Compressive Sensing, Lecture 2

Yesterday

- Undetermined systems of equations and ℓ_1 minimization
- Compressive sensing
  - Probabilistic approach: isotropy & incoherence
  - Deterministic approach: restricted isometry property

Today

- Ideas of the main proofs
- Main computational tool: convex optimization

Recap

Compressive sampling approach

- measure $b = A\bar{x}$
- obtain $\hat{x}$ via ℓ_1 minimization: $\hat{x} := \text{argmin}_x\{\|x\|_1 : Ax = b\}$.

Probabilistic approach

- fix $\bar{x} \in \mathbb{R}^n$ arbitrary
- randomize A
- with high probability $\hat{x}$ recovers $\bar{x}$ or $\bar{x}_s$

Deterministic approach: RIP

- find $m \times n$ matrix A satisfying RIP
- $\hat{x}$ recovers $\bar{x}$ or $\bar{x}_s$ for all $\bar{x} \in \mathbb{R}^n$

RIP and exact recovery

Recall

Given a sensing matrix A, δ_k is smallest δ such that

$$(1-\delta)\|x\|_2^2 \le \|Ax\|_2^2 \le (1+\delta)\|x\|_2^2$$

for all k-sparse vector x.

Observe

δ_{2s} is the smallest δ such that

$$(1-\delta)\|x_1 - x_2\|_2^2 \le \|A(x_1 - x_2)\|_2^2 \le (1+\delta)\|x_1 - x_2\|_2^2$$

for all s-sparse vectors x_1, x_2.

Therefore if $\delta_{2s} < 1$ in principle we can recover $\bar{x}$ from $b = A\bar{x}$, e.g., via ℓ_0 minimization.

RIP and signal recovery (special case)

Theorem

Assume $\bar{x} \in \mathbb{R}^n$ and A satisfies RIP with $\delta_{2s} \le \sqrt{2} - 1$. Then the ℓ_1 solution $\hat{x}$ satisfies

$$\|\hat{x} - \bar{x}\|_2 \le C \cdot \frac{\|\bar{x} - \bar{x}_s\|_1}{\sqrt{s}}$$

for some constant C.

Proof

Let $h := \hat{x} - \bar{x}$. Put $T_0 :=$ indexes of s largest entries of $|h|$, $T_1 :=$ indexes of s largest entries of $|h_{T_0^c}|$, etc.

Let $\Delta := \frac{\|\bar{x} - \bar{x}_s\|_1}{\sqrt{s}}$.

By construction of the T_js:

$$\sum_{j\geq 2}\|h_{T_j}\|_2 \leq \frac{\|h_{T_0^c}\|_1}{\sqrt{s}}.$$

By optimality of $\hat{x}$:

$$\|h_{T_0^c}\|_1 \leq \|h_{T_0}\|_1 + 2\cdot\sqrt{s}\cdot\Delta.$$

By RIP:

$$(1-\delta_{2s})\|h_{T_0\cup T_1}\|_2 \leq \sqrt{2}\cdot\delta_{2s}\cdot\sum_{j\geq 2}\|h_{T_j}\|_2.$$

Hence $\|h_{T_0\cup T_1}\|_2 \leq \frac{2\rho\cdot\Delta}{1-\rho}$ for $\rho := \frac{\sqrt{2}\cdot\delta_{2s}}{1-\delta_{2s}}$. Therefore

$$\|h\|_2 \leq \frac{2(1+\rho)}{1-\rho}\cdot\Delta$$

□

Gaussian matrices and RIP

Theorem

Let M be an $m\times n$ Gaussian matrix and $A := \frac{1}{\sqrt{m}}M$. If $m \geq \frac{k\log(en/k)}{\delta^2}$ for $\delta\in(0,1/3)$ and $1\leq k\leq n$, then with probability at least $1-2e^{-\delta^2 m}$

$$1-3\delta \leq \sigma_{\min}(A_T) \leq \sigma_{\max}(A_T) \leq 1+3\delta \text{ for all } |T| = k.$$

In particular, A satisfies RIP with high probability.

