Multi-unit Auctions with Budget Constraints*

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Abstract

Motivated by sponsored search auctions, we study multi-unit auctions with budget constraints. In the mechanism we propose, Sort-Cut, understating budgets or values is weakly dominated. Since Sort-Cut's revenue is increasing in budgets and values, all kinds of equilibrium deviations from true valuations turn out to be beneficial to the auctioneer. We show that the revenue of Sort-Cut can be an order of magnitude greater than that of the natural Market Clearing Price mechanism, and we discuss the efficiency properties of its ex-post Nash equilibrium.

Keywords: Multi-Unit Auctions, Budget Constraints, Sponsored Search

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1 Introduction

Consider the advertisement department of a computer manufacturer, who wants to appear in a particular Web search engine's query for "laptops." Search engines use complicated rules to determine the allocation[1] of these advertisements, or "sponsored links," and also their

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[1] By allocations, we mean which advertisements will be displayed, and in which order they will be displayed if there is more than one.

pricing rules. Roughly, the advertisers specify "a value per click" and a daily maximum budget. Allocation and pricing is then determined by a complex algorithm that makes sure that the advertisers are not charged more, per click, than their stated values and also are not charged more than their total budget per day.

Advertisers' true (estimated) values per click and daily budgets are, of course, their private own information, and given any allocation and pricing rule they will act strategically in bidding their values and budgets. It is then natural to ask whether there is any mechanism in which the participants would prefer to reveal their types truthfully: in this model, their per click values and daily budgets. Then there will not be any "gaming" of the mechanism and socially efficient allocations can be implemented. Second-price auctions in single-unit auction problems and different versions of Vickrey-Clark-Groves mechanisms in more general setups have been very successful in implementing socially efficient allocations in "dominant strategies." Unfortunately, a recent impossibility result (Dobzinski et al. 2008) precludes the existence of a truthful mechanism with Pareto-optimal allocations in this important setting.

In this paper, we propose a mechanism–*Sort-Cut*–which yields good revenue and Pareto optimality properties. In our mechanism, understating budgets or values is weakly dominated. Thus the only way a bidder can possibly benefit from lying in our mechanism is by overstating her values or budgets. We also show that the revenue of Sort-Cut is non-decreasing in budgets and values, which in turn yields high revenue for the auctioneer at equilibria.

The idea of Sort-Cut is very similar to the idea of a second-price auction. In second-price auctions without budget constraints, the highest bidder is allocated the object and pays the highest loser's bid to the auctioneer. Uniform-price auctions generalize this idea to multi-unit auctions. The idea is to charge the winners the opportunity cost: the losers' bids. When the bidders have budget constraints, however, losers might not be able to buy all the items if offered: they might simply not be able to afford it. Taking this into account, we modify the algorithm to charge the winners, per item, the value of the highest-value loser, but only up to this loser's budget. After the highest-value loser's budget is exhausted, she would not be able to afford any more items, so we start charging the winners the second-highest loser's value, up to her budget and so on. Given this pricing idea,[2] the winners and losers are determined via a cut-point to clear the market, i.e. all the available items are sold.

Sort-Cut has a number of desirable properties. First of all, it sells all the items so there is no inefficiency in that sense (whereas Borgs et al. 2005 and Goldberg et al. 2001 might leave some of the items unallocated). Second, bidders can only benefit by overstating their values or budgets, a deviation that is the most desirable for the auctioneer. Third, allocation in the equilibrium of Sort-cut is nearly Pareto optimal in the sense that all winners' values are greater than the cut-point bidder's announced value, and all losers' values are smaller than the cut-point bidder's announced value. And lastly, Sort-Cut reduces to a second-price auction when there are no budget constraints.[3]

We show that the revenue of Sort-cut differs at most by the budget of one bidder from the revenue of the "*market clearing price mechanism*." The market clearing mechanism determines a market clearing price and sells all the units for that price. This mechanism, however, is not truthful, and the bidders can benefit from understating their budgets (thereby decreasing the auctioneer's revenue).

