

On the block-structured distance to ill-posedness

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Well-posedness

Let X, Y be Euclidean spaces, and $C \subseteq X$ be a closed convex cone.

Definition: $A \in L(X, Y)$ is *well-posed* if

$$AC = Y.$$

Notice

- A well-posed iff

$$Ax = b, x \in C$$

feasible for all $b \in Y$.

- If $C = X = Y$ then

$$A \text{ well-posed} \Leftrightarrow A \text{ non-singular.}$$

Write

$A \in \mathcal{W}$ if A well-posed,
 $A \in \mathcal{I}$ if A ill-posed.

Thm 1 (Eckart-Young) Assume $A \in L(X, X)$ is non-singular. Then

$$\inf\{\|\Delta A\| : A + \Delta A \text{ is singular}\} = \frac{1}{\|A^{-1}\|} = \sup\{\delta : \delta\mathbb{B}_X \subseteq A\mathbb{B}_X\}.$$

($\mathbb{B}_X := \{x \in X : \|x\| \leq 1\}$.)

Thm 2 (Renegar) Assume $A \in \mathcal{W}$. Then

$$\inf\{\|\Delta A\| : A + \Delta A \in \mathcal{I}\} = \sup\{\delta : \delta\mathbb{B}_Y \subseteq A(\mathbb{B}_X \cap C)\}.$$

($\mathbb{B}_Y := \{y \in Y : \|y\| \leq 1\}$.)

Notice:

$$\sup\{\delta : \delta\mathbb{B}_Y \subseteq A(\mathbb{B}_X \cap C)\} = \frac{1}{\max_{v \in \mathbb{B}_Y} \inf\{\|x\| : Ax = v, x \in C\}}$$

sort of $\frac{1}{\|A^{-1}\|}$.

Problem:

Suppose $\Delta \subseteq L(X, Y)$ is given, e.g., some sparsity pattern.

Characterize

$$\text{dist}_{\Delta}(A, \mathcal{I}) := \inf\{\|B\|_{\Delta} : B \in \Delta, A + B \in \mathcal{I}\}.$$

Block-structure

Suppose $X_i \subseteq X$, $Y_i \subseteq Y$, $i = 1, \dots, k$. Let

$$\Delta := \left\{ B : B = \sum B_i, B_i \in L(X_i, Y_i) \right\},$$

and for $B = \sum B_i \in \Delta$, let

$$\|B\|_{\Delta} := \max_i \|B_i\|.$$

Thm 3 (P.) Assume $A \in \mathcal{W}$. Then

$$\text{dist}_{\Delta}(A, \mathcal{I}) = \frac{1}{\phi(A)},$$

where

$$\phi(A) = \max_{v^j \in \mathbb{B}_{Y_j}} \inf \left\{ \max_i \frac{\|x_i\|}{z_i} : z > 0, Ax = \sum z_j v^j, x \in C \right\}.$$

$\phi(A)$: sort of “ $\|A^{-1}\|$ ”.

Special case ($k = 1$):

$$\begin{aligned}\phi(A) &= \max_{v \in \mathbb{B}_{Y_1}} \inf \{ \|x_1\| : Ax = v, x \in C \} \\ &= \frac{1}{\sup\{\delta : \delta \mathbb{B}_{Y_1} \subseteq \{Ax : x \in C, \|x_1\| \leq 1\}\}}\end{aligned}$$

Thus,

$$\begin{aligned}\inf\{\|B\| : B \in L(X_1, Y_1), A + B \in \mathcal{I}\} &= \\ &= \sup\{\delta : \delta \mathbb{B}_{Y_1} \subseteq \{Ax : x \in C, \|x_1\| \leq 1\}\}.\end{aligned}$$

Can recover Thm 2 (Renegar):

$$\begin{aligned}\inf\{\|\Delta A\| : A + \Delta A \in \mathcal{I}\} &= \\ &= \sup\{\delta : \delta \mathbb{B}_Y \subseteq A(\mathbb{B}_X \cap C)\}.\end{aligned}$$

Sketch of the proof of Thm 3.

Special case $k = 1$.

Alternative (separation):

$$A \in \mathcal{I} \Leftrightarrow \exists y \neq 0 \text{ s.t. } A^*y \in C^*.$$

Norm-duality (following Borwein):

$$\phi(A) = \max_{v \in \mathbb{B}_{Y_1}} \inf \{ \|x_1\| : Ax = v, x \in C \} = \\ \sup_{u \in \mathbb{B}_{X_1^*}, u \neq 0} \sup \left\{ \frac{\|y_1\|}{\|u\|} : A^*y + u \in C^* \right\}.$$

RHS: sort of “ $\|A^{-*}\|$ ”.

Rank-one construction:

Come up with $u \in X_1^*$, $v \in Y_1$ such that

$$A + v\langle u, \cdot \rangle \in \mathcal{I}.$$

General case.

Alternative: $A \in \mathcal{I} \Leftrightarrow \exists y \neq 0$ s.t. $A^*y \in C^*$.

Structured norm-duality:

$$\phi(A) = \sup_{w^j \in \mathbb{B}_{X_j^*}} \sup \left\{ \min_{i, u^i \neq 0} \frac{\|y_i\|}{\|u^i\|} : A^*y + \sum u^j \in C^* \right\}.$$

RHS: sort of “ $\|A^{-*}\|$ ”.

