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Efficiency in coalition games with externalities

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Abstract

A natural extension of superadditivity is not sufficient to imply that the grand coalition is efficient when externalities are present. We provide a condition, analogous to convexity, that is sufficient for the grand coalition to be efficient and show that this also implies that the (appropriately defined) core is nonempty. Moreover, we propose a mechanism which implements the most efficient partition for all coalition formation games and characterizes the resulting payoff division.

JEL classification: C71; C72; D62

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1. Introduction

Much of the work on cooperative game theory tries to understand how coalitions behave in environments in which players can cooperate with each other. The central questions this body of literature asks are: first, which coalitions should form and second, how the gains of cooperation should be shared. Economic environments with no externalities (in which what a group of players can achieve by cooperating is independent of what other players do) are best modeled as *characteristic function games* (CFGs), introduced by Von Neumann and Morgenstern (1947). Coalition formation games in economic environments with externalities (in which what a group of players can achieve by cooperating depends on what other coalitions form) were first modeled by Lucas and Thrall (1963) as *partition function games* (PFGs).

By and large, the literature on both CFGs and PFGs has proceeded under the assumption that in cooperative settings, the *grand coalition*—the coalition of all players—will form. In this

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paper, we argue that in many economic environments, such an assumption is unnatural. The reason is that when there are externalities, the payoffs resulting from the formation of the grand coalition may be inefficient—total surplus may not be maximized. Maskin (2003) has also argued against the assumption that the grand coalition will form. His criticism, however, is not based on efficiency considerations but rather on the idea that when there are positive externalities, some players may be better off by remaining separate when others join a coalition. In other words, positive externalities may create an incentive for some players to free ride.

Bloch (1996), Ray and Vohra (1999) and Yi (1997) model economic environments with externalities and show that certain bargaining procedures might result in finer partitions than the grand coalition. However, they do not take the efficiency of the resulting partition into account; nor do they provide a characterization of the payoff division of the resulting games. Bloch (1996) assumes that the division of coalitional surplus is exogenously fixed: the game only determines the coalitional structures. He shows that any core stable allocation can be attained as a stationary perfect equilibrium of the game. Ray and Vohra (1999) consider a game in which the proposers offer a coalition and a contingent payoff division. They prove that there exists a stationary equilibrium of their game and provide an algorithm to determine an equilibrium partition. Yi (1997) characterizes and compares stable coalition structures under some different rules of coalition formation.

There is another strand of work which also models economic environments with externalities. Scarf (1971), Ichiishi (1981) and Zhao (1992) offer a synthesis of Nash equilibrium and the core. They model situations in which players behave cooperatively within each coalition and competitively across coalitions. Their goal is to provide sufficient conditions for the existence of an equilibrium in such settings.

Myerson (1977), Bolger (1989), Pham Do and Norde (2002), Macho-Stadler et al. (2004) and Albizuri et al. (2005) give axiomatic extensions of the Shapley value for CFGs to PFGs. They all assume that the grand coalition will form, even if it is not efficient. In the first part of their paper, de Clippel and Serrano (2006) require the efficiency of the grand coalition and provide upper and lower bounds for the players' payoffs. They also characterize a value by strengthening their marginality assumption.[1] In the second part of their paper, de Clippel and Serrano (2006) consider the case in which the grand coalition does not form and characterize a payoff configuration on the basis of Myerson's (1980) principle of balanced contributions. Their result is similar to Maskin's (2003): they argue that considerations of coalition formation may induce formation of finer partitions than the grand coalition, even if the grand coalition is efficient.

Maskin (2003) provides an axiomatic characterization of a generalized Shapley value and exhibits a mechanism that implements it. His mechanism has the interesting property that the grand coalition may not necessarily form. He axiomatizes the solution to the following noncooperative game: players enter a room sequentially and at stage k, player k enters the room and all players with the lowest index in coalitions in the room simultaneously bid for k. Player k either accepts one of the bids or makes his own (singleton) coalition and the game moves to the $(k+1)$st stage. At the end of the game, the lowest index players in each coalition distribute the promised bids and keep the rest for themselves. If there are negative externalities, the grand coalition always

[1] The value they characterize coincides with the value proposed in this paper when the game is fully cohesive (see Section 4).

forms, even though it may be inefficient. If there are positive externalities, this game might result in a partition finer than the grand coalition, but again this may be inefficient.

For CFGs, the assumption of superadditivity is commonly used. It says that what two coalitions can get by merging should not be less than the sum of what they get separately. This assumption implies the efficiency of the grand coalition in environments with no externalities. However, as we will show in Section 2, a straightforward extension of superadditivity does not imply efficiency of the grand coalition when externalities are present. Therefore, from an efficiency point of view, superadditive PFGs do not necessarily result in the grand coalition.

As an example, consider a symmetric Cournot oligopoly game with three firms. Assume that when two firms merge, because of cost reduction, they do better in the market. But since negative externalities are present, the other firm does worse. When the third firm also joins the coalition, superadditivity implies that their total payoff is no less than what they get separately. However, the members of the three-firm coalition do not necessarily gain over what they get when they were all separate, because of negative externalities.

In Section 2, we show that while superadditivity is not sufficient, a straightforward extension of the convexity assumption in CFGs to PFGs implies that the grand coalition is efficient. In Section 3, we then show that convex PFGs have a nonempty core (for a specific definition of the core). Note that for economic environments with externalities, there can be many definitions of the core. This is because after a deviation, the payoff of the deviating group depends on what the complementary coalition does.

In Section 4, we turn to the question of a noncooperative implementation of the efficient partition in PFGs. We propose a mechanism which gives an efficient predicted partition as well as a payoff division among the players. We also provide a characterization of this value (the resulting payoff division).

2. Efficiency, superadditivity and convexity in PFGs

The set of players is given by $N = \{1, 2, \dots, n\}$. In characteristic function games (CFGs), any coalition $S \subseteq N$ generates a value $v(S)$ and this value is independent of what other agents (not in S) do. In contrast, partition function games (PFGs) allow for externalities, and these are captured by writing v as a function of a coalition and a partition (which has that coalition as a member). That is, in PFGs, any coalition $S \subseteq N$ generates a value $v(S; \rho)$ where ρ is a partition of N with $S \in \rho$.

Formally, given a partition ρ of N and a coalition $S \in \rho$, the pair $(S; \rho)$ is called an *embedded coalition* of N. The set of all embedded coalitions is denoted by $EC(N)$. A PFG is a function v that assigns to every embedded coalition $(S; \rho) \in EC(N)$, a real number $v(S; \rho)$. By convention, $\emptyset \in \rho$ and $v(\emptyset; \rho) = 0$ for all partitions ρ of N.

