Computing arbitrage upper bounds on basket options in the presence of bid-ask spreads

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Abstract

We study the problem of computing the sharpest static-arbitrage upper bound on the price of a European basket option, given the bid-ask prices of vanilla call options in the underlying securities. We show that this semi-infinite problem can be recast as a linear program whose size is linear in the input data size. These developments advance previous related results, and enhance the practical value of static-arbitrage bounds as a pricing technique by taking into account the presence of bid-ask spreads. We illustrate our results by computing upper bounds on the price of a DJX basket option. The MATLAB code used to compute these bounds is available online at www.andrew.cmu.edu/user/jfp/.

Keywords: Option pricing, European options, incomplete markets, arbitrage bounds, linear programming

1. Introduction

Computing bounds for option prices under incomplete market conditions or under incomplete knowledge of the distribution of the price of the underlying assets is a widely studied pricing techniques, where in contrast to parametric pricing techniques, such as Monte Carlo simulations, strong assumptions about the underlying

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asset price distribution are not required. These type of bounds provide a mechanism for checking consistency of prices (see, e.g., De la Pena, Ibragimov, and Jordan (2004); Hobson, Laurence, and Wang (2005b,a)), and provide *robust* estimates for option prices in incomplete market conditions, or regardless of any model specifics. Also, these bounds are useful when the number of underlying assets makes the computation of parametric prices numerically challenging. Here, we study the problem of computing arbitrage bounds; that is, computing bounds on the price of an option given the only assumption of absence of arbitrage, and information about prices of other options on the same underlying assets. More specifically, we study the problem of computing the sharpest upper bound on the price of a European basket option, given the only assumption of absence of arbitrage, and information on the bid-ask prices of vanilla European call options on the same underlying assets and with the same maturity. Bounds of this type are called *static-arbitrage bounds.*

The computation of sharp static-arbitrage upper bounds can be formulated as the problem of finding the least expensive portfolio of cash and the options with known prices whose combined payoff super-replicates the payoff of the new basket option of interest (see, e.g. d'Aspremont and El Ghaoui (2006), Hobson et al. (2005b)). This problem has received a fair amount of attention in recent years. Of particular relevance to our work are the recent articles by Albrecher, Mayer, and Schoutens (2008); d'Aspremont and El Ghaoui (2006); Davis and Hobson (2007); Hobson et al. (2005b,a); Laurence and Wang (2005, 2008, 2009); and Peña, Vera, and Zuluaga (2010). In these articles, it is always assumed that the options can be bought and sold at the same price in the formulation of the static-arbitrage bound problem. In practice, the price at which an investor buys the option, i.e., the *ask* price, is higher than the price at which the investor can sell the option, i.e., the *bid* price. This gives rise to the so-called bid-ask spread. Using bid and ask prices in the computation of the super-replicating strategy gives a more practical value to the static-arbitrage pricing approach. In particular, this resolves a major limitation in previous approaches (see, e.g., d'Aspremont and El Ghaoui (2006), Hobson et al. (2005b)) that used mid-market prices (e.g., the average of the bid and ask prices) as the "nominal" option prices. Such approximation systematically underestimates the actual buying prices and overestimates the actual selling prices. It is then not surprising that the market data used in d'Aspremont and El Ghaoui (2006), Hobson et al. (2005b) requires a fair amount of "cleaning" to rule out apparent arbitrage opportunities created by these estimates (see Hobson et al. (2005b, Section 6.2)). By contrast, the arbitrage bound formulations considered here take into account bid-ask spreads and do not suffer from this limitation.

In this article we undertake a novel approach to the static-arbitrage upper bound problem based entirely on linear programming duality. The foundational block of our work is the construction of an efficient (linear-size) polyhedral description for the set of *super-replicating portfolios*, that is, the set of portfolios of cash and the given options whose payoff super-replicates the basket option's payoff. We show that the set of super-replicating portfolios is a projection of a polyhedron whose description only requires a number of variables and constraints that is linear in the number of given options (see Lemma 1). Although it is intuitively clear that the set of super-replicating portfolios admits a polyhedral description, straightforward attempts to do so yield intractable descriptions that require a number of constraints and variables that is exponentially large in the number of given option prices.

We note that the computation of static-arbitrage *lower* bounds poses a different set of challenges as the nature of *sub-replicating portfolios* is fundamentally different from that of the super-replicating portfolios. The different nature of the upper and lower bound computation has been recognized previously, as it was apparent that the computation of the upper bounds was more tractable (see d'Aspremont and El Ghaoui (2006); Hobson et al. (2005a,b)). In Peña et al. (2010), we present some results for the computation of static-arbitrage lower bounds that are similar in spirit to those discussed herein.

The paper is organized as follows. Section 2 formally presents the problem of computing sharp static-arbitrage upper bounds on a basket option, given the bid-ask prices of vanilla call options on the underlying assets. Also, we present the main building block of our approach; namely, an efficient polyhedral description of the super-replicating portfolios. The latter yields the first efficient linear programming formulation for the computation of static-arbitrage upper bounds that incorporates bid-ask spreads. In Section 3, we provide numerical experiments to illustrate some of our results; namely, we compute bounds of the price of a DJX basket option. The `MATLAB` code used to compute these bounds is available online at `www.andrew.cmu.edu/user/jfp/`. Finally, Section 4 presents the proofs of the results in the article.

2. Static-arbitrage upper bounds with bid-ask spreads

In this section we present an efficient linear programming formulation for the static-arbitrage bound problem that takes into account bid-ask spreads in the prices of the known options. Previous approaches to the computation of arbitrage bounds (see, e.g., d'Aspremont and El Ghaoui (2006); Hobson et al. (2005a,b)) ignore this important feature and simply assume that the known options can be bought and sold at a mid-market price. This constitutes a major practical limitation as these mid-market prices are rarely arbitrage-free. One of our numerical examples in Section 3 illustrates this phenomenon.