After discussing the related literature below, we introduce the model and our mechanism in Section 2. Section 3 discusses the truthfulness, revenue, and Pareto optimality properties of Sort-Cut. Section 4 compares the market clearing price mechanism with Sort-Cut. Finally, Section 5 discusses possible extensions of the current model.

[2] There is a caveat here, which is that the lowest-value winner might not be able to exhaust all of her budget. Then all higher-value bidders are charged first at the lowest-value winner's value up to her unused budget. This makes sense as the lowest-value winner is still a competitor with other winners to buy further items. The pricing for the lowest-value winner, for the same reason, starts from the highest-value loser. She cannot be a competitor with herself.

[3] Generalized second-price auctions studied by Edelman et al. (2007) also has a similar idea in multi-item auctions. In that mechanism the winner of the best item (first sponsored link) is charged the bid of the second-best item, the winner of the second-best item is charged the bid of the third-best item, and so on. In this environment there are no budget constraints and the second-highest bid is always the competitor of the highest value.

Related Literature The problem of multi-unit auctions with budget-constrained bidders was initiated by Borgs et al. (2005). Our model is similar to theirs except that we need not assume that utility is $-\infty$ when budget constraints are violated. They introduce a truthful mechanism that extracts a constant fraction of the optimal revenue asymptotically; however, their mechanism may leave some units unsold. The idea is to group the people randomly into two groups, and use the market clearing price of each group as a posted price for the other group. Another paper that uses the same model is by Abrams (2006), which uses techniques similar to Borgs et al. (2005) but improves upon them; however, it may still leave some units unsold.

In an important paper, Dobzinski et al. (2008) prove an impossibility result. They first assume that the budgets of all players are publicly known, and under this assumption, they present a truthful mechanism that is individually rational and Pareto-optimal. Their mechanism essentially applies Ausubel's multi-unit auction (Ausubel 2004) to this budgeted setting. Then they show that their mechanism is the unique mechanism that is both truthful and Pareto-optimal under the assumption of publicly known budgets. Finally, by showing that their mechanism is not truthful if budgets are private knowledge, they conclude that no mechanism for this problem can be individual rational, truthful, and Pareto-optimal.

Bhattacharya et al. (2010a) show that although the mechanism proposed for public budgets in Dobzinski et al. (2008) is not truthful, for lying to be beneficial the bidder must overstate her budget (value may be overstated or understated). This, together with the fact that the utility of a bidder who is charged more than her budget is $-\infty$, allows them to modify the non-truthful deterministic mechanism into a truthful randomized mechanism. For each bidder, instead of charging her the price specified by Dobzinski et al. (2008), they run a lottery (with appropriate probability) and charge her either 0, or all of her announced budget. Therefore, since a bidder has to pay all of her announced budget with positive probability, the expected utility of over stating the budget becomes $-\infty$. The assumption of the utility being $-\infty$ when the budget constraints are violated does not seem very realistic.

In our work, we drop that assumption by assuming that the utility of bidder who has to pay more than her budget is an arbitrary negative value. Furthermore, we avoid randomized pricing and allocation to guarantee ex-post individual rationality.

Ashlagi et al. (2010) look at budget constraints in position auctions; in their setting, bidders must be matched to slots where each slot corresponds to a certain fraction of the total supply. Bidders are profit maximizers who face budget constraints. They assume that a violation of budget constraints leads to zero utility for the bidder. They propose a modification of the Generalized Second Price mechanism that is Pareto-optimal and envy free. In their setting, the fraction of supply on each of the slots is fixed; this makes their problem more like a matching problem with a discrete structure. However, in our setting, the auctioneer has complete freedom regarding how much of the supply to give to each of the bidders.

Other papers have studied budget constraints in mechanism design, but in settings quite different from ours. Feldman et al. (2008) give a truthful mechanism for ad auctions with budget-constrained advertisers where there are multiple slots available for each query, and an advertiser cannot appear in more than one slot per query. They define advertisers to be click-maximizers; i.e. advertisers do not value their unused budget, they just want to maximize the amount of supply they get. However, in our model, advertisers are profit-maximizers.

