

Nonlinear Pricing with Resale*

Isa E. Hafalir[†]

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Abstract

We consider the problem of a monopolist—choosing an optimal nonlinear pricing scheme—facing two consumers who can resell some or all of the goods to each other in a secondary market. We suppose that the valuations of the consumers are drawn independently from a continuous distribution. We characterize the optimum direct mechanism and show that the monopolist can be better off or worse off as compared to the without resale case, depending on the specifics of the cost function of the monopolist and the utility functions of the consumers.

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1 Introduction

Consider the adverse selection problem of a transaction between a seller (the monopolist,) and buyers (consumers,) where the seller does not perfectly know how much the buyers are willing to pay for the goods. Suppose also that the seller sets the terms of the contract—i.e. a menu of quantities and prices. The problem of one principal facing one agent who has private information about his type was first analyzed by Mirrlees (1971). One monopolist and a consumer problem was then analyzed by Mussa and Rosen (1978), and Maskin and Riley (1984), among others. The reader is referred to Bolton and Dewatripont (2005), Laffont and Martimort (2002) and Wilson (1993) for a detailed discussion of economics of adverse selection.

All previous work on monopolist's problem of optimal nonlinear pricing focused either on a single consumer or on multiple consumers who cannot resell the goods to one another, except for Calzolari and Pavan (2006). Calzolari and Pavan (2006)

*PRELIMINARY AND INCOMPLETE, Comments are very welcome.

[†]Department of Economics, Penn State University, University Park, PA 16802, USA. E-mail: isaemin@psu.edu

recently consider a monopolist who designs an allocation rule and a disclosure policy that optimally fashion the beliefs of the participants in the secondary market. They consider two consumers with two types of valuations and show that it may be impossible to maximize the revenue with a deterministic selling procedure and disclosing only the decision to trade. In their model, the monopolist has one unit of the good at his hand. There is no question of how much to produce.

In this paper, we also consider two consumers who can resell the good to each other in the secondary market. However, we consider consumers drawing valuations from a *continuous* distribution, rather than focusing on a finite set of valuations. In order to be able to characterize optimal nonlinear pricing, we do not consider the disclosure policy of the monopolist. Specifically, we suppose that the monopolist proposes a *deterministic* menu of offers and no information is revealed after the trade between the monopolist and the consumers.

We consider two different models of the nonlinear pricing problem. In the first model, the monopolist is assumed to have a production technology of constant marginal cost, and the consumers are assumed to have concave utility functions. This model was analyzed by Maskin and Riley (1984) for a model without resale, and it results in quantity discounts, which is widely seen in practice. We find necessary and sufficient conditions for the optimal nonlinear pricing schemes of the model with resale. Moreover, we show that the maximal revenue achievable in an environment with resale is less than the maximal revenue achievable in an environment without resale. This follows from the observation that in an environment without resale, since the utilities of the consumers are concave, the monopolist can offer a certainty equivalent of the transactions of the model with resale at a lower price.

In the second model, the monopolist is assumed to have a convex cost function, and the consumers are assumed to have linear utility functions. For this model, the secondary market optimal behavior is easier to characterize. This is because consumers have linear utility functions and hence announcing a unit price is their optimal selling procedure in the secondary market.¹ This model results in premia rather than quantity discounts. We characterize the optimal menu for this model and show that the maximal revenue achievable in an environment with resale is more than the maximal revenue achievable in an environment without resale. This follows from the observation that in an environment with resale, by offering the same quantities as in without resale optimal menu, the monopolist can demand higher amounts of monetary transfers from the consumers.

We therefore conclude that the profitability of banning the resale (if the monopolist has a power to do so) depends on the specifics of the environment. If the utility functions of the consumers are concave, then the monopolist can gain by offering a certainty equivalent of the transactions in the secondary market. Therefore, the model without resale would give more revenue. If the cost function of the monopolist

¹Whereas in the first model, the consumers offer a nonlinear menu of quantities and prices to each other in the secondary market.

is convex, then the monopolist can benefit from selling to low value consumers and letting them to sell to high value consumers. Therefore, the model with resale would give more revenue.

2 Economic Environment

Consider a monopolist and two consumers, where the monopolist does not know how much the consumers are willing to pay for the good. Suppose that consumers' preferences depend on the preference characteristics θ . The characteristics θ are private information to the consumers. The monopolist and the other consumer only know the continuous cumulative distribution of θ , $F(\theta)$ on an interval $[\underline{\theta}, \bar{\theta}]$ —we suppose that F satisfies Myerson's regularity condition.² The consumers' preferences are represented by the utility function

$$u(\theta) = \theta v(q) - T,$$

where q is the number of units consumed, and T is the total amount paid. The monopolist's production cost is given by the function $K(q)$. The monopoly profit from selling q units against a sum of money T is then given by

$$\pi = T - K(q).$$

This paper considers the question of finding the profit maximizing pair (T, q) that the monopolist will be able to induce the consumers to choose, in an environment in which the consumers can resell some or all of the good that they have to each other. We suppose that resale takes place in an imperfect information setting. That is, no information about the values or actions taken in the first stage (buying from the monopolist stage) are revealed before the second stage (reselling to each other stage.) We consider no discounting of payoffs between the two stages.

We restrict analysis to direct revelation mechanisms $\{T(\theta), q(\theta)\}$ which are truthful. That is, the monopolist maximizes

$$\int_{\underline{\theta}}^{\bar{\theta}} [T(\theta) - K(q(\theta))] d\theta$$

subject to incentive compatibility and individual rationality constraints of the consumers.