Consider the problem of computing a sharp upper *static-arbitrage* bound on the price of a European basket option, given information on the bid-ask prices of European vanilla options, without making any assumptions other than the absence of arbitrage. This problem can be formulated as the following optimization problem:

$$
\begin{array}{ll}
\inf\limits_{z,y,\overline{y},\underline{y}} & z + \sum_{i=1}^{n}\sum_{j=0}^{m} \overline{p}_{ij}\overline{y}_{ij} - \sum_{i=1}^{n}\sum_{j=0}^{m} \underline{p}_{ij}\underline{y}_{ij} \\
\text{s.t.} & z + \sum_{i=1}^{n}\sum_{j=0}^{m} y_{ij}(s_i - K_{ij})^+ \geq \left(\sum_{i=1}^{n} \omega_i s_i - \kappa\right)^+ \quad \text{for all } s \in \mathbb{R}^n_+ \\
& y = \overline{y} - \underline{y} \\
& y \in \mathbb{R}^{n\times(m+1)} \\
& \overline{y}, \underline{y} \in \mathbb{R}^{n\times(m+1)}_+ \\
& z \in \mathbb{R}.
\end{array}
\tag{1}
$$

Above, the multidimensional variable s represents the possible prices of the n underlying assets (at maturity) in the basket. The constants $K_{ij} \in \mathbb{R}$, $i = 1, \ldots, n$, $j = 0, \ldots, m$, represent the strike price of the call options with payoff $(s_i - K_{ij})^+$ whose given ask (buying) and bid (selling) prices are $p^+_{ij} \geq p^-_{ij}$ respectively. The vector $\omega \in \mathbb{R}^n$ and constant $\kappa \in \mathbb{R}$ represent the weights and strike of the basket option with payoff $(\sum_{i=1}^{n} \omega_i s_i - \kappa)^+$ whose price we want to bound. Notice that the assumption on the same number of options m per asset can be made without loss of generality: If one of the assets has fewer than m options, we can artificially increase the number of known options to m by repeating one of the options.

Problem (1) has a natural financial interpretation: It finds the cheapest portfolio of positions in cash (z) and in call options (y_{ij}) with payoff $(s_i - K_{ij})^+$, $i = 1, \ldots, n$, $j = 0, \ldots, m$ that super-replicates the payoff of the basket option with payoff $(\sum_{i=1}^{n} \omega_i s_i - \kappa)^+$.

Following d'Aspremont and El Ghaoui (2006), we implicitly assume that all the options have the same maturity, and that the risk-free interest rate is zero; or equivalently, we compare the prices in the forward market (at maturity).

2.1. Super-replication of a linear payoff

Now we present the main building block of our approach; namely, an efficient polyhedral description of the super-replicating constraint (first constraint) in problem (1). The latter yields the first efficient linear programming formulation for the computation of static-arbitrage upper bounds that incorporates bid-ask spreads.

For ease of notation, let us first rewrite problem (1) in "vector form". That is,

$$
\begin{array}{ll}
\inf\limits_{z,y,\overline{y},\underline{y}} & z + \sum\limits_{j=0}^{m} \overline{p}^j \cdot \overline{y}^j - \sum\limits_{j=0}^{m} \underline{p}^j \cdot \underline{y}^j \\
\text{s.t.} & z + \sum\limits_{j=0}^{m} y^j \cdot (s - K^j)^+ \geq (\omega \cdot s - \kappa)^+ \quad \text{for all } s \in \mathbb{R}^n_+ \\
& y = \overline{y} - \underline{y} \\
& y \in \mathbb{R}^{n \times (m+1)} \\
& \overline{y}, \underline{y} \in \mathbb{R}^{n \times (m+1)}_+, \\
& z \in \mathbb{R}.
\end{array} \tag{2}
$$

where a^j denotes the vector $[a_{ij}]_{i=1,\ldots,n}$, and $(\cdot)$ denotes the dot (inner) product of vectors.

Now, assume $K = \begin{bmatrix} K^0 & K^1 & \cdots & K^m \end{bmatrix} \in \mathbb{R}^{n \times (m+1)}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$ are given. Define the set of *super-replicating strategies* $SR(K, b, c)$ as follows

$$
\begin{aligned}
SR(K, b, c) := \{(y, z) = (y^0, y^1, \ldots, y^m, z) \in \mathbb{R}^{n \times (m+1)} \times \mathbb{R} : \\
z + \sum_{j=0}^{m} y^j \cdot (s - K^j)^+ \geq b \cdot s - c \text{ for all } s \in \mathbb{R}^n_+\}.
\end{aligned} \tag{3}
$$

The set $SR(K, b, c)$ is the set of combinations of the call options $(s_i - K_{ij})^+$ and cash that super-replicate the linear payoff $b \cdot s - c$.