Pai and Vohra (2010) look at optimal auctions with budget constraints. In their setting, there is one indivisible good to be allocated, making the setting naturally different from ours. Moreover, they assume $-\infty$ utility if budget constraints are violated. In another paper, Malakhov and Vohra (2008) look at the divisible case; however, they assume that there are only two bidders, one of whom has no budget constraints while the budget constraint of the other one is publicly known. Kempe et al. (2009) look at budget constraints when the bidders have single-unit demand and items are heterogenous. Bhattacharya et al. (2010b) show that a sequential posted price can achieve a constant fraction of the optimal revenue in a budgeted setting with heterogeneous items when the budgets are common knowledge.

Both Borgs et al. (2005) and Dobzinski et al. (2008) argue that lack of quasi-linearity (because of hard budget constraints) is the most important difficulty of the problem. Still, some papers have tried to solve the problem by relaxing the hard budget constraints (Maskin 2000), or by modeling the budget constraint as an upper bound on the value obtained by the bidder rather than her payment (Mehta et al. 2007). It has also been shown (Borgs et al. 2005) that modeling budget constraints with quasi-linear functions can lead to arbitrarily low revenue.

Benoit and Krishna (2001) study an auction for selling two single items to budget-constrained bidders. They focus mainly on the effect of bidding aggressively on an unwanted item with the purpose of depleting the other bidder's budget. A similar effect arises in our model, but the focus of our work is generally very different from theirs. Another paper is that by Che and Gale (1996), which compares first-price and all-pay auctions in a budget-constrained setting and shows that the expected payoff of all-pay auctions is better under certain assumptions. However, they do not consider multi-unit items.

2 The Model and Sort-Cut

There are m divisible units of a good for sale. There are n bidders with a linear demand up to their budget limits. Specifically, each bidder $i \in N = \{1, ..., n\}$ has a two-dimensional type (b_i, v_i) where b_i denotes her budget limit and v_i denotes her private value. Bidder i's utility by getting q (possible fractional) units of the good and paying p is given by

$$u_i(q,p) = \begin{cases} qv_i - p & \text{if } p \leq b_i \\ -C & \text{if } p > b_i \end{cases}$$

where $\infty \geq C > 0$.

We are interested in mechanisms to sell m units to n bidders which have good incentive, efficiency, and revenue properties. The equilibrium concept we use is that of an *ex-post Nash* *equilibrium*. In an ex-post Nash equilibrium, no bidder wants to deviate after she observes all other players' strategies. We believe that this is an appropriate equilibrium concept, as we are motivated by sponsored search auctions. Typically, sponsored search auctions are dynamic auctions; and bids can be changed at any time. Therefore, it is reasonable that in a stable situation (steady state), no bidder will want to deviate even after the bids are revealed. Since the equilibrium concept is ex-post Nash, we do not need to assume strong conditions on private information. Specifically, we can allow for interdependency in two dimensional type within or across bidders.

We focus on direct mechanisms in which bidders announce their types (values and budgets). A mechanism consists of an allocation rule (how many units to allocate to each bidder) and a pricing rule (how much to charge each bidder). It takes the announcements as inputs and produces an allocation and a pricing scheme as an output. We consider mechanisms that satisfy two properties: (i) it must sell all m units, and (ii) it must satisfy individual rationality constraints (i.e. all bidders prefer to participate in the mechanism). Note that the latter condition implies that bidders who are not allocated any units (losers) cannot be charged a positive price. Bidders who are allocated nonzero units (winners), however, will be charged a positive price. Let us first introduce a general and an abstract *pricing rule*.

Definition 1 *The price is set according to a pricing function* $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, *if the marginal price of the next unit is* $\alpha(y)$ *dollars for a buyer who has already spent* y *dollars in the market. In other words, if the pricing of an item is set according to* α, *a buyer with* b *dollars can afford*

$$x(\alpha, b) = \int_0^b \frac{1}{\alpha(y)} dy$$

units of the item. We are interested in pricing rules $\alpha(\cdot)$ *that are nonincreasing and positive. Hence, we assume* $\alpha(y) \leq \alpha(y')$ *for all* $y \geq y'$ *and also* $\alpha(y) > 0$ *for all* y.