3 Linear Cost, Concave Utility Model

In this section, we suppose that the monopolist's production cost is linear and given by $K(q) = cq$ with the unit cost $c > 0$. Hence, monopolist's profit from selling q units

²Myerson's regularity condition is fairly common in nonlinear pricing literature.

against a sum of money T is given by

$$\pi = T - cq.$$

Moreover, we suppose that consumers' preferences are represented by the utility function which is concave in units consumed. Specifically,

$$u(\theta) = \theta v(q) - T,$$

where v satisfies $v(0) = 0$, $v'(0) = \infty$, and $v'(q) > 0$, $v''(q) < 0$ for all q .

We work backwards as usual. Given the menu offered by the monopolist, the consumers will behave optimally in the resale stage.

3.1 Behavior in the resale stage

Since the consumers' choices $\{T(\theta_i), q(\theta_i)\}$ are their private information in the second stage, the consumers will offer each other menus with imperfect information. Let us consider bidder 1 (bidder 2's problem is similar.) Consider the consumer 1 with preference parameter θ_1 , who announces his type as θ'_1 in the first stage and gets $q(\theta'_1)$ units of the product at the price of $T(\theta'_1)$. In the second stage, he will offer a menu $\{S_1(\cdot | \theta_1, \theta'_1), r_1(\cdot | \theta_1, \theta'_1)\}$ to consumer 2 where r_1 denotes the amount of the good transferred to consumer 2 and S_1 denotes the amount of the money transferred to consumer 1. Consumer 1's problem in the second stage is then

$$\max_{q(\theta'_1) \geq r_1(\cdot) \geq 0, S_1(\cdot)} \int_{\underline{\theta}}^{\bar{\theta}} [\theta_1 v(q(\theta'_1) - r_1(\theta_2)) + S_1(\theta_2)] f(\theta_2) d\theta_2$$

subject to

$$(IR2) \theta_2 [v(q(\theta_2) + r_1(\theta_2)) - v(q(\theta_2))] - S_1(\theta_2) \geq 0 \text{ for all } \theta_2 \in [\underline{\theta}, \bar{\theta}],$$

and

$$(IC2) \theta_2 v(q(\theta_2) + r_1(\theta_2)) - S_1(\theta_2) \geq \theta_2 v(q(\theta_2) + r_1(\theta'_2)) - S_1(\theta'_2) \text{ for all } (\theta_2, \theta'_2) \in [\underline{\theta}, \bar{\theta}]^2.$$

We denote the optimal $(S_1(\cdot), r_1(\cdot))$ by $(S(\cdot | \theta_1, \theta'_1), r(\cdot | \theta_1, \theta'_1))$ and the maximand by $C(\theta_1, \theta'_1)$. Note that when consumer 1 finds it profitable to sell some amount of the product he has to consumer 2, then consumer 2 cannot find it profitable to sell to consumer 1. The expected payoff of consumer 1 by buying from bidder 2 is given by

$$\int_{\underline{\theta}}^{\bar{\theta}} \max\{0, \max_{\theta''_1} (\theta_1 v(q(\theta'_1) + r_2(\theta''_1 | \theta_2, \theta_2)) - S_2(\theta''_1 | \theta_2, \theta_2))\} f(\theta_2) d\theta_2$$

Let us denote the maximand by $D(\theta_1, \theta'_1)$. Note that when $\theta'_1 = \theta_1$, then θ''_1 above should be also equal to θ_1 and when $\theta''_1 = \theta_1$, it gives a value not less than 0.

We can write consumer 1's payoff as

$$U(\theta_1, \theta'_1) = C(\theta_1, \theta'_1) + D(\theta_1, \theta'_1) - T(\theta'_1) \tag{1}$$

3.2 Optimal Menu

Therefore, the monopolist's problem for consumer 1 is given by (the problem for consumer 2 is similar)

$$\max_{q(\cdot), T(\cdot)} \int_{\underline{\theta}}^{\bar{\theta}} [T(\theta_1) - cq(\theta_1)] f(\theta_1) d\theta_1$$

subject to

$$(IR) \quad U(\theta_1, \theta_1) \geq 0 \text{ for all } \theta_1 \in [\underline{\theta}, \bar{\theta}],$$

and

$$(IC) \quad U(\theta_1, \theta'_1) \geq U(\theta_1, \theta_1) \text{ for all } (\theta_2, \theta'_2) \in [\underline{\theta}, \bar{\theta}]^2.$$

IC constraint implies that $C(\theta_1, \theta'_1) + D(\theta_1, \theta'_1) - T(\theta'_1)$ is maximized at $\theta'_1 = \theta_1$. Using envelope theorem ($v'(0) = \infty$ makes sure that solution for the maximization problem has an interior solution), this gives us the following FOC (let us denote $r_1(\theta_2 | \theta_1, \theta_1)$ by $r_1(\theta_2 | \theta_1)$ and $r_2(\theta_1 | \theta_2, \theta_2)$ by $r_2(\theta_1 | \theta_2)$.)

$$\theta_1 q'(\theta_1) \left(\int_{\underline{\theta}}^{\bar{\theta}} (v'(q(\theta_1) - r_1(\theta_2 | \theta_1)) + v'(q(\theta_1) + r_2(\theta_1 | \theta_2))) f(\theta_2) d\theta_2 \right) = T'(\theta_1)$$

Moreover, $U(\theta_1) = C(\theta_1) + D(\theta_1) - T(\theta_1)$ and by using envelope theorem, we obtain

$$\begin{aligned} U'(\theta_1) &= \int_{\underline{\theta}}^{\bar{\theta}} (v(q(\theta_1) - r_1(\theta_2 | \theta_1)) + v(q(\theta_1) + r_2(\theta_1 | \theta_2))) f(\theta_2) d\theta_2 \\ &\equiv m(\theta_1) \end{aligned}$$