Using the set $SR(K, b, c)$, the optimal super-replication problem (2) can be written as:

$$
\begin{array}{ll}
\inf\limits_{z,y,\overline{y},\underline{y}} & z + \sum\limits_{j=0}^{m} \overline{p}^j \cdot \overline{y}^j - \sum\limits_{j=0}^{m} \underline{p}^j \cdot \underline{y}^j \\
\text{s.t.} & (y, z) \in SR(K, \omega, \kappa) \\
& (y, z) \in SR(K, 0, 0) \\
& y = \overline{y} - \underline{y} \\
& \overline{y}, \underline{y} \in \mathbb{R}^{n \times (m+1)}_+
\end{array} \tag{4}
$$

The key in showing that the super-replication problem (4) can be rewritten using a number of variables and constraints that is linear in the number of given options is Lemma 1 below, which states that the set $SR(K, b, c)$ is a projection of the *lifted* polyhedron $LSR(K, b, c)$. The latter is a set in a higher dimensional space with an efficient polyhedral description. Define $LSR(K, b, c)$ as the set of points

$(y, z, \gamma, \beta, \xi) \in \mathbb{R}^{n\times(m+1)} \times \mathbb{R} \times \mathbb{R}_+^{n\times(m+1)} \times \mathbb{R}_+^{n\times m} \times \mathbb{R}^n$ that satisfy

$$\begin{aligned}
&\sum_{j=0}^{i} y^j - b = \gamma^i - \beta^i, && i = 0, \ldots, m-1 \\
&\sum_{j=0}^{m} y^j - b = \gamma^m \\
&\sum_{j=0}^{i} K^j \circ y^j \leq \xi + K^i \circ \gamma^i - K^{i+1} \circ \beta^i, && i = 0, \ldots, m-1 \\
&\sum_{j=0}^{m} K^j \circ y^j \leq \xi + K^m \circ \gamma^m \\
&-z - c \leq -e \cdot \xi.
\end{aligned} \tag{5}$$

Here $u \circ v \in \mathbb{R}^n$ denotes the *Hadamard product* of $u, v \in \mathbb{R}^n$, i.e., $(u \circ v)_i = u_i v_i,\ i = 1, \ldots, n$, and $e \in \mathbb{R}^n$ is the vector of all ones. Note that the number of variables and constraints in the description of $LSR(K, b, c)$ is proportional to mn, i.e., to the number of known option prices.

Lemma 1. *Assume $\vec{0} = K^0 \leq K^1 \leq \cdots \leq K^m \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$, $c \in \mathbb{R}$ are given. Then $(y, z) \in SR(K, b, c)$ if and only if there exist $\gamma \in \mathbb{R}_+^{n\times(m+1)}$, $\beta \in \mathbb{R}_+^{n\times m}$, and $\xi \in \mathbb{R}^n$ such that $(y, z, \gamma, \beta, \xi) \in LSR(K, b, c)$.*

Lemma 1 enables us to recast (4) as a linear program whose number of variables and constraints is proportional to mn, i.e., to the number of known option prices.

Theorem 2. *The optimal super-replication problem* (4) *can be rewritten as*

$$\begin{array}{ll}
\displaystyle\min_{z,y,\overline{y},\underline{y},\gamma,\beta,\xi,\tilde{\gamma},\tilde{\beta},\tilde{\xi}} & \displaystyle z + \sum_{j=0}^{m} \left(\overline{p}^j \cdot \underline{y}^j - \underline{p}^j \cdot \underline{y}^j\right) \\
\text{s.t.} & (y, z, \gamma, \beta, \xi) \in LSR(K, \omega, \kappa) \\
& (y, z, \tilde{\gamma}, \tilde{\beta}, \tilde{\xi}) \in LSR(K, 0, 0) \\
& y = \overline{y} - \underline{y} \\
& \overline{y}, \underline{y} \in \mathbb{R}_+^{n\times(m+1)}.
\end{array} \tag{6}$$

Remark 1. *The polyhedral description given by Lemma 1 allows the efficient modeling of other features in the optimal super-replication problem such as other types of transaction costs as well as restrictions on the positions of the super-replication strategy. Some of these features are illustrated in our numerical experiments in Section 3. In particular, proportional transaction costs (see, e.g., Cornuejols and T ut unc u (2007, Section 8.1.2)) can be taken into account in (6) by including the relevant transaction cost term in the objective function in* (6).

3. Some numerical results

We present a numerical example that takes into account the presence of bid-ask spreads in option prices. We also discuss the possibility of adding diversification constraints to the super-replication strategy problem. Although these features are prevalent in real pricing problems, they were beyond the scope of previous approaches to static-arbitrage bounds.

Related numerical results are presented in d'Aspremont and El Ghaoui (2006); Hobson et al. (2005b), where the authors provide extensive numerical experiments comparing static-arbitrage pricing techniques and parametric pricing techniques (such as Monte Carlo simulations) for basket options.

3.1. Bid-ask prices

In the literature related to the computation of static-arbitrage bounds (see, e.g., d'Aspremont and El Ghaoui (2006); Davis and Hobson (2007); Hobson et al. (2005a,b); Laurence and Wang (2005, 2008, 2009); and Peña et al. (2010)), it is assumed that the options can be bought and sold at the same price. In practice, the price at which an investor buys the option, i.e., the *ask* price, is higher than the price at which the investor can sell the option, i.e., the *bid* price. This gives rise to the so-called bid-ask spread as can be observed in Table 1 and Table 2, which lists the prices of vanilla options on stocks in the DJX index as traded on May 17th, 2004 on the June contracts with maturity on June 18th, 2004. This dataset is similar to that of Hobson et al. (2005b, Section 6.2, Table 2). However, we have only included traded contracts (with volume greater than zero), for liquidity considerations. With the data in Table 1 and Table 2, we can use the linear programming formulation (6) in Section 2 to compute the cheapest super-replicating strategy for the DJX basket call option with strike price 80.00 taking into account the bid-ask spread. We obtain the super-replication strategy given in Table 3, which yields an upper bound of 19.8872. From market data, the best bid price for this option was 18.7, and the best ask price was 19.5. Table 3 provides the long (buy) positions on the call options with position different from zero in the super-replicating portfolio. In this particular experiment, the super-replicating portfolio does not contain any short (sell) positions.