A PFG is said to have *positive externalities* if for any mutually disjoint $C, S, T \subseteq N$, and for any partition ρ of $N - (S \cup T \cup C)$, we have

$$v\big(C; \{S \cup T, C\} \cup \rho\big) > v\big(C; \{S, T, C\} \cup \rho\big).$$

Similarly a PFG is said to have *negative externalities* if

$$v\big(C; \{S \cup T, C\} \cup \rho\big) < v\big(C; \{S, T, C\} \cup \rho\big).$$

In words, a game has positive (negative) externalities if a merger between two coalitions makes other coalitions better (worse) off.

2.1. Superadditivity

It is well known that if a CFG is superadditive, then the grand coalition is efficient. That is, if for all $S, T \subseteq N$ with $S \cap T = \emptyset$,

$$v(S \cup T) \geqslant v(S) + v(T),$$

then $v(N) \geqslant \sum_{S \in \rho} v(S)$ for all partitions ρ of N.

A natural extension of superadditivity to PFGs used in Maskin (2003) and several others is as follows: A PFG is *superadditive* if for any $S, T \subseteq N$ with $S \cap T = \emptyset$, and any partition ρ of $N - (S \cup T)$,

$$v\big(S \cup T; \{S \cup T\} \cup \rho\big) \geqslant v\big(S; \{S, T\} \cup \rho\big) + v\big(T; \{S, T\} \cup \rho\big).$$

For notational convenience, let us denote $v(S \cup T; \{S \cup T\} \cup \rho)$ by $v_\rho(S \cup T; \{S \cup T\})$ and so on. With this notation, superadditivity can be written as follows: For any $S, T \subseteq N$ with $S \cap T = \emptyset$, and any partition ρ of $N - (S \cup T)$,

$$v_\rho\big(S \cup T; \{S \cup T\}\big) \geqslant v_\rho\big(S; \{S, T\}\big) + v_\rho\big(T; \{S, T\}\big).$$

In PFGs, superadditivity is not enough for the efficiency of the grand coalition, as the following example shows.

Example 1. Consider the following symmetric 3-player PFG: $N = \{1, 2, 3\}$

$$v\big(\{i\}; \big\{\{1\}, \{2\}, \{3\}\big\}\big) = 4 \quad \text{for } i = 1, 2, 3;$$
$$v\big(\{j, k\}; \big\{\{i\}, \{j, k\}\big\}\big) = 9 \quad \text{and} \quad v\big(\{i\}; \big\{\{i\}, \{j, k\}\big\}\big) = 1 \quad \text{for } \{i, j, k\} = N;$$
$$v\big(N; \{N\}\big) = 11.$$

This game is superadditive, but the grand coalition is not efficient since $v(N, \{N\}) = 11 < \sum_{i=1}^{i=3} v(\{i\}, \{\{1\}, \{2\}, \{3\}\}) = 12$.

In this game, the grand coalition is not efficient because there are negative externalities. Although the merging coalitions benefit from merging, others are worse off and the total payoff in the grand coalition is less then the total payoff in some other partition. It can be easily shown that if the externalities are positive and the game is superadditive, then the grand coalition is always efficient.

2.2. Convexity

A stronger assumption on value functions in CFGs is convexity, or supermodularity. Convexity assumption in CFGs implies not only that the merging of two coalitions is beneficial for them, but also that merging with bigger coalitions is more beneficial. That is, the game is convex if there are increasing returns to cooperation (see Moulin, 1988 for a detailed discussion). A natural extension of convexity to PFGs can be given as follows[2]: A PFG is *convex* if for any $S, T \subseteq N$ and any partition ρ of $N - (S \cup T)$,

[2] For more discussion of a different definition of convexity for PFGs, see Appendix A. We are grateful to Geoffroy de Clippel for suggesting this analysis.

$$v_\rho\big(S \cup T; \{S \cup T\}\big) + v_\rho\big(S \cap T; \{S \cap T, S - T, T - S\}\big)$$
$$\geqslant v_\rho\big(S; \{S, T - S\}\big) + v_\rho\big(T; \{T, S - T\}\big).$$

For CFGs, this definition reduces, of course, to the usual definition which is given in Appendix A. As shown by Example 1, superadditivity by itself does not imply the efficiency of the grand coalition. We will argue below that convexity implies that any coalition can achieve at least as much as the sum of what its parts can achieve (independent of whether game has positive, negative or mixed externalities) and in particular, it implies the efficiency of the grand coalition.

Proposition 1. *If a PFG is convex, then for any coalition C, any partition ρ of $N - C$ and ρ' of C,*

$$v_\rho(C) \geqslant \sum_{S \in \rho'} v_\rho(S; \rho').$$

Proof. Fix a coalition $C \subseteq N$ and a partition ρ of $N - C$. The proof is by induction on the cardinality of the partition ρ' of C. Let us denote ρ' by $\{C_1, C_2, \ldots, C_k\}$ with $k \leqslant |C|$ (suppose $C_i \neq \emptyset$).

For notational simplicity denote $C_i \cup C_{i+1} \cup \cdots \cup C_j$ by $\overline{S}_{ij}$ and $\{C_i, C_{i+1}, \ldots, C_j\}$ by $S_{i,j}$.

Induction hypothesis: For any $3 \leqslant l \leqslant k$ and any partition ρ'' of $C_{l+1} \cup C_{l+2} \cup \cdots \cup C_k$,

$$v_{\rho\cup\rho''}\big(\overline{S}_{1,l}; \{\overline{S}_{1,l}\}\big) \geqslant v_{\rho\cup\rho''}\big(\overline{S}_{1,l-2}; \{\overline{S}_{1,l-2}, C_{l-1}, C_l\}\big) + v_{\rho\cup\rho''}(C_{l-1}; S_{1,l})$$
$$+ v_{\rho\cup\rho''}(C_l; S_{1,l}).$$

Induction base: For $l = 3$, in the definition of convexity take $S = \overline{S}_{1,2}$ and $T = \overline{S}_{2,3}$ (so $S \cap T = \{C_2\}$), and get

$$v_{\rho\cup\rho''}\big(\overline{S}_{1,3}; \{\overline{S}_{1,3}\}\big) + v_{\rho\cup\rho''}\big(\{C_2\}; S_{1,3}\big)$$
$$\geqslant v_{\rho\cup\rho''}\big(\overline{S}_{1,2}; \{\overline{S}_{1,2}, C_3\}\big) + v_{\rho\cup\rho''}\big(\overline{S}_{2,3}; \{\overline{S}_{2,3}, C_1\}\big),$$

and by superadditivity applied to the right-hand side of the above inequality, obtain

$$v_{\rho\cup\rho''}\big(\overline{S}_{1,3}; \{\overline{S}_{1,3}\}\big) \geqslant v_{\rho\cup\rho''}(C_1; S_{1,3}) + v_{\rho\cup\rho''}(C_2; S_{1,3}) + v_{\rho\cup\rho''}(C_3; S_{1,3}).$$

Induction proof: Assume that the induction hypothesis is true for $l = t - 1$. We need to show that it is true for $l = t$ as well.