Related property of random projections

Theorem (Johnson-Lindenstrauss Lemma)

Assume $x_1,\ldots,x_n\in\mathbb{R}^d$. If $k \geq \frac{8\delta\log n}{\epsilon^2(1-2\epsilon/3)}$ for some $\epsilon\in(0,1)$ and $\delta\geq 1$, then a random projection $\Pi:\mathbb{R}^d\to\mathbb{R}^k$ satisfies

$$(1-\epsilon)\frac{k}{d}\|x_i-x_j\|_2^2 \leq \|\Pi x_i - \Pi x_j\|_2^2 \leq (1+\epsilon)\frac{k}{d}\|x_i-x_j\|_2^2,\ \forall i\neq j$$

with probability at least $1-\frac{n(n-1)}{n^{2\delta}}$.

Key lemmas (for both Johnson-Lindenstrauss and Gaussian RIP):

Lemma (Borell, Tsirelson-Ibragimov-Sudakov)

Let $X\sim N(0,I_d)$ and $f:\mathbb{R}^d\to\mathbb{R}$ be L-Lipschitz. Then for $t\geq 0$

$$\mathbb{P}\left(f(X)-\mathbb{E}[f(X)] > t\right) \leq e^{-t^2/2L^2}.$$

Lemma (Sudakov-Fernique)

Let $(X_t)_{t\in I}$ and $(Y_t)_{t\in I}$ be Gaussian processes. If $\mathbb{E}X_t = \mathbb{E}Y_t$ and $\mathbb{E}(X_s-X_t)^2 \leq \mathbb{E}(Y_s-Y_t)^2$ for all $s,t\in I$ then

$$\mathbb{E}\sup_{t\in I}X_t \leq \mathbb{E}\sup_{t\in I}Y_t.$$

Proof of Gaussian RIP Theorem

Assume $|T| = k$ and $t \geq 0$.

By Sudakov-Fernique:

$$\mathbb{E}(\sigma_{\max}(M_T)) \leq \sqrt{m} + \sqrt{k}$$

$$\mathbb{E}(\sigma_{\min}(M_T)) \geq \sqrt{m} - \sqrt{k}.$$

By Borell, Tsirelson-Ibragimov-Sudakov:

$$\mathbb{P}(\sigma_{\max}(M_T) \geq \sqrt{m} + \sqrt{k} + t) \leq e^{-t^2/2}$$

$$\mathbb{P}(\sigma_{\min}(M_T) \leq \sqrt{m} - \sqrt{k} - t) \leq e^{-t^2/2}.$$

Hence

$$\mathbb{P}\left(\max_{|T|=k} \sigma_{\max}(A_T) \geq 1 + \frac{\sqrt{k}+t}{\sqrt{m}}\right) \leq \binom{n}{k} e^{-t^2/2} \leq (en/k)^k e^{-t^2/2}$$
$$= \exp\left(k \log(en/k) - t^2/2\right).$$

To finish, take $t = 2\sqrt{m} \cdot \delta$. (Similar for $\sigma_{\min}$.) □

Signal recovery for probabilistic ("RIPless") approach

Probabilistic approach

- Suppose $\bar{x} \in \mathbb{R}^n$ is s-sparse.
- Pick $A \in \mathbb{R}^{m \times n}$ and measure $b = A\bar{x}$.

Question

How likely it is that the solution $\hat{x}$ to the ℓ_1 minimization problem

$$\begin{array}{ll} \min & \|x\|_1 \\ & Ax = b \end{array}$$

recovers $\bar{x}$?

Optimality conditions for ℓ_1 minimization

$$\begin{array}{ll} \min & \|x\|_1 \\ & Ax = b \end{array}$$

Optimality conditions

A feasible $x \in \mathbb{R}^n$ is optimal iff there exists $v = A^*\lambda$ such that

- $v_i = \text{sgn}(x_i)$ for $i \in T := \{i : x_i \neq 0\}$.
- $|v_i| \leq 1$ for $i \in T^c := \{i : x_i = 0\}$.

Sufficient condition for uniqueness

If, in addition, $|v_i| < 1$ for all $i \in T^c$ and A_T is full column rank then x is the unique solution.

Strategy to prove exact recovery via ℓ_1 minimization

Suppose $\bar{x}$ has support T, i.e., $T := \{i : x_i \neq 0\}$.

Take

$$v := A^* A_T (A_T^* A_T)^{-1} \text{sgn}(x_T).$$

By construction $v = A^*\lambda$ and $v_i = \text{sgn}(x_i)$ for $i \in T$.