As we mentioned in the Introduction, using bid and ask prices in the computation of the super-replicating strategy gives a more practical value to the static-arbitrage pricing approach. In particular, this resolves a major limitation in previous approaches d'Aspremont and El Ghaoui (2006); Hobson et al. (2005b) that used mid-market prices (e.g., the average of the bid and ask prices) as the "nominal" option prices. Such approximation systematically underestimates the actual buying prices and overestimates the actual selling prices. It is then not surprising that the market data used in d'Aspremont and El Ghaoui (2006); Hobson et al. (2005b) requires a fair amount of "cleaning" to rule out apparent arbitrage opportunities created by these estimates (see Hobson et al. (2005b, Section 6.2)). By contrast, the model herein that takes into account bid-ask spreads does not suffer from this limitation.

We note that although the super-replicating strategy in Table 3 contains only long positions, this does not mean that the bid-ask DJX option price upper bound of 19.8872 could be found by only using the ask (buy) prices as the option prices in the linear programming formulation of the problem (1) (i.e., by assuming that options can be both bought and sold at the ask prices). If this naive approach were attempted, the linear program would be unbounded, since the ask prices alone do not satisfy the arbitrage-free condition (see, e.g., Bertsimas and Popescu (2002)).

Figure 1 shows the resulting upper arbitrage bounds on the DJX basket option (taking into account the bid-ask spread) for other strike values, and compares the bounds with the best ask price for the options.

3.2. Diversifying the super-replicating strategy

Consider an investor looking at the strategy in Table 3, who wishes to create a super-replicating strategy that contains more positions in possible options, that is, a more diversified strategy. To obtain such a super-replicating strategy we add the following diversifying linear constraints to the LP formulation (6), which ensure that the portfolio will have a position in each *tier* of options:

$$e \cdot y^j \geq 0.05, j = 0, \ldots, m. \tag{7}$$

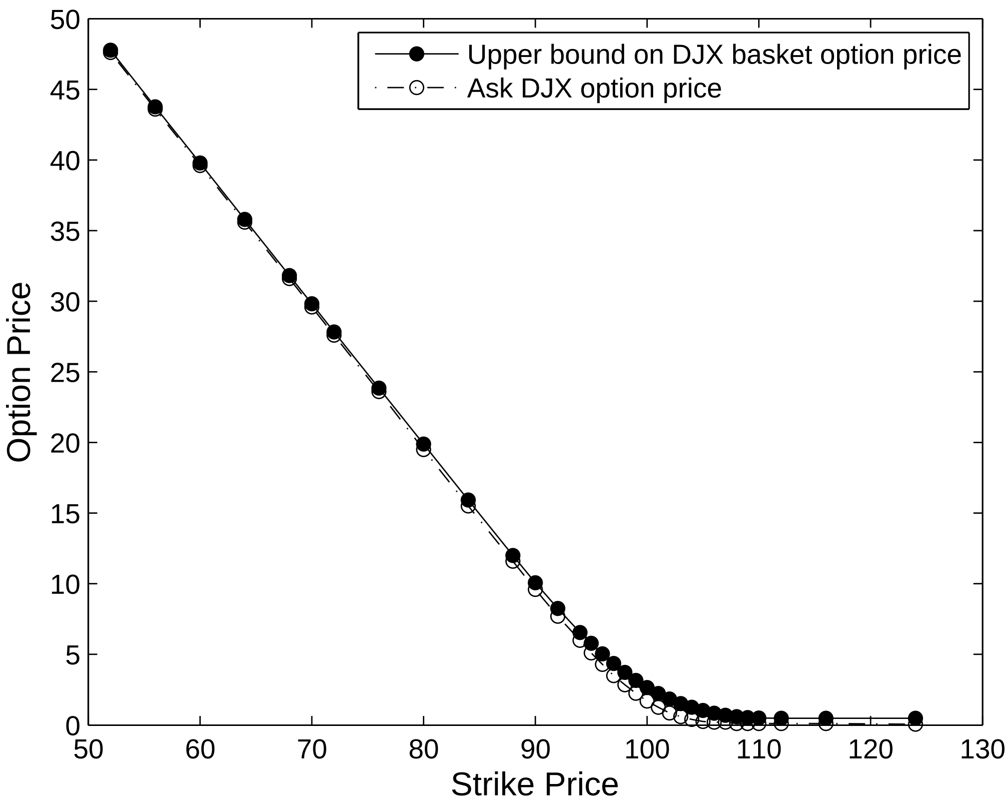
Above e represents the vector of all-ones. The solution to this *diversified* super-replicating strategy gives a portfolio whose cost is 19.9022, just 0.08% more expensive than the cheapest super-replicating strategy computed in Table 3. As Table 4 shows, such a strategy has the desired investor's diversification preference.

4. Proofs

4.1. Proof of Lemma 1.

The high level idea of the proof is to divide $\mathbb{R}^n_+$ in regions where each of the payoffs of the options in the problem is linear. Using Farkas' Lemma the set of super-replicating strategies in each region is a polyhedron. Thus the set of super-replicating strategies $SR(K, b, c)$ is just the intersection of all these polyhedra, again a polyhedron. Since the number of regions is exponential, the number of variables and constraints needed in a naive description of $SR(K, b, c)$ is exponential. The main point of the proof is that we can "collapse" first the variables and then the

Figure 1: The figure gives ask DJX index option prices trading on May 17th with maturity on June 18th, 2004, together with the corresponding upper bound option price computed using the data provided in Table 1.



constrains to obtain an efficient description, using only a linear number of variables and constraints.

Throughout this section we rely on the following convenient notation: Given a vector $v \in \mathbb{R}^n$ and a set of indices $I \subseteq \{1, \ldots, n\}$, we let v_I denote the subvector of v obtained by selecting the components of v indexed by I.