The following definition is also convenient for later discussions.

Definition 2 *(Shifted pricing) For a given pricing function* α *and a positive real number* z*, we define the pricing function* $\alpha^z(y)$ *as:*

$$\alpha^z(y) = \alpha(z + y).$$

Less formally, $\alpha^z(y)$ is the pricing function obtained by shifting $\alpha(y)$, z units to right. Note that we have, for any $z \in [0, b]$,

$$x(\alpha, b) = x(\alpha, z) + x(\alpha^z, b - z).$$

Throughout the proofs of our results, we sometimes make use of the terms "better (or worse) pricing function" and "getting to lower prices." We say that α is a *better pricing function* than α' for a bidder if $\alpha(y) \leq \alpha'(y)$ for all y. We say that α *gets to lower prices* than α' for a bidder with budget b if marginal payment at b is lower with α than with α'.

We are now ready to introduce a special class of pricing and allocation rules, which we name *Procedure Cut.*

Definition 3 *Procedure Cut takes budgets and values of the bidders* $(\mathbf{b}, \mathbf{v}) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_{++}$*, a pricing rule* $\alpha(\cdot)$*, and a real number* $c \in (0, \sum_{i=1}^n b_i]$ *as input. First, it sorts bid and value vectors* $(\mathbf{b}, \mathbf{v})$ *in nonascending*[4] *order of values and reindexes them so that* $v_1 \geq v_2 \geq ... \geq v_n$.[5] *Then, it picks* j *such that* $c \leq \sum_{i=1}^j b_i$ *and* $c > \sum_{i=1}^{j-1} b_i$*. Let* $s = \sum_{i=1}^j b_i - c$*. Procedure Cut sets the pricing function of bidders* $1, ..., j-1$ *to* α^c *and the pricing function of bidder* j *to* α^{c+s}*. The allocation of each bidder* $1, .., j-1$ *is such that she spends all her budget, i.e.* $x_i = x(\alpha^c, b_i)$ *for* $i = 1, .., j-1$*. The allocation of bidder* j *is such that she spends* $b_j - s$ *of her budget, i.e.* $x_j = x(\alpha^{c+s}, b_j - s)$*. Bidder* j*'s unused budget is denoted by* s*, where* $s \in [0, b_j)$*. All bidders* $j+1, ..., n$ *get no allocation and pay nothing.*

[4] It breaks ties among equal valued bidders arbitrarily.

[5] Note that after reindexing, budgets are not necessarily sorted in a descending way. A bidder with a high valuation could have a small budget.

Define $X(c,(\mathbf{b},\mathbf{v}))$ *to be the total number of units allocated to all bidders, i.e.* $X(c,(\mathbf{b},\mathbf{v})) = \sum_{i=1}^{j} x_i$*. Bidders* $1,...,j$ *are called full winners, bidder* j *is called a partial winner (or a cut-point bidder), and bidders* $j+1,...,n$ *are called losers.*

We consider pricing rules that are not too high, in the sense that they will be able to sell all the items if all budgets are exhausted. Hence we assume that for $B \equiv \sum_{i=1}^{n} b_i$

$$\alpha(B) \leq \frac{B}{m}.$$

With this assumption, we can easily conclude that $X(B,(\mathbf{b},\mathbf{v})) \geq m$. This is because when $c = B$, all bidders are full winners and their allocations satisfy

$$x\left(\alpha^B, b_i\right) \geq \frac{b_i}{\frac{\sum_{i=1}^{n} b_i}{m}}$$

and hence

$$X(B,(\mathbf{b},\mathbf{v})) = \sum\nolimits_{i=1}^{n} x\left(\alpha^B, b_i\right) \geq m$$

We are interested in rules that sell m units. In the following proposition, we show that for any procedure cut rule, $X(c,(\mathbf{b},\mathbf{v}))$ is strictly increasing and continuous in c. Together with the assumption that $X\left(\sum_{i=1}^{n} b_i,(\mathbf{b},\mathbf{v})\right) \geq m$, this will imply that there will be a unique c such that $X(c,(\mathbf{b},\mathbf{v})) = m$.