For the optimal menu, we must have $U(\underline{\theta}) = 0$, hence we obtain

$$U(\theta_1) = \int_{\underline{\theta}}^{\theta_1} m(\theta) d\theta$$

and

$$T(\theta_1) = C(\theta_1) + D(\theta_1) - \int_{\underline{\theta}}^{\theta_1} m(\theta) d\theta$$

Moreover,

$$\begin{aligned} C(\theta_1) + D(\theta_1) &= \theta_1 m(\theta_1) + \int_{\underline{\theta}}^{\bar{\theta}} S_1(\theta_2 | \theta_1) f(\theta_2) d\theta_2 - \int_{\underline{\theta}}^{\bar{\theta}} S_2(\theta_1 | \theta_2) f(\theta_2) d\theta_2 \\ &\equiv \theta_1 m(\theta_1) + l(\theta_1) \end{aligned}$$

Since the transfers between the bidders add up to zero, we have

$$\int_{\underline{\theta}}^{\bar{\theta}} l(\theta_1) f(\theta_1) d\theta_1 = 0$$

Thus, we obtain

$$\begin{aligned}
\int_{\underline{\theta}}^{\bar{\theta}} T(\theta_1) f(\theta_1) d\theta_1 &= \int_{\underline{\theta}}^{\bar{\theta}} \left[\theta_1 m(\theta_1) - \int_{\underline{\theta}}^{\theta_1} m(\theta) d\theta + l(\theta_1) \right] f(\theta_1) d\theta_1 \\
&= \int_{\underline{\theta}}^{\bar{\theta}} \left[\theta_1 m(\theta_1) - \int_{\underline{\theta}}^{\theta_1} m(\theta) d\theta \right] f(\theta_1) d\theta_1 \\
&= \int_{\underline{\theta}}^{\bar{\theta}} \theta_1 m(\theta_1) f(\theta_1) d\theta_1 - \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\theta_1} m(\theta) d\theta f(\theta_1) d\theta_1
\end{aligned}$$

and by integration by parts, we obtain

$$\begin{aligned}
\int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\theta_1} m(\theta) d\theta f(\theta_1) d\theta_1 &= (1 - F(\theta_1)) \int_{\underline{\theta}}^{\theta_1} m(\theta) d\theta \Big|_{\theta=\underline{\theta}}^{\theta=\bar{\theta}} - \int_{\underline{\theta}}^{\bar{\theta}} m(\theta_1) (1 - F(\theta_1)) d\theta_1 \\
&= \int_{\underline{\theta}}^{\bar{\theta}} m(\theta_1) (1 - F(\theta_1)) d\theta_1
\end{aligned}$$

Therefore, we obtain

$$\int_{\underline{\theta}}^{\bar{\theta}} T(\theta_1) f(\theta_1) d\theta_1 = \int_{\underline{\theta}}^{\bar{\theta}} \left(\theta_1 - \frac{1 - F(\theta_1)}{f(\theta_1)} \right) m(\theta_1) f(\theta_1) d\theta_1$$

Thus, monopolist's problem can be rewritten as

$$\max_{q(\cdot)} \int_{\underline{\theta}}^{\bar{\theta}} \left[\left(\theta - \frac{1 - F(\theta)}{f(\theta)} \right) m(\theta) - cq(\theta) \right] f(\theta) d\theta \quad (2)$$

subject to

$$q'(\theta) \geq 0$$

Maximizing (2) pointwise, and momentarily ignoring the constraint, we have the following condition for the best menu

$$c = \left(\theta - \frac{1 - F(\theta)}{f(\theta)} \right) m'(\theta) = \psi(\theta) m'(\theta)$$

if $\psi(\theta) \geq 0$, and $m'^{-1}\left(\frac{c}{\psi(\theta)}\right) \geq 0$, otherwise $q(\theta) = 0$. Moreover, $T(\theta) = \theta v(q(\theta)) - \int_{\underline{\theta}}^{\theta} v(q(\tau)) d\tau$.

3.3 Characterization of S , r and m

In this subsection, we find necessary and sufficient conditions for the optimal menu. By summing up IC2 constraint for (θ_2, θ'_2) and (θ'_2, θ_2) , we obtain

$$(\theta_2 - \theta'_2) [v(q(\theta_2) + r_1(\theta_2)) - v(q(\theta_2) + r_1(\theta'_2))] \geq 0$$

Therefore, IC2 constraint alone requires that $q + r_1$ has to be nondecreasing:

$$q'(\theta_2) + r_1'(\theta_2) \geq 0$$

IC2 constraint also implies the following first-order condition:

$$S_1'(\theta_2) - \theta_2 v'(q(\theta_2) + r_1(\theta_2)) r_1'(\theta_2) = 0$$

and it can be shown that above two conditions turns out to be sufficient for IC2 to hold.

Since $U_2(\theta_2) = \theta_2 v(q(\theta_2) + r_1(\theta_2)) - S_1(\theta_2)$, we obtain

$$U_2'(\theta_2) = v(q(\theta_2) + r_1(\theta_2)) + \theta_2 v'(q(\theta_2) + r_1(\theta_2)) q'(\theta_2) \equiv k(\theta_2)$$

and for the optimal menu, we must have $U_2(\underline{\theta}) = \underline{\theta} v(q(\underline{\theta}))$, supposing $v(q(\underline{\theta})) = 0$ (we confirm this later), we obtain

$$U_2(\theta_2) = \int_{\underline{\theta}}^{\theta_2} k(\theta) d\theta$$

and

$$S_1(\theta_2) = \theta_2 v(q(\theta_2) + r_1(\theta_2)) - \int_{\underline{\theta}}^{\theta_2} k(\theta) d\theta$$