Proof of Lemma 1. Define the set of partitions $\mathcal{P}(n,m)$ of $\{1,\ldots,n\}$ as follows:

$$\mathcal{P}(n,m) := \left\{ \left(J^0, J^1, \ldots, J^m\right) : \bigcup_{i=0}^{m} J^i = \{1,\ldots,n\}, J^i \cap J^j = \emptyset \text{ for } i \neq j \right\}.$$

Given $J \in \mathcal{P}(n,m)$, define

$$P_J := \left\{ s : K^i_{J^i} \leq s_{J^i} \leq K^{i+1}_{J^i} \text{ for } i = 0,1,\ldots,m-1, \text{ and } s_{J^m} \geq K^m_{J^m} \right\}.$$

Since $\{P_J : J \in \mathcal{P}(n,m)\}$ is a partition of $\mathbb{R}^n_+$, it follows that $(y,z) \in SR(K,b,c)$ if and only if

$$z + \sum_{j=0}^{m} y^j \cdot (s - K^j)^+ \geq b \cdot s - c \text{ for all } s \in P_J \text{ for all } J \in \mathcal{P}(n,m). \tag{8}$$

From the construction of $\{P_J : J \in \mathcal{P}(n,m)\}$, it follows that each $(s_i - K^j_i)^+$ is linear on each P_J. Indeed, for $s \in P_J$ we have

$$\sum_{j=0}^{m} y^j \cdot (s - K^j)^+ = \sum_{i=0}^{m} \sum_{j=0}^{i} y^j_{J^i} \cdot (s_{J^i} - K^j_{J^i}).$$

Therefore, (8) is equivalent to

$$\sum_{i=0}^{m} \left(-b_{J^i} + \sum_{j=0}^{i} y^j_{J^i} \right) \cdot s_{J^i} \geq \sum_{i=0}^{m} \sum_{j=0}^{i} y^j_{J^i} \cdot K^j_{J^i} - z - c \text{ for all } s \in P_J \text{ for all } J \in \mathcal{P}(n,m).$$

By Farkas Lemma (see, e.g. Ziegler (1998, Proposition 1.9(i))), the latter holds if and only if for each $J \in \mathcal{P}(n,m)$ there exist $\gamma^{i,J}, \beta^{i,J} \in \mathbb{R}^{J^i}_+$, $i = 0,\ldots,m-1$, $\gamma^{m,J} \in \mathbb{R}^{J^m}_+$ such that

$$\begin{aligned}
& -b_{J^i} + \sum_{j=0}^{i} y^j_{J^i} = \gamma^{i,J} - \beta^{i,J}, && i = 0,\ldots,m-1 \\
& -b_{J^m} + \sum_{j=0}^{m} y^j_{J^m} = \gamma^{m,J} \\
& \sum_{i=0}^{m} \left(\sum_{j=0}^{i} y^j_{J^i} \cdot K^j_{J^i} \right) - z - c \leq \sum_{i=0}^{m-1} \left(K^i_{J^i} \cdot \gamma^{i,J} - K^{i+1}_{J^i} \cdot \beta^{i,J} \right) + K^m_{J^m} \cdot \gamma^{m,J}, && J \in \mathcal{P}(n,m).
\end{aligned} \tag{9}$$

We will first reduce the number of variables used in this description of $SR(K,b,c)$:

Claim 1. *Assume* $(y_0,\dots,y_m,z)\in\mathbb{R}^n\times\cdots\times\mathbb{R}^n\times\mathbb{R}$ *is given. Then there exist* $\gamma^{i,J},\beta^{i,J}\in\mathbb{R}_+^{J^i}$, $i=0,\dots,m-1$, $\gamma^{m,J}\in\mathbb{R}_+^{J^m}$ *for each* $J\in\mathcal{P}(n,m)$ *such that* (9) *holds if and only if there exist* $\gamma^i,\beta^i\in\mathbb{R}_+^n$, $i=0,\dots,m-1$ *and* $\gamma^m\in\mathbb{R}_+^n$ *such that*

$$-b+\sum_{j=0}^{i} y^j=\gamma^i-\beta^i, \qquad i=0,\dots,m-1$$
$$-b+\sum_{j=0}^{m} y^j=\gamma^m$$

$$\sum_{i=0}^{m}\left(\sum_{j=0}^{i} y^j_{J^i}\cdot K^j_{J^i}\right)-z-c\le\sum_{i=0}^{m-1}\left(K^i_{J^i}\cdot\gamma^i_{J^i}-K^{i+1}_{J^i}\cdot\beta^i_{J^i}\right)+K^m_{J^m}\cdot\gamma^m_{J^m},\quad J\in\mathcal{P}(n,m). \tag{10}$$

Then, we will reduce the number of constrains used in (10):

Claim 2. *Assume* $(y,z,\gamma,\beta)\in\mathbb{R}^{n\times(m+1)}\times\mathbb{R}\times\mathbb{R}_+^{n\times(m+1)}\times\mathbb{R}_+^{n\times m}$ *is given. Then* (10) *holds if and only if there exists* $\xi\in\mathbb{R}^n$ *such that* (5) *holds.*

Claim 1 and Claim 2 show that $(y,z)\in SR(K,b,c)$ if and only if there exist $\gamma\in\mathbb{R}_+^{n\times(m+1)}$, $\beta\in\mathbb{R}_+^{n\times m}$, and $\xi\in\mathbb{R}^n$ such that $(y,z,\gamma,\beta,\xi)\in LSR(K,b,c)$. $\square$

Now we present the proofs of Claim 1 and Claim 2.