Fix a partition ρ'' of $C_{t+1} \cup C_{t+2} \cup \cdots \cup C_k$. For $S = \overline{S}_{1,t-1}$ and $T = \overline{S}_{2,t}$ (so $S \cap T = \overline{S}_{2,t-1}$), from convexity, obtain

$$v_{\rho\cup\rho''}\big(\overline{S}_{1,t}; \{\overline{S}_{1,t}\}\big) + v_{\rho\cup\rho''}\big(\overline{S}_{2,t-1}; \{\overline{S}_{2,t-1}, C_1, C_t\}\big)$$
$$\geqslant v_{\rho\cup\rho''}\big(\overline{S}_{1,t-1}; \{\overline{S}_{1,t-1}, C_t\}\big) + v_{\rho\cup\rho''}\big(\overline{S}_{2,t}; \{\overline{S}_{2,t}, C_1\}\big). \tag{1}$$

Again from convexity, obtain (by $S = \overline{S}_{1,t-2}$ and $T = \overline{S}_{2,t-1}$, so $S \cap T = \overline{S}_{2,t-2}$)

$$v_{\rho\cup\rho''}\big(\overline{S}_{1,t-1}; \{\overline{S}_{1,t-1}, C_t\}\big) + v_{\rho\cup\rho''}\big(\overline{S}_{2,t-2}; \{\overline{S}_{2,t-2}, C_1, C_{t-1}, C_t\}\big)$$
$$\geqslant v_{\rho\cup\rho''}\big(\overline{S}_{1,t-2}; \{\overline{S}_{1,t-2}, C_{t-1}, C_t\}\big) + v_{\rho\cup\rho''}\big(\overline{S}_{2,t-1}; \{\overline{S}_{2,t-1}, C_1, C_t\}\big). \tag{2}$$

Add up (1) and (2) to obtain

$$v_{\rho\cup\rho''}\big(\overline{S}_{1,t}; \{\overline{S}_{1,t}\}\big) + v_{\rho\cup\rho''}\big(\overline{S}_{2,t-2}; \{\overline{S}_{2,t-2}, C_1, C_{t-1}, C_t\}\big)$$
$$\geqslant v_{\rho\cup\rho''}\big(\overline{S}_{1,t-2}; \{\overline{S}_{1,t-2}, C_{t-1}, C_t\}\big) + v_{\rho\cup\rho''}\big(\overline{S}_{2,t}; \{\overline{S}_{2,t}, C_1\}\big). \tag{3}$$

Use the induction hypothesis for at $l = t - 1$ to obtain

$$v_{\rho\cup\rho''\cup\{C_1\}}\big(\bar{S}_{2,t};\{\bar{S}_{2,t}\}\big) \geqslant v_{\rho\cup\rho''\cup\{C_1\}}\big(\bar{S}_{2,t-2};\{\bar{S}_{2,t},C_{t-1},C_t\}\big) + v_{\rho\cup\rho''\cup\{C_1\}}(C_{t-1};S_{2,t}) + v_{\rho\cup\rho''\cup\{C_1\}}(C_t;S_{2,t}). \tag{4}$$

Use (3), (4), and the induction hypothesis to obtain

$$v_{\rho\cup\rho''}\big(\bar{S}_{1,t};\{\bar{S}_{1,t}\}\big) \geqslant v_{\rho\cup\rho''}\big(\bar{S}_{1,t-2};\{\bar{S}_{1,t-2},C_{t-1},C_t\}\big) + v_{\rho\cup\rho''}(C_{t-1};S_{1,t}) + v_{\rho\cup\rho''}(C_t;S_{1,t}),$$

which completes the induction proof.

Thus, conclude that for any partition ρ' of C, we have

$$v_\rho(C) \geqslant \sum_{S\in\rho'} v_{\rho\cup\rho'}(S). \qquad \square$$

Now, we can state an immediate corollary of the above proposition, which states that convexity implies the efficiency of the grand coalition.

Corollary 1. *If a PFG is convex, then for any partition ρ of N,*

$$v\big(N;\{N\}\big) \geqslant \sum_{S\in\rho} v_\rho(S).$$

It should be noted that convex PFGs do not necessarily have positive externalities. Consider Example 1, with the difference that $v(N;\{N\}) = 15$ instead of 11. This game is convex, yet has negative externalities.

Let us define the PFGs with the property that any coalition can achieve at least as much as the sum of what its parts can achieve by fully cohesive[3] PFGs. Formally, a PFG is *fully cohesive* if for any coalition C, any partition ρ of $N - C$ and ρ' of C,

$$v_\rho(C) \geqslant \sum_{S\in\rho'} v_\rho(S;\rho').$$

In other words, a fully cohesive PFG assigns more to the subset $C \subseteq N$, than to any of its partitions, for any partition of the set $N - C$. Proposition 1 shows that convexity implies full cohesiveness. One might wonder if convexity is too strong to guarantee full cohesiveness. The example in Appendix A, however, shows that a weaker definition of convexity (which is, for CFGs, equivalent to the convexity definition given in this paper) is not enough to guarantee full cohesiveness.

3. The core

In this section, we focus on convex PFGs. Hence, in the games we consider here, the grand coalition is the most efficient partition. Therefore, any other partition can be Pareto improved by making appropriate side-transfers.

3 This term was first defined by Currarini (2003).

For CFGs, a vector of payoffs $x = (x_1, x_2, \dots, x_n)$ is in the core if for all $S \subset N$, we have

$$\sum_{i \in S} x_i \geqslant v(S).$$

A nice feature of convex CFGs is that they have a nonempty core. In CFGs, when group of agents is deciding whether or not to deviate, they do not consider what other agents would do (a coalition's value is independent of what other coalitions form). However, this is not the case in PFGs. In PFGs, one has to make assumptions about what a deviating coalition conjectures about the reaction of the others while defining the core. Hence, there can be many definitions of the core. One simple definition of the core can be given by supposing that the agents in the deviating coalition S presume that agents in $N - S$ will form singletons after the deviation.

Definition 1. A vector of payoffs $x = (x_1, x_2, \dots, x_n)$ is in the *core with singleton expectations*, named *s-core*, if for all $S \subset N$, we have

$$\sum_{i \in S} x_i \geqslant v\big(S; \{S\} \cup [N - S]\big),$$

where $[N - S]$ denotes the partition of $N - S$ to singletons.

The next proposition shows that any convex game has a nonempty s-core.