Proposition 1 $X(c,(\mathbf{b},\mathbf{v}))$ *is strictly increasing and continuous in* c*.*

Proof. In the Appendix. ■

As noted above, an important corollary of Proposition 1 is that there will be a unique c^* that will satisfy $X(c^*,(\mathbf{b},\mathbf{v})) = m$.

Definition 4 *We call the unique* c^* *with* $X(c^*,(\mathbf{b},\mathbf{v})) = m$ *to be the* cut-point*. Given pricing function* $\alpha(\cdot)$ *and vectors* $(\mathbf{b},\mathbf{v})$*, we name Procedure Cut that sells* m *items (with* $c = c^*$*) to be the* m-Procedure Cut*.*

We now can introduce our new mechanism that we call the *Sort-Cut Mechanism.*

Definition 5 *Sort-Cut is a m-Procedure Cut mechanism in which* $\alpha(\cdot)$ *is a step function defined by (reindexed)* $(\mathbf{b}, \mathbf{v})$*:* $\alpha(y) = v_i$ *for* $y \in (\sum_{k=1}^{i-1} b_k, \sum_{k=1}^{i} b_k]$.[6]

In other words, Sort-Cut takes the vectors $(\mathbf{b}, \mathbf{v})$ and sorts them in nonascending order of values, calculates the unique cut-point c^* according to the pricing function that each full winner (bidders $1, .., j-1$) pays v_j per unit up to a budget of s, then pays v_{j+1} per unit up to a budget of b_{j+1}, then pays v_{j+2} per unit up to a budget of b_{j+2}, and so on, until their budgets are exhausted; the partial winner (bidder j) pays v_{j+1} per unit up to a budget of b_{j+1}, then pays v_{j+2} per unit up to a budget of b_{j+2}, and so on, until she spends $b_j - s$.

Let us the denote the Sort-Cut revenue by $R^S(\mathbf{b}, \mathbf{v})$ (Note that $R^S(\mathbf{b}, \mathbf{v}) = c^*$ where $X(c^*, (\mathbf{b}, \mathbf{v})) = m$). Next, we show that the revenue of Sort-Cut is nondecreasing in the budget and value announcements of the bidders.

Proposition 2 $R^S(\mathbf{b}, \mathbf{v})$ *is nondecreasing in* $\mathbf{b}$ *and* $\mathbf{v}$*.*

Proof. In the Appendix. ■

3 Truthfulness, Revenue, and Near Pareto Optimality

3.1 Truthfulness

In this section, we show that Sort-Cut has good incentive properties. More specifically, we show that no bidder benefits from understating her value or budget. First we argue that three deviations that understate value or budget or both are weakly dominated in ex-post equilibria. Then we consider two other deviations that might potentially decrease revenue and argue that either they are not reasonable or they result in higher revenue.

[6] And also $\alpha(y) = \varepsilon > 0$ for $y \in (B, \infty)$

Proposition 3 *For any bidder i with types* (b_i, v_i), *bidding* (b_i, v_i) *weakly dominates bidding* (b_i, v_i^-) *for* $v_i^- < v_i$.

Proof. Consider any $(\mathbf{b}_{-i}, \mathbf{v}_{-i})$. First of all, if i becomes a loser by bidding (b_i, v_i^-), her utility cannot increase with this deviation. This is because losers' utilities are zero, and by construction, a bidder with type (b_i, v_i) achieves a nonnegative utility by bidding (b_i, v_i). We will look at the possible cases one by one.

- If i is a loser by bidding (b_i, v_i), then she will be a loser by bidding (b_i, v_i^-) (since the pricing function gets better for the winners). Hence her utility cannot increase by this deviation.

- If i is a partial winner by bidding (b_i, v_i) and bidding (b_i, v_i^-) makes her a partial winner, then she will have the same pricing function but she will be able to use less of her budget (since the pricing function for full winners becomes better); hence her utility cannot increase. Bidder i cannot become a winner by bidding (b_i, v_i^-), when she is a partial winner by bidding (b_i, v_i).

