdot q(x),$$

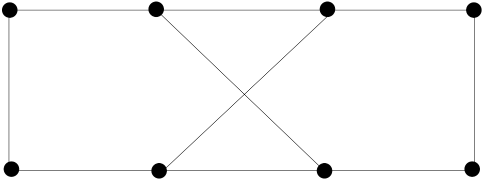
  where $q(x)$ has non-negative coefficients. $\square$

Examples



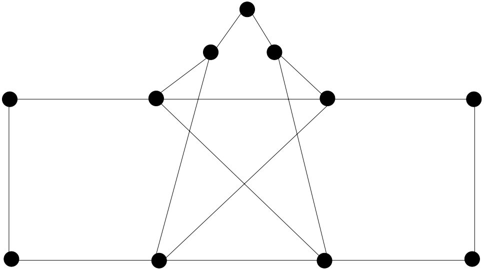
Smallest graph G with $\nu^{(0)}(G) = \vartheta^{(0)}(G) > \alpha(G) = 2$

Examples



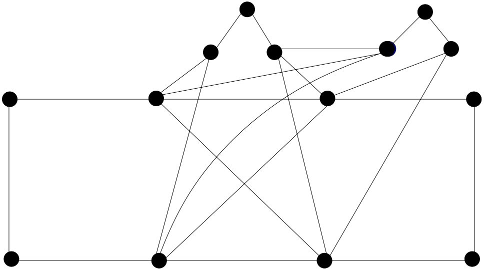
Smallest graph G with $\nu^{(1)}(G) = \vartheta^{(1)}(G) > \alpha(G) = 3$

Examples



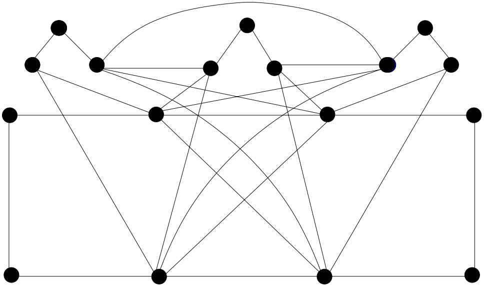
Smallest graph G with $\nu^{(2)}(G) > \alpha(G) = 4$

Examples



Smallest graph G with $\nu^{(3)}(G) > \alpha(G) = 5$

Examples



Smallest graph G with $\nu^{(4)}(G) > \alpha(G) = 6$?

At least $\vartheta^{(2)}(G) > \alpha(G)$.

Concluding Remarks

- Formulation of $\alpha(G)$ in terms of $\mathcal{C}_n$
- Approximations for $\alpha(G)$ via approximations for $\mathcal{C}_n$
- Results on the speed of convergence of these approximations
- Conjecture on the rank of the SDP approximations still open.