Proposition 2. *If a PFG is convex, then the s-core is nonempty*

Proof. Define the following CFG with $\hat{v}(S) = v(S; \{S\} \cup [N - S])$. We claim that the CFG $\hat{v}$ is convex.

Take any $S, T \subset N$, with $|T - S| = |S - T| = 1$, then from definition of convexity[4] in the PFG (with $\rho = [N - (S \cup T)]$)

$$v_\rho\big(S \cup T; \{S \cup T\}\big) + v_\rho\big(S \cap T; \{S \cap T, S - T, T - S\}\big) \\ \geqslant v_\rho\big(S; \{S, T - S\}\big) + v_\rho\big(T; \{T, S - T\}\big)$$

or in characteristic function notation

$$\hat{v}(S \cup T) + \hat{v}(S \cap T) \geqslant \hat{v}(S) + \hat{v}(T). \tag{5}$$

Now, by using (5) we need to show that for all S' and T' (without the restriction $|T' - S'| = |S' - T'| = 1$), (5) is true. This follows from the fact that weak convexity is equivalent to convexity for CFGs (see Moulin, 1988).

A very well-known result of convex CFG tells us that $x_i = \hat{v}(\{1, \dots, i\}) - \hat{v}(\{1, \dots, i - 1\})$ is in the core of the game (Shapley, 1971). Hence we obtain that the s-core is nonempty for convex PFGs, since

$$x_i = v\big(\{1, \dots, i\}; \{1, \dots, i\}, \{i + 1\}, \dots, \{n\}\big) \\ - v\big(\{1, \dots, i - 1\}; \{1, \dots, i - 1\}, \{i\}, \dots, \{n\}\big)$$

is in the core of the PFG. □

[4] Note that we are using the definition of convexity only for sets with $|T - S| = |S - T| = 1$. Therefore, this proposition would be true for weakly convex games (discussed in Appendix A) as well.

Note that the s-core definition assumes that the deviating coalitions are very pessimistic in PFGs with positive externalities but are very most optimistic in PFGs with negative externalities (which makes it very easy to deviate). Therefore, we immediately have the following remark.

Remark 1. A convex PFG with negative externalities has a nonempty core (independent of agents' conjectures about what will happen after the deviation).

Another natural core specification is given by the following definition.

Definition 2. A vector of payoffs $x = (x_1, x_2, \dots, x_n)$ is in the *core with cautious expectations*, named *c-core*, if for all $S \subset N$, we have

$$\sum_{i \in S} x_i \geqslant v_{\rho^*}\big(S, \{S\}\big), \quad \text{where } \rho^* = \arg\min_{\rho} v_{\rho}\big(S, \{S\}\big).$$

This definition of the core is analogous to the definition of *α-core* in the literature on CFGs. It is easy to see that if the game has positive externalities, then $\rho^* = [N - S]$ and if the game has negative externalities, then $\rho^* = \{N - S\}$. The following corollary is an implication of Proposition 2.

Corollary 2. *If a PFG game is convex, then the c-core is nonempty.*

There can be other definitions of the core. Maskin (2003) makes the assumption that any deviating coalition S presumes the complementary coalition $N - S$.

Definition 3. A vector of payoffs $x = (x_1, x_2, \dots, x_n)$ is in the *core with merging expectations*, named *m-core*, if for all $S \subset N$, we have

$$\sum_{i \in S} x_i \geqslant v\big(S; \{S, N - S\}\big).$$

The following example shows that convexity does not imply a nonempty m-core.

Example 2. Consider the following symmetric 3-player PFG: $N = \{1, 2, 3\}$

$$v\big(\{i\}; \big\{\{1\}, \{2\}, \{3\}\big\}\big) = 4 \quad \text{for } i = 1, 2, 3;$$
$$v\big(\{j, k\}; \big\{\{i\}, \{j, k\}\big\}\big) = 9 \quad \text{and} \quad v\big(\{i\}; \big\{\{i\}, \{j, k\}\big\}\big) = 6 \quad \text{for } \{i, j, k\} = N;$$
$$v\big(N; \{N\}\big) = 16.$$

One can easily verify that above game is convex: Take $S = \{1, 2\}$ and $T = \{2, 3\}$, we have

$$\begin{aligned} 20 &= v\big(N; \{N\}\big) + v\big(\{2\}; \big\{\{1\}, \{2\}, \{3\}\big\}\big) \\ &> v\big(\{1, 2\}; \big\{\{1, 2\}, \{3\}\big\}\big) + v\big(\{2, 3\}; \big\{\{2, 3\}, \{1\}\big\}\big) = 18. \end{aligned}$$

However, this game has an empty m-core. When a singleton deviates, he presumes that the other two will make a coalition, so he can get a payoff of 6 by deviating. This implies the grand coalition should allocate at least 18, hence the m-core is empty.

Note that the above example has positive externalities. Moreover, Maskin's (2003) definition of the core relies on a very optimistic conjecture about the reactions of other agents when the externalities are positive.

One natural expectation of the deviating agents is that others will take this deviation as given and try to maximize their own payoff. We call this *rational expectations.*

Definition 4. A vector of payoffs $x = (x_1, x_2, \dots, x_n)$ is in the *core with rational expectations*, named *r-core*, if for all $S \subset N$, we have

$$\sum_{i \in S} x_i \geqslant v_{\rho^*}\big(S; \{S\}\big), \quad \text{where } \rho^* = \arg\max_{\rho} \sum_{C \in \rho} v_\rho\big(C; \{S\}\big).$$

Although natural, convex PFGs might have an empty core with rational expectations. The game specified in Example 2 has an empty r-core. When a singleton i deviates, the best j and k can achieve is obtained by forming a coalition since they can get 9 rather than a total of 8. This then implies any core allocation should allocate at least 18, which is not possible.[5]

4. A noncooperative implementation

In this section, we do not impose any restrictions on PFGs. Specifically, we do not require v to have superadditivity, convexity, or positive or negative externalities.

For PFGs, Myerson (1977), Bolger (1989), Pham Do and Norde (2002), Maskin (2003), Macho-Stadler et al. (2004), Albizuri et al. (2005) and de Clippel and Serrano (2006) give extensions of Shapley value of CFGs to PFGs. Except for Maskin (2003) and de Clippel and Serrano (2006), other papers propose that the grand coalition will form. Most of these models consider environments in which the grand coalition is the most efficient partition. Their values and implementations are either not applicable to environments in which the grand coalition is not the most efficient or they result in inefficient partitions. Maskin (2003) proposes that for superadditive games, the grand coalition will form with negative externalities (where the grand coalition is not necessarily efficient), but might not form with positive externalities (where the grand coalition is efficient).