Proof of Claim 1. It is straightforward to check (10) implies (9). Assume (9) holds. Define $\gamma^m\in\mathbb{R}_+^n$ and $\gamma^i,\beta^i\in\mathbb{R}_+^n$, $i=0,\dots,m-1$ as follows. Let $\bar{J}=(\emptyset,\dots,\emptyset,\{1,\dots,n\})$ and put

$$\gamma^m:=\gamma^{m,\bar{J}}.$$

Then the second equation in (10) holds. Consequently, for any $J\in\mathcal{P}(n,m)$ we have

$$\gamma^m_{J^m}=-b_{J^m}+\sum_{j=0}^{m} y^j_{J^m}=\gamma^{m,J}. \tag{11}$$

Next, fix $i\in\{0,\dots,m-1\}$. For each $\ell\in\{1,\dots,n\}$ define the partition $J[i,\ell]$ by

$$J[i,\ell]:=\operatorname*{arg\,max}_{\{J\in\mathcal{P}(n,r):l\in J^i\}}\left(K^i_\ell\gamma^{i,J}_\ell-K^{i+1}_\ell\beta^{i,J}_\ell\right).$$

Let $\gamma^i, \beta^i \in \mathbb{R}^n_+$ be defined by $\gamma^i_\ell = \gamma^{i,J[i,\ell]}_\ell$ and $\beta^i_\ell = \beta^{i,J[i,\ell]}_\ell$, $\ell \in \{1, \ldots, n\}$. From the first identity in (9), applied to $J = J[i, \ell]$, we get

$$-b_{J[i,\ell]^i} + \sum_{j=0}^{i} y^j_{J[i,l]^i} = \gamma^{i,J[i,l]} - \beta^{i,J[i,l]}.$$

In particular,

$$-b_\ell + \sum_{j=0}^{i} y^j_\ell = \gamma^{i,J[i,\ell]}_\ell - \beta^{i,J[i,\ell]}_\ell = \gamma^i_\ell - \beta^i_\ell.$$

This holds for $i \in \{0, \ldots, m-1\}$ and $\ell \in \{1, \ldots, n\}$ thus the first equation in (10) follows. It only remains to prove the last inequality in (10). To that end, fix $J \in \mathcal{P}(n,m)$. For $i = 0, \ldots, m-1$ and $\ell \in J^i$, the construction of $J[i,\ell]$ implies that

$$K^i_\ell \gamma^{i,J}_\ell - K^{i+1}_\ell \beta^{i,J}_\ell \le K^i_\ell \gamma^{i,J[i,\ell]}_\ell - K^{i+1}_\ell \beta^{i,J[i,\ell]}_\ell = K^i_\ell \gamma^i_\ell - K^{i+1}_\ell \beta^i_\ell.$$

Thus

$$K^i_{J^i} \cdot \gamma^{i,J} - K^{i+1}_{J^i} \cdot \beta^{i,J} \le K^i_{J^i} \cdot \gamma^i_{J^i} - K^{i+1}_{J^i} \cdot \beta^i_{J^i}.$$

Hence from the last inequality in (9) and (11) we get

$$\begin{aligned} \sum_{i=0}^{m} \left(\sum_{j=0}^{i} y^j_{J^i} \cdot K^j_{J^i} \right) - z - c \quad &\le \quad \sum_{i=0}^{m-1} \left(K^i_{J^i} \cdot \gamma^{i,J} - K^{i+1}_{J^i} \cdot \beta^{i,J} \right) + K^m_{J^m} \cdot \gamma^{m,J} \\ &\le \quad \sum_{i=0}^{m-1} \left(K^i_{J^i} \cdot \gamma^i_{J_i} - K^{i+1}_{J^i} \cdot \beta^i_{J_i} \right) + K^m_{J^m} \cdot \gamma^m_{J^m}, \end{aligned}$$

completing the proof of the claim. ☐

Proof of Claim 2. Assume (5) holds. Let $J \in \mathcal{P}(n,m)$ be given. From the third inequality in (5) we have

$$\sum_{j=0}^{i} K^j_{J^i} \cdot y^j_{J^i} \le e_{J^i} \cdot \xi_{J^i} + K^i_{J^i} \cdot \gamma^i_{J^i} - K^{i+1}_{J^i} \cdot \beta^i_{J^i},\ i = 0, \ldots, m-1.$$

Likewise, form the fourth inequality in (5) we have

$$\sum_{j=0}^{m} K^j_{J^m} \cdot y^j_{J^m} \le e_{J^m} \cdot \xi_{J^m} + K^m_{J^m} \cdot \gamma^m_{J^m}.$$

Adding all of these inequalities and rearranging terms, we get

$$\sum_{i=0}^{m}\left(\sum_{j=0}^{i} y^j_{J^i}\cdot K^j_{J^i}\right) - e\cdot\xi \le \sum_{i=0}^{m-1}\left(K^i_{J^i}\cdot\gamma^i_{J^i} - K^{i+1}_{J^i}\cdot\beta^i_{J^i}\right) + K^m_{J^m}\cdot\gamma^m_{J^m}.$$

From (5) $-z-c \le -e\cdot\xi$, and we get (10).

Now assume that (10) holds. For $\ell = 1,\dots,n$ let

$$\xi_\ell := \max\left\{\sum_{j=0}^{m} K^j_\ell y^j_\ell - K^m_\ell\gamma^m_\ell, \sum_{j=0}^{i} K^j_\ell y^j_\ell - K^i_\ell\gamma^i_\ell + K^{i+1}_\ell\beta^i_\ell : i=0,\dots,m-1\right\}.$$

This choice of ξ ensures that the first four constraints in (5) hold. Let $\bar{J}\in\mathcal{P}(n,m)$ be such that

$$\xi_{\bar{J}^m} = \sum_{j=0}^{m} K^j_{\bar{J}^m}\circ y^j_{\bar{J}^m} - K^m_{\bar{J}^m}\circ\gamma^m_{\bar{J}^m},$$

and

$$\xi_{\bar{J}^i} = \sum_{j=0}^{i} K^j_{\bar{J}^i}\circ y^j_{\bar{J}^i} - K^i_{\bar{J}^i}\gamma^i_{\bar{J}^i} + K^{i+1}_{\bar{J}^i}\beta^i_{\bar{J}^i} \text{ for } i=0,\dots,m-1.$$