- If i is a winner by bidding (b_i, v_i) and bidding (b_i, v_i^-) makes her a winner, her utility does not change. This is because Sort-cut pricing ignores the value of winners in the pricing calculation. If i is a winner by bidding (b_i, v_i) and bidding (b_i, v_i^-) makes her a partial winner, then the original partial winner j (with an unused budget s) has to be a winner after the deviation. We argue that i's utility decreases. It is true that i would get the items at a lower per-unit price after the deviation, but at the same time she is using less of her budget. The argument is that, by this deviation i cannot get to lower-priced items, and this follows from the fact that revenue of Sort-cut cannot decrease after the deviation. More formally, let us denote the unused budget of i after the deviation by s'. We know that $s' \geq s$ (because revenue cannot increase). Bidder i's utility difference with the deviation can be shown to be nonpositive (where α and

c are defined with respect to $(\mathbf{b}, \mathbf{v})$)

$$\begin{aligned}
& \left(x\left(\alpha^{c+s}, b_i - s'\right) v_i - (b_i - s')\right) - \left(x\left(\alpha^c, b_i\right) v_i - b_i\right) \\
= & \left(x\left(\alpha^{c+s}, b_i - s'\right) - x\left(\alpha^c, b_i\right)\right) v_i + s' \\
\leq & \left(x\left(\alpha^{c+s'}, b_i - s'\right) - x\left(\alpha^c, b_i\right)\right) v_i + s' \\
= & \left(x\left(\alpha^{c+s'}, b_i - s'\right) - \left(x\left(\alpha^c, s'\right) + x\left(\alpha^{c+s'}, b_i - s'\right)\right)\right) v_i + s' \\
= & s' - x\left(\alpha^c, s'\right) v_i \\
\leq & s' - \frac{s'}{v_i} v_i \\
= & 0
\end{aligned}$$

where the first inequality follows from $s' \geq s$ and the second inequality follows from $\alpha^c(y) \leq v_i$.

■

Proposition 4 *For any bidder i with type (b_i, v_i), bidding (b_i, v_i) weakly dominates bidding $\left(b_i^-, v_i\right)$ for $b_i^- < b_i$.*

Proof. Consider any $(\mathbf{b}_{-i}, \mathbf{v}_{-i})$: First of all, as in the previous proof, if i becomes a loser by bidding $\left(b_i^-, v_i\right)$, her utility cannot increase with this deviation. We look at the possible cases one by one.

- If i is a loser by bidding (b_i, v_i), then she will be a loser by bidding $\left(b_i^-, v_i\right)$ (since the pricing function gets better for the winners).

- If i is a partial winner by bidding (b_i, v_i) and bidding $\left(b_i^-, v_i\right)$ makes her a partial winner, then she will have the same pricing function but will be able to use less of her budget (since the pricing function for winners becomes better), hence her utility cannot increase. Bidder i cannot become a winner by bidding $\left(b_i, v_i^-\right)$, when she is a partial winner by bidding (b_i, v_i).

- If i is a winner by bidding (b_i, v_i) and bidding $\left(b_i^-, v_i\right)$ makes her a partial winner, then i would be worse off with this deviation. This is because (i) she is using less of her budget, and (ii) her pricing got worse. If i is a full winner by bidding (b_i, v_i) and bidding $\left(b_i^-, v_i\right)$ leaves her a full winner, we can argue that her utility decreases. It is true that i may get the items at a lower per-unit price after the deviation, but at the same time she is using less of her budget. The argument is that by this deviation i cannot get to lower-priced items, which follows from the fact that the revenue of Sort-cut cannot increase after the deviation. More formally, bidder i's utility difference with the deviation can be shown to be nonpositive as follows. Here α and c are defined with respect to $(\mathbf{b}, \mathbf{v})$ and c' $(\leq c)$ is the Sort-cut revenue after deviation.