Both Maskin (2003) and de Clippel and Serrano (2006) argue that inefficient outcomes may emerge in superadditive games if one introduces considerations of coalition formation, while we argue in this paper that the "nonformation" of the grand coalition might emerge from efficiency considerations. In an environment in which side payments are allowed, formation of inefficient partitions is implausible since a Pareto superior allocation would be available to the agents.

Below, we propose a game form with the property that all subgame perfect equilibria result in an efficient partition. Moreover, all subgame perfect equilibria result in the same payoff division. Consider the following simple game.[6]

Take any ordering of the players (we consider the natural ordering, since the game will be the same for any other permutation σ of the players): players enter a room sequentially and at

[5] One could also look for a *consistent core* notion. Specifically, assume that a deviating coalition S expects that the players in $N - S$ would form a partition with an associated allocation which is in the core of the reduced game among $N - S$. For the game in Example 2, however, the consistent core is also empty. This is because when a singleton deviates, the core of the two person game contains only the grand coalition of two players.

[6] We provide an alternative game and a characterization in Appendix B.

stage k, agent k enters the room. Suppose that before he enters, agents $1, \dots, k-1$ have already formed a partition of $K-1 = \{1, \dots, k-1\}$ and there are three kinds of people in the room: a boss, dependents and independents. Let $\langle\langle b, D, I \rangle\rangle$ denote the *state* at the end of period $k-1$.

At stage k, agent k proposes a partition of $K = \{1, \dots, k\}$ and a payoff division for all agents in $K-1$. After the proposal, there can be two outcomes:

- If everybody accepts the proposal, then the proposed partition forms, k becomes the new boss and all others become dependents. In this case, the state at the end of period k is given by $\langle\langle k, D \cup I \cup \{b\}, \emptyset \rangle\rangle$.
- If anybody rejects, the old partition and a singleton coalition of k forms and k becomes an independent. In this case, the state at the end of period k is given by $\langle\langle b, D, I \cup \{k\} \rangle\rangle$.

At the end of the game (stage n) independents receive their payoff from the resulting partition.[7] The boss gets the rest of the total payoffs, distributes the promised payoffs to the dependents and gets what is left. This game is played for all possible orderings of players. The *efficient generalized Shapley value* (EGSV) of players is the average of their payoffs obtained for different orderings.

For the above game, it is not difficult to see that at every stage, the newcomer will propose an acceptable offer and become the new boss. Before showing why this is the case, let us introduce some notation.

Let ρ^k denote a partition of K. Define $V(\rho^k)$ as follows:

$$V(\rho^k) = \sum_{C \in \rho^k} v\big(C; \rho^k \cup [N-K]\big),$$

where $[N-K]$ denotes the partition of $N-K$ to singletons.

Let $\bar{\rho}^k$ be defined as follows:

$$\bar{\rho}^k = \arg\max_{\rho^k} V\big(\rho^k\big).$$

That is, $\bar{\rho}^k$ is the most efficient partition given that agents $k+1, \dots, n$ remain singletons.[8] Let $\tilde{\rho}^k$ be the proposed partition at stage k of the game. Finally, let p_i^k be the promised payoff to i at stage k.

Consider the last stage. If any of the agents reject agent n's offer, then the final partition $\rho^{n-1} \cup \{n\}$ forms (where ρ^{n-1} is equal to $\tilde{\rho}^{n-1}$ if $(n-1)$'s offer was accepted at stage $n-1$), independents receive their resulting payoffs, dependents receive what has been promised, and the boss gets what is left. Then, for an independent agent i in $N-1$ to accept the offer, he needs to be promised at least $v(\{i\}; \rho^{n-1} \cup \{n\})$. For a dependent agent i to accept the offer, he needed to be promised at least p_i^{n-1} and the boss needs to be promised at least $V(\rho^{n-1})$ minus what independents and dependents get after a rejection. Hence, agent n can propose an acceptable offer for a sum of $V(\rho^{n-1})$. If his offer is rejected, n receives $v(\{n\}; \rho^{n-1} \cup \{n\})$. Whereas, by proposing the partition $\bar{\rho}^n$, he can get the payoff $V(\bar{\rho}^n) - V(\rho^{n-1})$ which is never less than $v(\{n\}; \rho^{n-1} \cup \{n\})$ (note that $\bar{\rho}^n$ is the most efficient partition of N). Therefore, n's (weakly)

[7] Note that independents form singleton coalitions.

[8] Note that $\bar{\rho}^k$ is not necessarily unique for general class of PFGs, however $V(\bar{\rho}^k)$ is unique. If there is more than one efficient partition, let $\bar{\rho}^k$ denote a selection.

best strategy is to offer $\tilde{\rho}^n = \overline{\rho}^n$ and promise to each agent what he gets if he rejects the offer. This actually proves that this game always results in the most efficient partition.

At stage $n-1$, if the proposal is rejected then partition $\rho^{n-2} \cup \{n-1\}$ forms, then (from backward induction) an independent agent i receives $v(\{i\}; \rho^{n-2} \cup \{n-1\} \cup \{n\})$, dependents receive what has been promised, and the boss receives what is left from $V(\rho^{n-2} \cup \{n-1\})$. Agent $n-1$ can then propose an acceptable offer at a sum of $V(\rho^{n-2})$. By proposing the partition $\overline{\rho}^{n-1}$, he can get the payoff $V(\overline{\rho}^{n-1}) - V(\rho^{n-2})$ which is never less than $v(\{n-1\}; \rho^{n-2} \cup \{n-1\} \cup \{n\})$ (note that $\overline{\rho}^{n-1}$ is the most efficient partition of $N-1$ when n remains singleton). Therefore, $(n-1)$'s best strategy is to offer $\tilde{\rho}^{n-1} = \overline{\rho}^{n-1}$ and promise each agent what he gets (at the end of the game) if he rejects.

Continuing in this fashion, and using backward induction, we can conclude that agent k at stage k proposes the partition $\overline{\rho}^k$ and promises payoffs which add up to $V(\overline{\rho}^{k-1})$ Hence, at the end of the game agent k gets a payoff of $m^k = V(\overline{\rho}^k) - V(\overline{\rho}^{k-1})$ (for the natural ordering) and the most efficient partition is the result of the game. Then EGSV of player i, which is denoted by $\psi_i^{\mathrm{Sh}}(v)$ is the average of these marginal contributions over all possible permutations σ of the players. Formally,[9] for any permutation σ of the players, let ρ_σ^k denote a partition of the set formed by the predecessors in σ, $P_\sigma^k = \{\sigma(j)\colon\ j \in N \text{ and } j \leqslant \sigma^{-1}(k)\}$. Then,

$$V\left(\rho_\sigma^k\right) = \sum_{C \in \rho_\sigma^k} v\left(C; \rho_\sigma^k \cup \left[N - P_\sigma^k\right]\right),$$

and let $\overline{\rho}_\sigma^k$ be defined by

$$\overline{\rho}_\sigma^k = \arg\max_{\rho_\sigma^k} V\left(\rho_\sigma^k\right).$$

Then, we obtain

$$m_\sigma^k = V\left(\overline{\rho}_\sigma^k\right) - V\left(\overline{\rho}_\sigma^{\sigma(\sigma^{-1}(k)-1)}\right)$$

and EGSV is given by

$$\psi_i^{\mathrm{Sh}}(v) = \frac{1}{n!} \sum_\sigma m_\sigma^i.$$

It is easy to see that this value reduces to Shapley value (Shapley, 1953) for superadditive CFGs.