Then, from (10) (applied to the partition $\bar{J}$) we have

$$\begin{aligned}
-z-c &\le -\sum_{i=0}^{m}\left(\sum_{j=0}^{i} y^j_{\bar{J}^i}\cdot K^j_{\bar{J}^i}\right) + \sum_{i=0}^{m-1}\left(K^i_{\bar{J}^i}\cdot\gamma^i_{\bar{J}^i} - K^{i+1}_{\bar{J}^i}\cdot\beta^i_{\bar{J}^i}\right) + K^m_{\bar{J}^m}\cdot\gamma^m_{\bar{J}^m} \\
&= -\sum_{i=0}^{m-1}\left(\sum_{j=0}^{i} y^j_{\bar{J}^i}\cdot K^j_{\bar{J}^i} - K^i_{\bar{J}^i}\cdot\gamma^i_{\bar{J}^i} + K^{i+1}_{\bar{J}^i}\cdot\beta^i_{\bar{J}^i}\right) - \sum_{j=0}^{m} y^j_{\bar{J}^m}\cdot K^j_{\bar{J}^m} + K^m_{\bar{J}^m}\cdot\gamma^m_{\bar{J}^m} \\
&= -\sum_{i=0}^{m}\sum_{\ell\in\bar{J}^i}\xi_\ell \\
&= -e\cdot\xi.
\end{aligned}$$

Hence the last constraint in (5) holds as well. This completes the equivalence between (5) and (10). $\square$

4.2. Proof of Theorem 2.

The optimal super-replication problem (2) can be written as:

$$\begin{array}{ll} \min\limits_{z,y,\overline{y},\underline{y}} & z + \sum_{j=0}^{m} (\overline{p}^j \cdot \overline{y}^j - \underline{p}^j \cdot \underline{y}^j) \\ \text{s.t.} & (y,z) \in SR(K,\omega,\kappa) \\ & (y,z) \in SR(K,0,0) \\ & y = \overline{y} - \underline{y} \\ & \overline{y}, \underline{y} \in \mathbb{R}_+^{n\times(m+1)}. \end{array} \tag{12}$$

From Lemma 1 it follows that (12) is equivalent to the following linear program

$$\begin{array}{ll} \min\limits_{z,y,\overline{y},\underline{y},\gamma,\beta,\xi,\tilde{\gamma},\tilde{\beta},\tilde{\xi}} & z + \sum_{j=0}^{m} (\overline{p}^j \cdot \overline{y}^j - \underline{p}^j \cdot \underline{y}^j) \\ \text{s.t.} & (y,z,\gamma,\beta,\xi) \in LSR(K,\omega,\kappa) \\ & (y,z,\tilde{\gamma},\tilde{\beta},\tilde{\xi}) \in LSR(K,0,0) \\ & y = \overline{y} - \underline{y} \\ & \overline{y}, \underline{y} \in \mathbb{R}_+^{n\times(m+1)}. \end{array}$$

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Table 1: CBOE data from May 17th, 2004 on June 2004 contracts expiring June 18. The table gives prices of call options traded (volume greater than zero) on May 17th for the 30 stocks underlying the DJX index basket option. For every stock, the first row corresponds to the different strike prices, and the second and third rows correspond to the respective ask and bid prices. The entry of 0.00 for each stock gives the close price of the stock, which can be considered as the forward option (call option with strike price zero) price.

	0.00	20.00	22.50	25.00	27.50			
MSFT	25.79	5.70	3.20	1.15	0.20			
	25.42	5.50	3.10	1.05	0.15			
	0.00	25.00	27.50	30.00	32.50	35.00		
AA	29.70	4.10	2.20	0.95	0.30	0.15		
	28.60	3.90	2.05	0.85	0.20	0.05		
	0.00	65.00	70.00	75.00				
AIG	70.15	5.40	2.10	0.45				
	69.22	5.30	2.00	0.40				
	0.00	47.50	50.00					
AXP	49.30	2.20	0.80					
	48.20	2.05	0.70					
	0.00	40.00	42.50	45.00				
BA	43.61	3.10	1.35	0.40				
	42.49	2.90	1.25	0.30				
	0.00	35.00	37.50					
VZ	36.74	1.50	0.40					
	35.68	1.40	0.30					
	0.00	60.00	70.00	75.00	80.00	85.00		
CAT	74.45	13.80	4.90	1.95	0.60	0.20		
	72.70	13.60	4.80	1.90	0.50	0.10		
	0.00	40.00	42.50	45.00				
DD	41.48	2.00	0.70	0.15				
	41.01	1.80	0.55	0.10				
	0.00	20.00	22.50	25.00	27.50			
DIS	22.99	3.00	1.05	0.20	0.10			
	22.69	2.95	0.90	0.15	0.00			
	0.00	25.00	27.50	30.00	32.50	35.00	37.50	
GE	30.06	5.10	2.70	0.85	0.15	0.05	0.05	
	29.68	4.90	2.60	0.75	0.10	0.00	0.00	
	0.00	47.50	50.00	55.00	60.00	65.00	70.00	
WMT	55.25	7.40	5.10	1.45	0.15	0.05	0.05	
	54.14	7.20	4.90	1.30	0.10	0.00	0.00	
	0.00	35.00	40.00	42.50	45.00	47.50	50.00	55.00
GM	43.90	8.70	4.20	2.30	1.05	0.40	0.15	0.05
	42.88	8.60	4.00	2.20	0.95	0.30	0.10	0.00
	0.00	30.00	32.50	35.00	37.50	40.00		
HD	33.75	3.80	1.85	0.60	0.15	0.05		
	33.07	3.60	1.70	0.55	0.10	0.00		
	0.00	30.00	32.50	35.00	37.50	40.00		
HON	33.43	2.85	1.15	0.30	0.10	0.05		
	32.44	2.70	1.00	0.20	0.00	0.00		
	0.00	15.00	17.50	20.00	22.50			
HPQ	19.70	4.60	2.30	0.70	0.15			
	19.21	4.50	2.20	0.65	0.10			

Table 2: Continuation of Table 1.