Rank- k construction:

Come up with $u^i \in X_i^*$, $v^i \in Y_i$ such that

$$A + \sum v^i \langle u^i, \cdot \rangle \in \mathcal{I}.$$

Componentwise distance

Assume $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, $E \in \{0, 1\}^{m \times n}$ and

$$\Delta = \{B \in \mathbb{R}^{m \times n} : B \text{ has sparsity pattern } E\}.$$

For $B \in \Delta$, let $\|B\|_{\Delta} = \max |B_{ij}|$.

Can sharpen Thm 3.

Thm 4 (P.) Assume $A \in \mathcal{W}$. Then

$$\text{dist}_{\Delta}(A, \mathcal{I}) = \frac{1}{\Phi(A)},$$

where

$$\Phi(A) = \max_S \inf_{x, z} \left\{ \max_{i=1 \dots n} \frac{|x_i|}{z_i} : Ax = SBz, x \in C, z > 0 \right\},$$

max over signature matrices

$$\{S \in \mathbb{R}^{m \times m} : |S| = I_m\}.$$

More abstract setting

Let $F : X \rightrightarrows Y$ be a set-valued mapping.

Define $\text{graph}(F) \subseteq X \times Y$, $F^{-1} : Y \rightrightarrows X$, and $\|F\|$ as follows

$$\text{graph}(F) := \{(x, y) : y \in F(x)\},$$

$$x \in F^{-1}(y) \Leftrightarrow y \in F(x),$$

$$\|F\| = \sup_{x \in \mathbb{B}_X} \inf\{\|y\| : y \in F(x)\}.$$

($\mathbb{B}_X := \{x \in X : \|x\| \leq 1\}$.)

A set-valued mapping F is *sublinear* if $\text{graph}(F)$ is a convex cone.

- Sublinear mappings a.k.a. *convex processes*.
- Sublinear mappings subsume conic systems.

Thm 5 (Lewis) Assume F is a sublinear mapping with closed graph. Then

$$\inf\{\|G\| : G \in L(X, Y), F+G \text{ not surj}\} = \frac{1}{\|F^{-1}\|}.$$

Block-structured version:

Thm 6 (Lewis, also P.) Assume F is a sublinear mapping with closed graph. Then

$$\inf\{\|B\|_{\Delta} : B \in \Delta, F + B \text{ not surj}\} = \frac{1}{\|F^{-1}\|_{\Delta}^{-}}.$$

where

$$\|F^{-1}\|_{\Delta}^{-} = \sup_{v^j \in \mathbb{B}_{Y_j}} \inf \left\{ \max_i \frac{\|x_i\|}{z_i} : z > 0, x \in F^{-1}\left(\sum z_j v^j\right) \right\}.$$

Adjoint: Define $F^* : Y^* \rightrightarrows X^*$ by

$$u \in F^*(v) \Leftrightarrow \langle v, y \rangle \geq \langle u, x \rangle \quad \forall y \in F(x).$$

Sketch of the proof of Thm 6.

Alternative (separation):

$$F \text{ not surjective} \Leftrightarrow \exists v \neq 0 \text{ s.t. } 0 \in F^*(v).$$

Structured norm-duality (following Borwein):

Thm 7 (P.) Assume F a sublinear mapping with closed graph. Then

$$\|F^{-1}\|_{\Delta}^{-} = \|F^{-*}\|_{\Delta}^{+},$$

where

$$\|F^{-*}\|_{\Delta}^{+} = \sup_{u^j \in \mathbb{B}_{X_j^*}} \sup \left\{ \min_{i, u^i \neq 0} \frac{\|v_i\|}{\|u^i\|} : z > 0, v \in F^{-*} \left(\sum z_j u^j \right) \right\}.$$

Rank- k construction.

For X, Y Banach spaces.

Good news:

Structured norm-duality holds.

Bad news:

Alternative step fails:

F not surj $\not\Rightarrow \exists v \neq 0$ s.t. $0 \in F^*(v)$.

Thm 6 (structured case) does not hold.

Good news:

Dual counterpart holds.

Definition:

$$F^* \text{ is } \Delta\text{-singular iff } \|F^{-*}\|_{\Delta}^{\dagger} = \infty.$$

For X, Y Banach spaces:

Thm 8 (P.) *Assume $F^* : Y^* \rightrightarrows X^*$ is not Δ -singular. Then*

$$\inf\{\|B^*\|_{\Delta} : F^* + B^* \text{ is } \Delta\text{-sing}\} = \frac{1}{\|F^{-*}\|_{\Delta}^{\dagger}}.$$

Good news:

Under suitable circumstances

$$F \text{ is not surj} \Leftrightarrow F^* \text{ is } \Delta\text{-singular.}$$

Thus, amended version of Thm 6 holds for Banach spaces X, Y .

In particular, Thm 5 (unstructured case) holds for Banach spaces X, Y (Lewis).

Conclusions

- Eckart-Young identity extends to:
 - * conic systems and sublinear mappings
 - * block-structured perturbations.
- Three main ideas:
 - * alternative
$$A \in \mathcal{I} \Leftrightarrow \exists y \neq 0 \text{ s.t. } A^*y \in C^*$$
 - * norm-duality
$$\|A^{-1}\| = \|A^{-*}\|$$
 - * low-rank construction.
- Subtleties in infinite-dimensions.
- Future work:
 - * other types of structure, e.g., Toeplitz (cf. Rump)
 - * structured metric regularity (cf. Dontchev et al.)