4.1. Characterization of payoffs

Before we state our axiomatic results, let us define what a *value* means.

Definition 5. A *value* is a function ψ that assigns to every PFG v, a unique utility vector $\psi(v) \in \mathbb{R}^n$.

When we consider a fully cohesive PFG, then $\tilde{\rho}^k = \overline{\rho}^k = K$ and the payoff division of above game coincides with the value given by Pham Do and Norde (2002) and de Clippel and Serrano (2006). It is not difficult to see that for arbitrary games the value ψ^{Sh} is not additive. This is true even for (nonsuperadditive) CFGs. Consider the following example.

[9] We are grateful to an anonymous referee for suggesting this notation.

Example 3. Consider the following two symmetric 3-player CFGs: $v(\{i\}) = 2$, $v(\{i, j\}) = 1$, $v(\{1, 2, 3\}) = 4$ and $w(\{i\}) = 1$, $w(\{i, j\}) = 5$, $w(\{1, 2, 3\}) = 4$. Any value which is efficient and symmetric should give $(2, 2, 2)$ to the players in both of the games. However, in the game $v + w$ this value should give $(3, 3, 3)$. Hence, the value is not additive.

On the other hand, ψ^{Sh} is efficient-cover additive. That is, when we consider the efficient cover (or fully-cohesive cover) of two games, then EGSV is additive. More formally, let the *efficient cover* of the game v be defined as follows: for any $S \subset N$ and any partition ρ of $N - S$,

$$\bar{v}_\rho\big(S; \{S\}\big) = \max_{\rho': \text{ partition of } S} \sum_{C \in \rho'} v_\rho(C; \rho').$$

Definition 6. A value ψ is *efficient-cover additive* if $\psi_i(\bar{v}) + \psi_i(\bar{w}) = \psi_i(\bar{v} + \bar{w})$ and *fully efficient* if

$$\sum_{i \in N} \psi_i(v) = V\big(\bar{\rho}^n\big).$$

The other two axioms that will characterize ψ^{Sh} are null player property and efficient-cover anonymity.

Definition 7. A value ψ satisfies the *null-player property* if the following holds: if $V(\bar{\rho}^k_\sigma) - V(\bar{\rho}_\sigma^{\sigma(\sigma^{-1}(k)-1)})$ for all permutations, then $\psi_k(v) = 0$.

Definition 8. A value is *efficient-cover anonymous* if the following holds: For any permutation σ, $\psi(\sigma(\bar{v})) = \sigma(\psi(\bar{v}))$, where $\sigma(\bar{v})(S, \rho) = \bar{v}(\sigma(S); \{\sigma(T)\colon T \in \rho\})$ for each embedded coalition (S, ρ) and $\sigma(x)_i = x_{\sigma(i)}$ for each $x \in \mathbb{R}^n$ and each $i \in N$.

Now, we can introduce the characterization for EGSV.

Proposition 3. *A value is efficient-cover additive, fully efficient, efficient-cover anonymous and satisfies null-player property if and only if it is efficient generalized Shapley value,* ψ^{Sh}.

The proof of this proposition follows from the above observations and Proposition 3 in de Clippel and Serrano (2006).

4.2. An example

In this section, we consider an example to illustrate the difference between our results and results of de Clippel and Serrano's (2006) and Maskin's (2003) in terms of the resulting partition and payoff division. Consider the following PFG, which was considered in both de Clippel and Serrano (2006) and Maskin (2003).

Example 4. Consider the following 3-player PFG: $N = \{1, 2, 3\}$

$$v\big(\{i\}; \big\{\{1\}, \{2\}, \{3\}\big\}\big) = 0 \quad \text{for } i = 1, 2, 3;$$
$$v\big(\{i\}; \big\{\{i\}, \{j, k\}\big\}\big) = 9 \quad \text{for } \{i, j, k\} = N;$$

$$v(\{1,2\};\{\{1,2\},\{3\}\}) = 12;$$
$$v(\{1,3\};\{\{1,3\},\{2\}\}) = 13;$$
$$v(\{2,3\};\{\{2,3\},\{1\}\}) = 14;$$
$$v(N;\{N\}) = 24.$$

Note that in this game the most efficient partition is the grand coalition. However, both the balanced contributions approach of de Clippel and Serrano (2006) and the coalition formation game of Maskin (2003) result in a partition of a singleton and a coalition of two. For this game, the resulting value (the payoff division) is given by $(\frac{43}{6}, \frac{44}{6}, \frac{45}{6})$ in de Clippel and Serrano (2006) and $(7, \frac{22}{3}, \frac{25}{3})$ in Maskin (2003). Note that both these payoff vectors add up to less than 24, the total attainable by the grand coalition.

Let us now apply our implementation. Consider the ordering 1, 2, 3. In the case that agent 2 is independent when agent 3 enters the room, if either agent 1 or agent 2 rejects 3's offer, they will get 0 payoff. Therefore, agent 3 will offer the partition of $\{N\}$ and will offer a payoff of 0 to both agent 1 and 2. Given this, agent 2 will offer agent 1 a payoff of 0 in the second stage (because if agent 1 rejects 2's offer, agent 2 will be an independent and agent 1 will have 0 payoff at the end of the game) and the partition of $\{1,2\}$. Therefore, 3 will face the partition $\{1,2\}$ in the third stage and offer payoff of 0 payoff to agent 1, payoff of 12 to agent 2 and the partition of $\{N\}$. We then can conclude that for the ordering $1,2,3$ the grand coalition is the resulting partition and payoff divisions are $(0, 12, 12)$. For the other orderings we can confirm that the grand coalition will form and payoff divisions are given by $(0, 11, 13)$ if the ordering is $1, 3, 2$; $(12, 0, 12)$ if the ordering is 2, 1, 3; $(10, 0, 14)$ if the ordering is 2, 3, 1; $(13, 11, 0)$ if the ordering is 3, 1, 2; and $(10, 14, 0)$ if the ordering is 3, 2, 1. We therefore conclude that EGSV for above PFG is given by $(\frac{15}{2}, 8, \frac{17}{2})$.