Thus, IC implies

$$\begin{aligned} \int_{\underline{\theta}}^{\bar{\theta}} S_1(\theta_2) f(\theta_2) d\theta_2 &= \int_{\underline{\theta}}^{\bar{\theta}} \left[\theta_2 v(q(\theta_2) + r_1(\theta_2)) - \int_{\underline{\theta}}^{\theta_2} k(\theta) d\theta \right] f(\theta_2) d\theta_2 \\ &= \int_{\underline{\theta}}^{\bar{\theta}} \theta_2 v(q(\theta_2) + r_1(\theta_2)) f(\theta_2) d\theta_2 - \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\theta_2} k(\theta) d\theta f(\theta_2) d\theta_2 \end{aligned}$$

and by integration by parts, we obtain

$$\begin{aligned} - \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\theta_2} k(\theta) d\theta f(\theta_2) d\theta_2 &= (1 - F(\theta_2)) \int_{\underline{\theta}}^{\theta_2} k(\theta) d\theta \Big|_{\theta_2=\underline{\theta}}^{\theta_2=\bar{\theta}} - \int_{\underline{\theta}}^{\bar{\theta}} k(\theta_2) (1 - F(\theta_2)) d\theta_2 \\ &= - \int_{\underline{\theta}}^{\bar{\theta}} k(\theta_2) (1 - F(\theta_2)) d\theta_2 \end{aligned}$$

Therefore, we obtain

$$\int_{\underline{\theta}}^{\bar{\theta}} S_1(\theta_2) f(\theta_2) d\theta_2 = \int_{\underline{\theta}}^{\bar{\theta}} \left(\theta_2 v(q(\theta_2) + r_1(\theta_2)) - \frac{1 - F(\theta_2)}{f(\theta_2)} k(\theta_2) \right) f(\theta_2) d\theta_2$$

Thus, bidder 1's problem can be rewritten as

$$\max_{r_1(\theta_2)} \int_{\underline{\theta}}^{\bar{\theta}} \left[\theta_1 v(q(\theta'_1) - r_1(\theta_2)) + \left(\begin{array}{l} \left(\theta_2 - \frac{1-F(\theta_2)}{f(\theta_2)} \right) v(q(\theta_2) + r_1(\theta_2)) \\ - \frac{1-F(\theta_2)}{f(\theta_2)} \theta_2 v'(q(\theta_2) + r_1(\theta_2)) q'(\theta_2) \end{array} \right) \right] f(\theta_2) d\theta_2$$

subject to

$$r_1(\theta_2) \geq 0$$

Maximizing (2) pointwise with respect to $r_1(\theta_2)$, and momentarily ignoring the constraint, we have the following condition for the best menu

$$\theta_{21} v'(q(\theta'_2) - r_1(\theta_2)) = \left[\begin{array}{l} \left(\theta_2 - \frac{1-F(\theta_2)}{f(\theta_2)} \right) v'(q(\theta_2) + r_1(\theta_2)) \\ - \frac{1-F(\theta_2)}{f(\theta_2)} \theta_2 q'(\theta_2) v''(q(\theta_2) + r_1(\theta_2)) \end{array} \right]$$

We denote the optimal menu satisfying above necessary and sufficient condition by $(S_1(\cdot | \theta_1, \theta'_1), r_1(\cdot | \theta_1, \theta'_1))$. $m(\theta_1)$ is then given by $v(q(\theta_1) - r_1(\theta_2 | \theta_1)) + v(q(\theta_1) + r_2(\theta_1 | \theta_2))$. Note that r_1 and r_2 (and S_1 and S_2) are always positive. And if one of them is strictly positive, the other should be zero. Let us introduce the following notation for $\{i, j\} = \{1, 2\}$

$$r(\theta_i | \theta_j) = \begin{cases} -r_j(\theta_i | \theta_j) & \text{if } r_j(\theta_i | \theta_j) > 0 \\ r_i(\theta_j | \theta_i) & \text{if } r_i(\theta_j | \theta_i) > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$S(\theta_i | \theta_j) = \begin{cases} S_j(\theta_i | \theta_j) & \text{if } S_j(\theta_i | \theta_j) > 0 \\ -S_i(\theta_j | \theta_i) & \text{if } S_i(\theta_j | \theta_i) > 0 \\ 0 & \text{otherwise} \end{cases}$$

which will prove useful for the next subsection.

3.4 Revenue Superiority of Without Resale Model

Given the menu offered by the monopolist, the consumers will behave optimally in the resale stage. Consider a consumer 1 with value θ_1 who announced his type as θ'_1 in the first (buying from the monopoly) stage. Let us denote the amount of the good the consumer 1 transfers from consumer 2 with type θ_2 by $r(\theta_1, \theta'_1 | \theta_2)$ (which can be negative) and the amount of the money the consumer 1 transfers to consumer 2 with type θ_2 by $S(\theta_1, \theta'_1 | \theta_2)$ (which can be negative). Note that these functions depends on $\{T(\theta), q(\theta)\}$. However, for notational convenience, we compress this relation. Moreover, again for notational convenience, let us denote $r(\theta_1, \theta_1 | \theta_2)$ by $r(\theta_1 | \theta_2)$ and $S(\theta_1, \theta_1 | \theta_2)$ by $S(\theta_1 | \theta_2)$. Given this notation, we can write the monopolist's problem (for one consumer) as

$$\max_{q(\cdot), T(\cdot)} \int_{\underline{\theta}}^{\bar{\theta}} [T(\theta_1) - cq(\theta_1)] f(\theta_1) d\theta_1$$