	0.00	80.00	85.00	90.00	95.00	100.00			
IBM	86.03 85.15	6.30 6.10	2.65 2.50	0.70 0.65	0.20 0.15	0.05 0.00			
	0.00	27.50	30.00	32.50	35.00	37.50	40.00	42.50	45.00
JPM	35.47 34.75	8.00 7.80	5.60 5.40	3.30 3.10	1.45 1.35	0.45 0.35	0.10 0.05	0.10 0.00	0.05 0.00
	0.00	47.50	50.00	55.00					
KO	50.12 49.51	2.70 2.55	1.00 0.85	0.05 0.00					
	0.00	40.00	42.50	45.00					
XOM	43.54 43.01	3.40 3.20	1.45 1.35	0.40 0.30					
	0.00	20.00	22.50	25.00	27.50	30.00			
INTC	27.30 26.44	6.90 6.80	4.50 4.30	2.30 2.25	0.80 0.70	0.20 0.10			
	0.00	50.00	55.00						
JNJ	55.10 54.13	5.00 4.80	1.10 1.05						
	0.00	80.00	85.00	90.00					
UTX	82.80 81.50	3.60 3.40	1.20 1.10	0.30 0.20					
	0.00	80.00	85.00	90.00					
MMM	83.89 82.75	4.20 4.00	1.30 1.15	0.25 0.15					
	0.00	45.00	47.50	50.00	55.00	60.00			
MO	50.00 48.50	4.90 4.70	2.80 2.65	1.20 1.15	0.15 0.10	0.10 0.00			
	0.00	45.00	47.50	50.00					
MRK	46.89 46.00	2.15 1.95	0.70 0.60	0.15 0.10					
	0.00	30.00	35.00	37.50	40.00	42.50			
PFE	35.91 35.00	5.70 5.50	1.30 1.20	0.30 0.25	0.10 0.05	0.05 0.00			
	0.00	90.00	95.00	100.00	105.00	110.00	115.00		
PG	107.15 105.81	16.50 16.30	11.70 11.40	7.10 6.90	3.30 3.10	1.00 0.90	0.25 0.20		
	0.00	25.00							
SBC	24.49 24.11	0.40 0.35							
	0.00	20.00	25.00	27.50	30.00				
MCD	26.05 25.50	5.90 5.80	1.40 1.30	0.35 0.25	0.05 0.05				
	0.00	30.00	35.00	40.00	42.50	45.00	47.50	50.00	55.00
C	45.30 44.83	15.00 14.80	10.00 9.80	5.20 5.10	3.00 2.90	1.30 1.25	0.40 0.35	0.15 0.05	0.05 0.00

Table 3: Strike 80.00 DJX basket super-replicating strategy from LP formulation (eq. (6)). Option price upper bound = 19.8872. A positive value in the "Position" represents a long position in the corresponding asset's call option with that "Strike".

Ticker	Strike	Position	Ticker	Strike	Position	Ticker	Strike	Position
DD	0.00	0.071	WMT	47.50	0.071	UTX	80.00	0.017
UTX	0.00	0.054	GM	35.00	0.071	MMM	80.00	0.071
SBC	0.00	0.071	HD	30.00	0.071	MO	45.00	0.071
AA	25.00	0.071	HON	30.00	0.071	MRK	45.00	0.071
AIG	65.00	0.071	HPQ	15.00	0.071	PFE	30.00	0.071
AXP	47.50	0.071	IBM	80.00	0.071	PG	90.00	0.071
BA	40.00	0.071	JPM	27.50	0.071	MCD	20.00	0.071
VZ	35.00	0.071	KO	47.50	0.071	MSFT	22.50	0.071
CAT	60.00	0.071	XOM	40.00	0.071	C	35.00	0.071
DIS	20.00	0.071	INTC	20.00	0.071			
GE	25.00	0.071	JNJ	50.00	0.071			

Table 4: Strike 80.00 DJX basket super-replicating strategy from LP formulation (eq. (6)) plus diversification constraints (eq. (7)). Option price upper bound = 19.9022. A positive value in the "Position" represents a long position in the corresponding asset's call option with that "Strike".

Ticker	Strike	Position	Ticker	Strike	Position	Ticker	Strike	Position
DD	0.00	0.071	HD	30.00	0.071	PFE	30.00	0.071
UTX	0.00	0.054	HON	30.00	0.071	PG	90.00	0.071
SBC	0.00	0.071	HPQ	15.00	0.071	MCD	20.00	0.071
AA	25.00	0.071	IBM	80.00	0.071	MSFT	22.50	0.071
AIG	65.00	0.071	JPM	27.50	0.071	C	35.00	0.071
AXP	47.50	0.071	KO	47.50	0.071	KO	55.00	0.050
BA	40.00	0.071	XOM	40.00	0.071	MCD	30.00	0.050
VZ	35.00	0.071	INTC	20.00	0.071	GE	35.00	0.050
CAT	60.00	0.071	JNJ	50.00	0.071	WMT	70.00	0.050
DIS	20.00	0.071	UTX	80.00	0.017	GM	55.00	0.050
GE	25.00	0.071	MMM	80.00	0.071	JPM	45.00	0.050
WMT	47.50	0.071	MO	45.00	0.071			
GM	35.00	0.071	MRK	45.00	0.071			