We would be done if we can show that $|v_i| < 1$ for $i \in T^c$.

An easy probabilistic result:

Theorem

For A Gaussian, achieve exact recovery with probability at least $1 - 3/\sqrt{n}$ provided $m \geq 4s \log n$.

Proof of Theorem

Put $w := A_T(A_T^* A_T)^{-1}\mathrm{sgn}(x_T)$.

Observe: w, A_i are independent for $i \in T^c$.

Thus $v_i | w \sim \mathcal{N}(0, \|w\|_2^2)$. Hence

$$\mathbb{P}(|v_i| \geq 1 | w) \leq 2e^{-1/2\|w\|_2^2}.$$

On the other hand, as in the RIP Theorem,

$$\mathbb{P}(\sigma_{\min}(A_T) \leq \sqrt{m} - \sqrt{s} - t) \leq e^{-t^2/2}.$$

Therefore with probability at least $1 - e^{-t^2/2}$

$$\|w\| \leq \frac{\sqrt{s}}{\sqrt{m} - \sqrt{s} - t} := B.$$

Consequently

$$\mathbb{P}\left(\max_{i \in T^c} |v_i| \geq 1\right) \leq 2ne^{-1/2B^2} + e^{-mt^2/2}.$$

To finish, take $t := \sqrt{\log n}$.

Optimality of Compressive Sensing

Back to Gaussian sensing

m Gaussian measurements and ℓ_1 decoding:

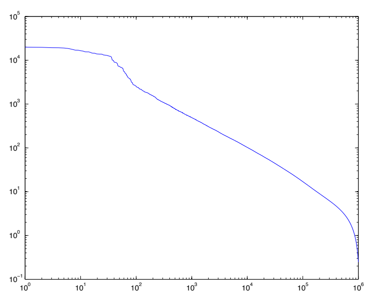
$$\|\hat{x} - x\|_2 \lesssim \frac{\|x - x_s\|_1}{\sqrt{s}}, \; s \approx \frac{m}{\log(n/m)}.$$

Question

Can we do better with other measurements or other algorithms?

Signals with power law

image



sorted wavelet coefficients

Power law decay: $|x|_{(1)} \geq |x|_{(2)} \geq \cdots \geq |x|_{(n)}$

$$|x|_{(k)} \leq \frac{C}{k^p}$$

Model

ℓ_p ball $\mathcal{B}_p := \{x : \|x\|_p \leq 1\}$.
Discuss case $p = 1$ but same discussion applies to $0 \leq p \leq 1$.

Recovery of ℓ_1 ball $\mathcal{B}$

Gaussian sensing

- Suppose unknown vector is in $\mathcal{B}$
- Take m Gaussian measurements

$$\|\hat{x} - x\|_2 \lesssim \sqrt{\frac{\log(n/m) + 1}{m}}.$$

Ideal sensing

Best we can hope from m linear measurements:

$$E_m(\mathcal{B}) = \inf_{D,F} \sup_{x \in \mathcal{B}} \|x - D(F(x))\|.$$

Gelfand widths

Theorem (Donoho)

$$d_m(\mathcal{B}) \le E_m(\mathcal{B}) \le C \cdot d_m(\mathcal{B}),$$

where $d_m(\mathcal{B})$ is the m-width of $\mathcal{B}$:

$$d_m(\mathcal{B}) := \inf_V \left\{ \sup_{x \in \mathcal{B}} \|P_V x\|_2 : \textit{codim}(V) < m \right\}$$

Theorem (Kashin, Garnaev-Gluskin)

For ℓ_1 ball

$$C_1 \cdot \sqrt{\frac{\log(n/m)+1}{m}} \le d_m(\mathcal{B}) \le C_2 \cdot \sqrt{\frac{\log(n/m)+1}{m}}.$$

Compressive sensing achieves the limits of performance.

Convex functions and sets

Subdifferential

Assume $f : \mathbb{R}^n \to \mathbb{R}$ convex and $x \in \mathbb{R}^n$. A vector $g \in \mathbb{R}^n$ is a subgradient of f at x if

$$f(y) \ge f(x) + \langle g, y - x \rangle, \quad \text{for all } y \in \mathbb{R}^n.$$

Subdifferential $\partial f(x) := \{g : g \text{ subgradient of } f \text{ at } x\}$.