subject to

$$\begin{aligned}
(IC) \quad & \theta_1 \int_{\underline{\theta}}^{\bar{\theta}} v(q(\theta_1) + r(\theta_1 | \theta_2)) f(\theta_2) d\theta_2 - \int_{\underline{\theta}}^{\bar{\theta}} (T(\theta_1) + S(\theta_1 | \theta_2)) f(\theta_2) d\theta_2 \\
\geq \quad & \theta_1 \int_{\underline{\theta}}^{\bar{\theta}} v(q(\theta'_1) + r(\theta_1, \theta'_1 | \theta_2)) f(\theta_2) d\theta_2 - \int_{\underline{\theta}}^{\bar{\theta}} (T(\theta'_1) + S(\theta_1, \theta'_1 | \theta_2)) f(\theta_2) d\theta_2
\end{aligned}$$

and

$$(IR) \quad \theta_1 \int_{\underline{\theta}}^{\bar{\theta}} v(q(\theta'_1) + r(\theta_1, \theta'_1 | \theta_2)) f(\theta_2) d\theta_2 - \int_{\underline{\theta}}^{\bar{\theta}} (T(\theta'_1) + S(\theta_1, \theta'_1 | \theta_2)) f(\theta_2) d\theta_2 \geq 0.$$

Let us introduce the following handy notation

$$\begin{aligned}
\int_{\underline{\theta}}^{\bar{\theta}} (T(\theta'_1) + S(\theta_1, \theta'_1 | \theta_2)) f(\theta_2) d\theta_2 &= \tilde{T}(\theta_1, \theta'_1) \\
\int_{\underline{\theta}}^{\bar{\theta}} (q(\theta'_1) + r(\theta_1, \theta'_1 | \theta_2)) f(\theta_2) d\theta_2 &= \tilde{q}(\theta_1, \theta'_1) \\
\int_{\underline{\theta}}^{\bar{\theta}} v(q(\theta'_1) + r(\theta_1, \theta'_1 | \theta_2)) f(\theta_2) d\theta_2 &= \tilde{v}(\theta_1, \theta'_1)
\end{aligned}$$

and

$$\tilde{T}(\theta_1, \theta_1) = \tilde{T}(\theta_1), \quad \tilde{q}(\theta_1, \theta_1) = \tilde{q}(\theta_1), \quad \tilde{v}(\theta_1, \theta_1) = \tilde{v}(\theta_1).$$

Given this new notation, the monopolist's problem can be written as

$$\max_{q(\cdot), T(\cdot)} \int_{\underline{\theta}}^{\bar{\theta}} [T(\theta_1) - cq(\theta_1)] f(\theta_1) d\theta_1$$

subject to

$$(IC) \quad \theta_1 \tilde{v}(\theta_1) - \tilde{T}(\theta_1) \geq \theta_1 \tilde{v}(\theta_1, \theta'_1) - \tilde{T}(\theta_1, \theta'_1)$$

and

$$(IR) \quad \theta_1 \tilde{v}(\theta_1) - \tilde{T}(\theta_1) \geq 0.$$

Consider the optimal menu of the monopolist (with a slight abuse of notation) $\{T(\theta), q(\theta)\}$ and corresponding $\{\tilde{v}(\theta_1, \theta'_1), \tilde{T}(\theta_1, \theta'_1)\}$. We will argue below that without resale, the monopolist can make a bigger profit.

Since behaving as if his value is θ'_1 in both stages would give less revenue to consumer 1 compared to announcing his type as θ'_1 and behaving optimally in the second stage, we obtain

$$\theta_1 \tilde{v}(\theta_1, \theta'_1) - \tilde{T}(\theta_1, \theta'_1) \geq \theta_1 \tilde{v}(\theta'_1) - \tilde{T}(\theta'_1).$$

This, together with IC constraint, then implies

$$\theta_1 \tilde{v}(\theta_1) - \tilde{T}(\theta_1) \geq \theta_1 \tilde{v}(\theta'_1) - \tilde{T}(\theta'_1).$$

Moreover, since v is concave, from Jensen's inequality, we obtain

$$v(\tilde{q}(\theta_1)) \geq \tilde{v}(\theta_1)$$

which implies that there is some $\hat{q}(\theta_1) \leq \tilde{q}(\theta_1)$ with

$$v(\hat{q}(\theta_1)) = \tilde{v}(\theta_1).$$

Now, we claim that monopoly without resale would give a higher revenue than monopoly with resale. The monopolist can obtain a higher revenue by offering the menu $\{\hat{q}(\theta_1), \tilde{T}(\theta_1)\}$. This is because, the transfers among the two consumers add up to zero, and therefore

$$\int_{\underline{\theta}}^{\bar{\theta}} \hat{q}(\theta_1) f(\theta_1) d\theta_1 \leq \int_{\underline{\theta}}^{\bar{\theta}} \tilde{q}(\theta_1) f(\theta_1) d\theta_1 = \int_{\underline{\theta}}^{\bar{\theta}} q(\theta_1) f(\theta_1) d\theta_1$$

and

$$\int_{\underline{\theta}}^{\bar{\theta}} \tilde{T}(\theta_1) f(\theta_1) d\theta_1 = \int_{\underline{\theta}}^{\bar{\theta}} T(\theta_1) f(\theta_1) d\theta_1.$$

Hence, we have

$$\int_{\underline{\theta}}^{\bar{\theta}} \left(\tilde{T}(\theta_1) - c\hat{q}(\theta_1) \right) f(\theta_1) d\theta_1 \geq \int_{\underline{\theta}}^{\bar{\theta}} \left(T(\theta_1) - cq(\theta_1) \right) f(\theta_1) d\theta_1$$

and

$$\begin{aligned} \theta_1 v(\hat{q}(\theta_1)) - \tilde{T}(\theta_1) &\geq \theta_1 v(\hat{q}(\theta'_1)) - \tilde{T}(\theta'_1) \\ \theta_1 v(\hat{q}(\theta_1)) - \tilde{T}(\theta_1) &\geq 0 \end{aligned}$$

Therefore, the menu $\{\hat{q}(\theta_1), \tilde{T}(\theta_1)\}$ satisfies IC and IR constraints of the without resale problem, and it gives a revenue bigger than the optimal revenue of the with resale model. The inequality is strict unless $q(\theta'_1) + r(\theta_1 | \theta_2)$ is constant for all θ_2 (otherwise Jensen's inequality would give a strict inequality.) Since this would not be true generically, we can conclude that without resale model is revenue superior to with resale model. We hence obtain the following proposition.