5. Conclusion

When externalities are present, the assumption that "two coalitions together can do better than what they can do separately" (superadditivity) is not enough to conclude that the grand coalition is the most efficient partition. We have identified a natural extension of convexity (supermodularity) to be a sufficient condition implying that "any number of coalitions together can do better than what they can do separately." We have also shown that convexity implies that a particular definition of the core is nonempty. As a remark, we noted that convex PFGs with negative externalities always have a nonempty core and the core with cautious expectations is also nonempty.

There have been different extensions of the Shapley value to PFGs that have been proposed, but except for Maskin (2003) and de Clippel and Serrano (2006), all implicitly or explicitly assume that the grand coalition will form. We have proposed a mechanism which always results in an *efficient* partition and provided a characterization of the resulting payoff division.

Applications of our game to noncooperative setups, such as bidding rings and distribution of payoffs after the bidding in the auction theory setup, are left for future work.

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Appendix A. Different convexity definitions in PFGs

For CFGs, the convexity definition "$v(S \cup T) + v(S \cap T) \geqslant v(S) + v(T)$ for all $S, T \subseteq N$" is equivalent to the definition "$v(S \cup T) + v(S \cap T) \geqslant v(S) + v(T)$ for all $S, T \subseteq N$ with $|S - T| = |T - S| \leqslant 1$." The proof of this argument is not difficult (see Moulin, 1988, p. 112). A straightforward extension of the latter definition to PFGs can be given as: A PFG is *weakly convex* if for any $S, T \subseteq N$ with $|S - T| = |T - S| \leqslant 1$ and any partition ρ of $N - (S \cup T)$,

$$\begin{aligned} &v_\rho\big(S \cup T; \{S \cup T\}\big) + v_\rho\big(S \cap T; \{S \cap T, S - T, T - S\}\big) \\ &\quad \geqslant v_\rho\big(S; \{S, T - S\}\big) + v_\rho\big(T; \{T, S - T\}\big) \end{aligned}$$

or, if the game is superadditive and for any $S \subseteq N - \{i, j\}$ and any partition of $N - (S \cup \{i, j\})$,

$$\begin{aligned} &v_\rho\big(S \cup \{i, j\}; \big\{S \cup \{i, j\}\big\}\big) - v_\rho\big(S \cup \{i\}; \big\{S \cup \{i\}, \{j\}\big\}\big) \\ &\quad \geqslant v_\rho\big(S \cup \{j\}; \big\{S \cup \{j\}, \{i\}\big\}\big) - v_\rho\big(S; \big\{S, \{j\}, \{i\}\big\}\big). \end{aligned}$$

The following example shows that weak convexity is not equivalent to convexity. Moreover, it shows that weak convexity does not imply the efficiency of the grand coalition.

Example 5. Consider the following symmetric 5-player PFG: $N = \{1, 2, 3, 4, 5\} = \{i, j, k, l, m\}$

$$\begin{aligned} &v\big(\{i\}; [N]\big) = 3 \quad \text{for } i \in N; \\ &v\big(\{i, j\}; \big\{\{i, j\}, \{k\}, \{l\}, \{m\}\big\}\big) = 7 \quad \text{and} \quad v\big(\{k\}; \big\{\{i, j\}, \{k\}, \{l\}, \{m\}\big\}\big) = 3; \\ &v\big(\{i, j\}; \big\{\{i, j\}, \{k, l\}, \{m\}\big\}\big) = 9 \quad \text{and for} \quad v\big(\{m\}; \big\{\{i, j\}, \{k, l\}, \{m\}\big\}\big) = 8; \\ &v\big(\{i, j, k\}; \big\{\{i, j, k\}, \{l\}, \{m\}\big\}\big) = 12 \quad \text{and} \quad v\big(\{l\}; \big\{\{i, j, k\}, \{l\}, \{m\}\big\}\big) = 3; \\ &v\big(\{i, j, k\}; \big\{\{i, j, k\}, \{l, m\}\big\}\big) = 17 \quad \text{and} \quad v\big(\{l, m\}; \big\{\{i, j, k\}, \{l, m\}\big\}\big) = 6; \\ &v\big(\{i, j, k, l\}; \big\{\{i, j, k, l\}, \{m\}\big\}\big) = 18 \quad \text{and} \quad v\big(\{m\}; \big\{\{i, j, k, l\}, \{m\}\big\}\big) = 3; \\ &v\big(N; \{N\}\big) = 25. \end{aligned}$$

We can confirm that the above game is weakly convex. First, note that the game is superadditive. Next, check the inequalities required by weak convexity. If $S = \{1\}$, weak convexity implies

$$\begin{aligned} 5 &= v\big(\{1, 2, 3\}; \big\{\{1, 2, 3\}, \{4\}, \{5\}\big\}\big) - v\big(\{1, 2\}; \big\{\{1, 2\}, \{3\}, \{4\}, \{5\}\big\}\big) \\ &> v\big(\{1, 3\}; \big\{\{1, 3\}, \{2\}, \{4\}, \{5\}\big\}\big) - v\big(\{1\}; \big\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\big\}\big) = 4, \end{aligned}$$

and

$$\begin{aligned} 8 &= v\big(\{1, 2, 3\}; \big\{\{1, 2, 3\}, \{4, 5\}\big\}\big) - v\big(\{1, 2\}; \big\{\{1, 2\}, \{3\}, \{4, 5\}\big\}\big) \\ &\geqslant v\big(\{1, 3\}; \big\{\{1, 3\}, \{2\}, \{4, 5\}\big\}\big) - v\big(\{1\}; \big\{\{1\}, \{2\}, \{3\}, \{4, 5\}\big\}\big) = 6. \end{aligned}$$

If $S = \{1, 2\}$, weak convexity implies

$$\begin{aligned} 6 &= v\big(\{1, 2, 3, 4\}; \big\{\{1, 2, 3, 4\}, \{5\}\big\}\big) - v\big(\{1, 2, 3\}; \big\{\{1, 2, 3\}, \{4\}, \{5\}\big\}\big) \\ &\geqslant v\big(\{1, 2, 4\}; \big\{\{1, 2, 4\}, \{3\}, \{5\}\big\}\big) - v\big(\{1, 2\}; \big\{\{1, 2\}, \{3\}, \{4\}, \{5\}\big\}\big) = 5. \end{aligned}$$