Normal cone

Assume $S \subseteq \mathbb{R}^n$ is convex and $x \in S$.
Normal cone to S at x:

$$N_S(x) := \{d : \langle d, y - x \rangle \le 0 \ \text{ for all } \ y \in S\}.$$

Convex optimization

Problem of the form

$$\begin{array}{ll} \min & f(x) \\ & x \in S. \end{array}$$

where f and S are convex.

Sufficient optimality conditions

A point $\bar{x} \in S$ is a solution to the above problem if

$$-\partial f(\bar{x}) \cap N_S(\bar{x}) \ne \emptyset.$$

Special convex optimization problems

Linear programming

Objective function is linear: $f(x) = \langle c, x \rangle$
Constraint set is polyhedral: $S = \{x : Ax = b,\ Bx \ge d\}$.

Optimality conditions for linear programming

$$c = A^* y + B^* z,\ z \ge 0,\ \langle z, Bx - d \rangle = 0.$$

Special convex optimization problems

Consider the vector space $\mathbb{S}^n$: $n \times n$ symmetric matrices with inner product

$$\langle X, Z \rangle = \text{trace}(XZ).$$

Cone of positive semidefinite matrices

$$\mathbb{S}^n_+ := \{X \in \mathbb{S}^n : \lambda(X) \geq 0\} = \{X : u^\mathsf{T} X u \geq 0 \; \forall u \in \mathbb{R}^n\}.$$

Write $X \succeq Z$ for $X - Z \in \mathbb{S}^n_+$.

Semidefinite programming

Objective function: $f(X) = \langle C, X \rangle$
Constraint set: $S = \{X \in \mathbb{S}^n : \mathcal{A}(X) = b, \, \mathcal{B}(X) \succeq D\}$ for some linear maps $\mathcal{A}, \mathcal{B}$.

Sufficient optimality conditions for semidefinite programming

$$C = \mathcal{A}^*(y) + \mathcal{B}^*(Z), \; Z \succeq 0, \; \langle Z, \mathcal{B}(X) - D \rangle = 0.$$

What is so special about linear and semidefinite programming?

- They have powerful duality properties
- They can be solved efficiently (via interior-point methods)
- Popular matlab-based solvers: `SeDuMi, SDPT3`
- Matlab toolbox CVX serves as a wrapper for these solvers

CVX examples

To solve

$$\min_x \; \|x\|_1 \quad Ax = b \quad \rightsquigarrow \quad \min_{x,t} \; \sum_{i=1}^n t_i \quad x \leq t, \; -x \leq t, \; Ax = b$$

Use CVX code

```
cvx_begin
  variable x(n);
  variable t(n);
  minimize(sum(t));
  subject to
    x <= t;
    -x <= t;
    A*x == b;
cvx_end
```


CVX does some standard transformations.

To solve

$$\min \; \|x\|_1 \quad Ax = b$$

Use CVX code

```
cvx_begin
  variable x(n);
  minimize(norm(x,1));
  subject to
    A*x == b;
cvx_end
```

More CVX examples

To solve

$$\begin{aligned} \min \quad & \langle I, X \rangle \\ & \langle A, X \rangle = b \\ & X \succeq 0 \end{aligned}$$

use CVX code

```
cvx_begin
  variable X(n,n) symmetric;
  minimize( trace( I * X ) );
  subject to
    trace( A * X ) == b;
    X == semidefinite(n);
cvx_end
```


Main references for today's material

- Slides for this minicourse:
  `http://andrew.cmu.edu/user/jfp/UNencuentro`
- E. Candès, "The restricted isometry property and its implications for compressed sensing," *C. R. Acad. Sci. Paris,* Ser. I 346, pp. 589-592, 2008.
- R. Baraniuk, M. Davenport, R. DeVore, M. Wakin, "A Simple Proof of the Restricted Isometry Property for Random Matrices," *Constructive Approximation,* 2008.
- D. Donoho, "Compressed sensing," *IEEE Trans. Inf. Theory,* vol 52, no. 4, pp. 1289–1306, April 2006.
- M. Grant and S. Boyd, `CVX`*: Matlab Software for Disciplined Convex Programming,* `http://cvxr.com/cvx/`