Proposition 1 *If the monopolist has a constant marginal cost, and the consumers have concave utility functions, then the maximal revenue achievable in an environment with resale is less than the maximal revenue achievable in an environment without resale.*

4 Convex Cost, Linear Utility Model

In this section, we suppose that consumers' preferences are linear in units consumed, and represented by the utility function

$$u(\theta) = \theta q - T$$

Assuming that the monopolist's production cost is given by the convex function $K(q)$ with $K(0) = 0$, $K'(q) > 0$ and $K''(q) > 0$ for all q , his profit from selling q units against a sum of money T is then given by

$$\pi = T - K(q)$$

We consider the question of finding the profit maximizing pair (T, q) that the monopolist will be able to induce the consumers to choose. We find the optimal menu for both without resale case and with resale case³.

4.1 Without resale, 2 consumers

We restrict analysis to direct revelation mechanisms $\{T(\theta), q(\theta)\}$ which are truthful. The monopolist's problem can be written as

$$\max_{q(\theta), T(\theta)} \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\bar{\theta}} [T(\theta_1) + T(\theta_2) - K(q(\theta_1) + q(\theta_2))] f(\theta_1) f(\theta_2) d\theta_1 d\theta_2$$

subject to

$$(IR) \quad u(\theta) \equiv \theta q(\theta) - T(\theta) \geq 0$$

and

$$(IC) \quad \theta q(\theta) - T(\theta) \geq \theta q(\theta') - T(\theta').$$

By summing up IC constraint for (θ, θ') and (θ', θ) , we obtain

$$(\theta - \theta')(q(\theta) - q(\theta')) \geq 0$$

Therefore, IC constraint alone requires that q has to be nondecreasing:

$$q'(\theta) \geq 0 \tag{3}$$

IC constraint also implies the following first-order condition:

$$T'(\theta) - \theta q'(\theta) = 0 \tag{4}$$

and (4) together with (3) turns out to be sufficient for IC to hold.

³The solution for without resale case for this model was not given in the literature for two consumers case.

Since $u(\theta) = \theta q(\theta) - T(\theta)$, we obtain

$$u'(\theta) = q(\theta)$$

and for the optimal menu, we must have $u(\underline{\theta}) = 0$, hence we obtain

$$u(\theta) = \int_{\underline{\theta}}^{\theta} q(\tau) d\tau$$

and

$$T(\theta) = \theta q(\theta) - \int_{\underline{\theta}}^{\theta} q(\tau) d\tau$$

Moreover, IC implies

$$\begin{aligned} \int_{\underline{\theta}}^{\bar{\theta}} T(\theta) f(\theta) d\theta &= \int_{\underline{\theta}}^{\bar{\theta}} \left[\theta q(\theta) - \int_{\underline{\theta}}^{\theta} q(\tau) d\tau \right] f(\theta) d\theta \\ &= \int_{\underline{\theta}}^{\bar{\theta}} \theta q(\theta) f(\theta) d\theta - \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\theta} q(\tau) d\tau f(\theta) d\theta \end{aligned}$$

and by integration by parts, we obtain

$$\begin{aligned} - \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\theta} q(\tau) d\tau f(\theta) d\theta &= (1 - F(\theta)) \int_{\underline{\theta}}^{\theta} q(\tau) d\tau \Big|_{\theta=\underline{\theta}}^{\theta=\bar{\theta}} - \int_{\underline{\theta}}^{\bar{\theta}} q(\theta) (1 - F(\theta)) d\theta \\ &= - \int_{\underline{\theta}}^{\bar{\theta}} q(\theta) (1 - F(\theta)) d\theta \end{aligned}$$

Therefore, we obtain

$$\int_{\underline{\theta}}^{\bar{\theta}} T(\theta) f(\theta) d\theta = \int_{\underline{\theta}}^{\bar{\theta}} \left(\theta - \frac{1 - F(\theta)}{f(\theta)} \right) q(\theta) f(\theta) d\theta$$

and let $\int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\bar{\theta}} [T(\theta_1) + T(\theta_2)] f(\theta_1) f(\theta_2) d\theta_1 d\theta_2 = A$, then

$$\begin{aligned} A &= \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\bar{\theta}} T(\theta_1) f(\theta_1) d\theta_1 f(\theta_2) d\theta_2 + \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\bar{\theta}} T(\theta_2) f(\theta_2) d\theta_2 f(\theta_1) d\theta_1 \\ &= \int_{\underline{\theta}}^{\bar{\theta}} T(\theta_1) f(\theta_1) d\theta_1 \int_{\underline{\theta}}^{\bar{\theta}} f(\theta_2) d\theta_2 + \int_{\underline{\theta}}^{\bar{\theta}} T(\theta_2) f(\theta_2) d\theta_2 \int_{\underline{\theta}}^{\bar{\theta}} f(\theta_1) d\theta_1 \\ &= 2 \int_{\underline{\theta}}^{\bar{\theta}} \left(\theta - \frac{1 - F(\theta)}{f(\theta)} \right) q(\theta) f(\theta) d\theta \end{aligned}$$

Thus, monopolist's problem can be rewritten as

$$\max_{q(\theta)} \left[2 \int_{\underline{\theta}}^{\bar{\theta}} \psi(\theta) q(\theta) f(\theta) d\theta - \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\bar{\theta}} K(q(\theta_1) + q(\theta_2)) f(\theta_1) f(\theta_2) d\theta_1 d\theta_2 \right]$$

subject to

$$q'(\theta) \geq 0$$

$$\text{where } \psi(\theta) \equiv \theta - \frac{1-F(\theta)}{f(\theta)}.$$

The constraint $q'(\theta) \geq 0$ holds as long as $\psi(\theta)$ is nondecreasing (Myerson's regularity condition).