If $S = \{1, 2, 3\}$, weak convexity implies

$$\begin{aligned} 7 &= v\big(\{1,2,3,4,5\}; \big\{\{1,2,3,4,5\}\big\}\big) - v\big(\{1,2,3,4\}; \big\{\{1,2,3,4\},\{5\}\big\}\big) \\ &\geqslant v\big(\{1,2,3,5\}; \big\{\{1,2,3,5\},\{4\}\big\}\big) - v\big(\{1,2,3\}; \big\{\{1,2,3\},\{4\},\{5\}\big\}\big) = 6. \end{aligned}$$

Since all inequalities are satisfied, the game is weakly convex. However, the grand coalition is not efficient, since $2v(\{1,2\}; \{\{1,2\},\{3,4\},\{5\}\}) + v(\{5\}; \{\{1,2\},\{3,4\},\{5\}\}) = 26 > 25 = v(N; \{N\})$. We can also confirm that this game is not convex, since for $S = \{1,2,3,4\}$ and $T = \{3,4,5\}$ we have

$$\begin{aligned} 34 &= v\big(S \cup T; \{S \cup T\}\big) + v\big(S \cap T; \{S \cap T, S - T, T - S\}\big) \\ &< v\big(S; \{S, T - S\}\big) + v\big(T; \{T, S - T\}\big) = 35. \end{aligned}$$

Appendix B. An alternative implementation

The game we provide here is the same with the game in Section 4, except for the last stage: Consider the natural ordering of the players. At stage k, agent k enters the room and proposes a partition of $K = \{1, \ldots, k\}$ and a payoff division for all agents in $K - 1$. If everybody accepts the proposal then the proposed partition forms, k becomes the new boss and all others become dependents. If anybody rejects, the old partition and singleton coalition of k forms (k becomes an independent).

At the end of the game (stage n) independents randomly form a partition among themselves with each partition forming with equal probability. The last agent, n, becomes the boss of independents and receives the total of payoffs from the resulting partition among independents. The boss of dependents gets the total payoff from partition among dependents and himself, distributes the promised payoffs to the dependents and receives what is left. This game is played for all possible orderings of players. The *efficient generalized Shapley value2* (EGSV2) of players is the average of their payoffs obtained for different orderings.

For the above game, a newcomer will propose an acceptable offer and become the new boss.

Let $p(k)$ represents the number of partitions of a set with cardinality k. Let ρ^k denote a typical partition of K. Define $V_2(\rho^k)$ as follows:

$$V_2\big(\rho^k\big) = \frac{1}{p(n-k)} \sum_{\rho:\ \text{partition of}\ N-K} \sum_{C \in \rho^k} v\big(C; \rho^k \cup \rho\big).$$

Let $\bar{\rho}^k$ be defined as follows:

$$\bar{\rho}_2^k = \arg\max_{\rho^k} V_2\big(\rho^k\big).$$

That is, $\bar{\rho}_2^k$ is the most efficient partition given that agents $k+1, \ldots, n$ form a random partition among themselves, all being formed with equal probability. Let $\tilde{\rho}_2^k$ be the proposed partition at stage k of the game.

Consider the last stage. The last stage in this game is the same as the last stage in the game given in Section 4. Therefore, the weakly best option is to offer $\tilde{\rho}_2^n = \bar{\rho}_2^n$ and promise to each agent what they get if they reject the offer.

At stage $n-1$, if the proposal is rejected then the partition $\rho^{n-2} \cup \{n-1\}$ forms and agent $n-1$ becomes an independent, then (from backward induction) an independent agent i gets 0 payoff, dependents receive what has been promised, and the boss gets what is left from

$$V_2\big(\rho^{n-2} \cup \{n-1\}\big) = \frac{1}{2}\left(\sum_{C \in \rho^{n-2}} v_{\rho^{n-2}}\big(C; \{n-1\} \cup \{n\}\big) + v_{\rho^{n-2}}\big(C; \{n-1, n\}\big)\right).$$

Agent $n-1$ can then propose an acceptable offer at a sum of $V_2(\rho^{n-2})$. By proposing an offer which will not be accepted, agent $n-1$ will get 0 payoff. Therefore, $(n-1)$'s best strategy is to offer $\tilde{\rho}_2^{n-1} = \bar{\rho}_2^{n-1}$ and promise each agent what they get (at the end of the game, in expected terms) if they reject.

Continuing in this fashion, we can conclude that agent k at stage k proposes the partition $\bar{\rho}_2^k$ and the acceptable promises of payoffs which add up to $V_2(\bar{\rho}_2^{k-1})$. Hence, at the end of the game agent k gets a payoff of $l^k = V_2(\bar{\rho}_2^k) - V_2(\bar{\rho}_2^{k-1})$ (for the natural ordering) and the most efficient partition is the result of the game. Then EGSV2 of player i, which is denoted by $\varphi_i^{\mathrm{Sh}}(v)$ is the average of these marginal contributions. That is,

$$\varphi_i^{\mathrm{Sh}}(v) = \frac{1}{n!}\sum_{\sigma} l_\sigma^i,$$

where l_σ^i is defined similar to m_σ^i.

When we consider a fully cohesive PFG, then $\tilde{\rho}^k = \bar{\rho}^k = K$ and the payoff division of this game coincides with value given by Albizuri et al. (2005). In addition to the axioms of efficient-cover additivity, full efficiency and efficient-cover anonymity we add the following extensions of two axioms which were first given by Albizuri et al. (2005).

A value ψ satisfies the (*efficient-cover*) *oligarchy axiom* if the following holds. If there exists $S \subseteq N$ such that

$$\bar{v}_\rho(C) = \begin{cases} \bar{v}(N; N) & \text{if } C \supseteq S, \\ 0 & \text{otherwise,} \end{cases}$$

then, $\sum_{i \in S} \psi_i(v) = \bar{v}(N; \{N\})$.

A value satisfies (*efficient-cover*) *embedded coalition anonymity* if $\psi_i(\zeta_S \bar{v}) = \psi_i(\bar{v})$ for all bijections ζ_S on $\{(T; \rho)\colon T = S\}$ and for all $i \in N$. According to this axiom, only worths of different embedded coalitions are important, not which embedded coalitions correspond to those worths.

Now, we can introduce the characterization for EGSV2.

Proposition 4. *A value is efficient-cover additive, fully efficient, efficient-cover anonymous and satisfies* (*efficient-cover*) *oligarchy and* (*efficient-cover*) *embedded coalition anonymity axioms if and only if it is EGSV2,* φ^{Sh}.

The proof of this proposition follows from Section 4 of this paper and Theorem 3 in Albizuri et al. (2005).

The difference between the payoff configurations of the game introduced above and the game in Section 4 is that the value given by the above game can be obtained as the Shapley value of an expected CFG (with all partitions being formed with equal probability), whereas the value given by the game in Section 4 can be obtained as the Shapley value of a fictitious CFG (with $\tilde{v}(S) = v(S; \{S\} \cup [N - S])$).

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