4.2 With resale, 2 consumers

We suppose that the consumers choices $(T(\theta), q(\theta))$ are their private information after the first stage—the stage they interact with the monopolist. Therefore, consumers will offer each other the products they have with imperfect information. Let us consider bidder 1 with value θ . When he gets $q(\theta'_1)$ amount of the product in the first stage, he will set a price $p(\theta_1)$ to maximize the following:

$$q(\theta'_1) \max_p (1 - F(p)) (p - \theta_1).$$

similarly, bidder 2 will set a price too. However, at most one of the prices will be accepted (since $p(\theta_1) = \psi^{-1}(\theta_1) > \theta_1$.)

Then bidder 1's overall payoff is

$$\begin{aligned} u(\theta_1, \theta'_1) &= q(\theta'_1) [\theta_1 + (1 - F(p(\theta_1))) (p(\theta_1) - \theta_1)] \\ &\quad + \int_{\underline{\theta}}^{\bar{\theta}} \max\{0, q(\theta_2) (\theta_1 - p(\theta_2))\} f(\theta_2) d\theta_2 - T(\theta'_1). \end{aligned}$$

Let us denote $\theta_1 + (1 - F(p(\theta_1))) (p(\theta_1) - \theta_1)$ by $h(\theta_1)$ and $\int_{\underline{\theta}}^{\bar{\theta}} \max\{0, q(\theta_2) (\theta_1 - p(\theta_2))\} f(\theta_2) d\theta_2$ by $g(\theta_1)$. Note that $h(\theta_1)$ and $g(\theta_1)$ are increasing functions of θ . With this new notation, bidders' payoffs are given by

$$u(\theta_1, \theta'_1) = q(\theta'_1) h(\theta_1) + g(\theta_1) - T(\theta'_1).$$

Moreover, since

$$\int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\bar{\theta}} [T(\theta_1) + T(\theta_2)] f(\theta_1) f(\theta_2) d\theta_1 d\theta_2 = 2 \int_{\underline{\theta}}^{\bar{\theta}} T(\theta) f(\theta) d\theta$$

the monopolist's problem is given by

$$\max_{q(\theta), T(\theta)} 2 \int_{\underline{\theta}}^{\bar{\theta}} T(\theta) f(\theta) d\theta - \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\bar{\theta}} K(q(\theta_1) + q(\theta_2)) f(\theta_1) f(\theta_2) d\theta_1 d\theta_2$$

subject to

$$(IR) \ u(\theta) \equiv u(\theta, \theta) \geq 0$$

and

$$(IC) \ u(\theta) \geq u(\theta, \theta').$$

IC can be written as

$$h(\theta)q(\theta) - T(\theta) \geq h(\theta)q(\theta') - T(\theta').$$

IC constraint therefore requires that q has to be nondecreasing. Moreover, it implies the following first-order condition:

$$T'(\theta) - h(\theta)q'(\theta) = 0$$

Integration of the above equality and then integration by parts gives us

$$\begin{aligned} T(\theta) &= \int_{\underline{\theta}}^{\theta} h(\tau)q'(\tau) d\tau + T(\underline{\theta}) \\ &= h(\theta)q(\theta) - h(\underline{\theta})q(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} h'(\tau)q(\tau) d\tau + T(\underline{\theta}) \\ &= h(\theta)q(\theta) - \int_{\underline{\theta}}^{\theta} h'(\tau)q(\tau) d\tau \end{aligned}$$

where the last equality follows from $T(\underline{\theta}) = h(\underline{\theta})q(\underline{\theta})$ (this is because $g(\underline{\theta}) = 0$ and for the optimal menu, we must have $u(\underline{\theta}) = 0$).

Therefore, IC implies

$$u(\theta) = g(\theta) + \int_{\underline{\theta}}^{\theta} h'(\tau)q(\tau) d\tau$$

and

$$\begin{aligned} \int_{\underline{\theta}}^{\bar{\theta}} T(\theta) f(\theta) d\theta &= \int_{\underline{\theta}}^{\bar{\theta}} \left[h(\theta)q(\theta) - \int_{\underline{\theta}}^{\theta} h'(\tau)q(\tau) d\tau \right] f(\theta) d\theta \\ &= \int_{\underline{\theta}}^{\bar{\theta}} h(\theta)q(\theta) f(\theta) d\theta - \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\theta} h'(\tau)q(\tau) d\tau f(\theta) d\theta \end{aligned}$$

and by integration by parts, we obtain

$$\begin{aligned} - \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\theta} h'(\tau)q(\tau) d\tau f(\theta) d\theta &= (1 - F(\theta)) \int_{\underline{\theta}}^{\theta} h'(\tau)q(\tau) d\tau \Big|_{\theta=\underline{\theta}}^{\theta=\bar{\theta}} - \int_{\underline{\theta}}^{\bar{\theta}} q(\theta) h'(\theta) (1 - F(\theta)) d\theta \\ &= - \int_{\underline{\theta}}^{\bar{\theta}} q(\theta) h'(\theta) (1 - F(\theta)) d\theta \end{aligned}$$

Hence, we obtain

$$\int_{\underline{\theta}}^{\bar{\theta}} T(\theta) f(\theta) d\theta = \int_{\underline{\theta}}^{\bar{\theta}} \left(\frac{h(\theta)}{h'(\theta)} - \frac{1-F(\theta)}{f(\theta)} \right) h'(\theta) q(\theta) f(\theta) d\theta$$

From envelope theorem, we obtain $h'(\theta) = F(p(\theta))$ and

$$\left(\frac{h(\theta)}{h'(\theta)} - \frac{1-F(\theta)}{f(\theta)} \right) h'(\theta) = h(\theta) - \frac{1-F(\theta)}{f(\theta)} F(p(\theta)) \equiv \varphi(\theta)$$

The monopolist's problem can be rewritten as

$$\max_{q(\theta)} \left[2 \int_{\underline{\theta}}^{\bar{\theta}} \varphi(\theta) q(\theta) f(\theta) d\theta - \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\bar{\theta}} K(q(\theta_1) + q(\theta_2)) f(\theta_1) f(\theta_2) d\theta_1 d\theta_2 \right]$$

subject to

$$q'(\theta) \geq 0$$

The constraint $q'(\theta) \geq 0$ will hold as long as $\varphi(\theta)$ is nondecreasing.

4.3 Revenue Superiority of With Resale Model

Noting $\theta - \frac{1-F(\theta)}{f(\theta)} = \psi(\theta)$ and $p(\theta) = \psi^{-1}(\theta)$, $\varphi(\theta)$ can be written as

$$\begin{aligned} \varphi(\theta) &= (1 - F(p(\theta)))p(\theta) + F(p(\theta))\theta - \frac{1-F(\theta)}{f(\theta)}F(p(\theta)) \\ &= (1 - F(p(\theta)))p(\theta) + F(p(\theta))\psi(\theta) \\ &= (1 - F(\psi^{-1}(\theta)))\psi^{-1}(\theta) + F(\psi^{-1}(\theta))\psi(\theta) \end{aligned}$$

Since $\psi(\theta) < \theta < \psi^{-1}(\theta)$ for $\bar{\theta} > \theta > \underline{\theta}$, we obtain

$$\varphi(\theta) > \psi(\theta)$$

and revenue for the monopolist is given by

$$\max_{q(\theta)} \int_{\underline{\theta}}^{\bar{\theta}} \psi(\theta) q(\theta) f(\theta) d\theta - \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\bar{\theta}} K(q(\theta_1) + q(\theta_2)) f(\theta_1) f(\theta_2) d\theta_1 d\theta_2$$

in without resale model and

$$\max_{q(\theta)} \int_{\underline{\theta}}^{\bar{\theta}} \varphi(\theta) q(\theta) f(\theta) d\theta - \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\bar{\theta}} K(q(\theta_1) + q(\theta_2)) f(\theta_1) f(\theta_2) d\theta_1 d\theta_2$$

in with resale model. This implies that revenue of the monopolist in the former model is strictly greater since revenue maximizing q in without resale model would give more revenue in with resale model. We hence obtain the following proposition.

Proposition 2 *If the monopolist has a convex cost function, and the consumers have linear utility functions, then the maximal revenue achievable in an environment with resale is more than the maximal revenue achievable in an environment without resale.*

4.3.1 An Example

Let us consider the example where $K(q) = q^2$ and $F(\theta) = \theta$ over $\theta \in [0, 1]$. We obtain, for $A = \int_0^1 K(q(\theta_1) + q(\theta_2)) d\theta_1$ and $B = \int_0^1 \int_0^1 K(q(\theta_1) + q(\theta_2)) d\theta_1 d\theta_2$

$$\begin{aligned} A &= \int_0^1 (q(\theta_1)^2 + q(\theta_2)^2 + 2q(\theta_1)q(\theta_2)) d\theta_1 \\ &= \int_0^1 q(\theta)^2 d\theta + q(\theta_2)^2 + 2q(\theta_2) \int_0^1 q(\theta) d\theta \end{aligned}$$

and

$$\begin{aligned} B &= \int_0^1 \left(\int_0^1 q(\theta)^2 d\theta + q(\theta_2)^2 + 2q(\theta_2) \int_0^1 q(\theta) d\theta \right) d\theta_2 \\ &= \int_0^1 q(\theta)^2 d\theta + \int_0^1 q(\theta_2)^2 d\theta_2 + 2 \int_0^1 q(\theta) d\theta \int_0^1 q(\theta_2) d\theta_2 \\ &= 2 \left(\int_0^1 q(\theta)^2 d\theta + \left(\int_0^1 q(\theta) d\theta \right)^2 \right) \end{aligned}$$

Then the maximization problem of the monopolist without resale is

$$2 \max_{q(\theta)} \left[\int_0^1 (2\theta - 1) q(\theta) d\theta - \int_0^1 q(\theta)^2 d\theta - \left(\int_0^1 q(\theta) d\theta \right)^2 \right]$$

which is maximized at $q(\theta) = \max\{0, \frac{2\theta - 4 + 2\sqrt{2}}{2}\}$ and gives a revenue of 6.209×10^{-2} .

The maximization problem of the monopolist with resale is

$$2 \max_{q(\theta)} \left[\int_0^1 \frac{(\theta + 1)(3\theta - 1)}{4} q(\theta) d\theta - \int_0^1 q(\theta)^2 d\theta - \left(\int_0^1 q(\theta) d\theta \right)^2 \right]$$

which is maximized at $q(\theta) = \max\{0, \frac{4(\theta+1)(3\theta-1)-3}{32}\}$ and gives a revenue of 6.653×10^{-2}

5 Conclusion

In this paper, we have solved for the optimal mechanisms for a monopolist who expects that consumers would resell in a secondary market. Abstaining from the possible disclosure policy of the monopolist, we focused on the nonlinear pricing menus that the monopolist would be able to implement. We have worked on the continuous distributions of the values of the consumers, which has never done before for the monopoly with resale problem. We have shown that the profitability of banning the resale depends on the specifics of the environment.

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