$$
\begin{aligned}
& \left(x\left(\alpha^{c'+b_i-b_i^-}, b_i^-\right) v_i - b_i^-\right) - \left(x\left(\alpha^c, b_i\right) v_i - b_i\right) \\
= & \left(x\left(\alpha^{c'+b_i-b_i^-}, b_i^-\right) - x\left(\alpha^c, b_i\right)\right) v_i + b_i - b_i^- \\
\leq & \left(x\left(\alpha^{c+b_i-b_i^-}, b_i^-\right) - x\left(\alpha^c, b_i\right)\right) v_i + b_i - b_i^- \\
= & \left(x\left(\alpha^{c+b_i-b_i^-}, b_i^-\right) - \left(x\left(\alpha^c, b_i - b_i^-\right) + x\left(\alpha^{c+b_i-b_i^-}, b_i^-\right)\right)\right) v_i + b_i - b_i^- \\
= & \ b_i - b_i^- - x\left(\alpha^c, b_i - b_i^-\right) v_i \\
\leq & \ b_i - b_i^- - \frac{b_i - b_i^-}{v_i} v_i \\
= & \ 0
\end{aligned}
$$

  where the first inequality follows from $c \geq c'$ and the second inequality follows from $\alpha^c(y) \leq v_i$ for all y.[7]

■

Similarly, we can argue that bidding $\left(b_i^-, v_i^-\right)$ for $b_i^- < b_i$ and $v_i' < v_i^-$ is weakly dominated by bidding (b_i, v_i). This follows from the proofs above. The two previous propositions imply

[7] To see why bidder i's pricing after the deviation is according to $\alpha^{c'+b_i-b_i^-}$, note that her pricing is according to $\alpha^{c'}$ according to types after the deviation, and this translates to $\alpha^{c'+b_i-b_i^-}$ with the original types.

that both $\left(b_i^-, v_i\right)$ and $\left(b_i, v_i^-\right)$ dominate $\left(b_i^-, v_i^-\right)$ when $b_i^- < b_i$ and $v_i^- < v_i$. Applying either of them one more time, we have the following result.

Proposition 5 *For any bidder i with type (b_i, v_i), bidding (b_i, v_i) weakly dominates bidding $\left(b_i^-, v_i^-\right)$ for $b_i^- < b_i$ and $v_i^- < v_i$.*

Propositions 3, 4, and 5 establish that these-revenue decreasing deviations should not occur in equilibrium (they are weakly dominated). There are two deviations, however, that may increase or decrease the revenue. These deviations are "understating budget and overstating value" and "overstating budget and understating value." We now show that the former deviation is not reasonable in the sense that it could be a best response only when the utility with that strategy is zero. Then we show that the latter deviation could happen in equilibrium, yet whenever it is a (strict) profitable deviation from truthful revelation, the revenue increases with the deviation.

Proposition 6 *For any bidder i with type (b_i, v_i), for $b_i^- < b_i$ and $v_i^+ > v_i$, bidding $\left(b_i^-, v_i^+\right)$ can never be in the set of best responses unless bidder i's utility in her best response is* 0.

Proof. Given $(\mathbf{b}_{-i}, \mathbf{v}_{-i})$, suppose that $\left(b_i^-, v_i^+\right)$ is a best response for i where $b_i^- < b_i$ and $v_i^+ > v_i$. Since bidding (b_i, v_i) would give nonnegative utility to bidder i, the utility by bidding $\left(b_i^-, v_i^+\right)$ has to be nonnegative. We claim that bidding $\left(b_i, v_i^+\right)$ is a better response than $\left(b_i^-, v_i^+\right)$, and it is strictly better when the utility by bidding $\left(b_i, v_i^+\right)$ is strictly positive. This implies $\left(b_i^-, v_i^+\right)$ could be a best response only when bidder i's utility in her best response is 0.

Suppose that utility by bidding $\left(b_i^-, v_i^+\right)$ is nonnegative, and consider the utility difference between bidding $\left(b_i^-, v_i^+\right)$ versus bidding $\left(b_i, v_i^+\right)$. The utility difference is clearly zero if i is a loser in both cases. For all other cases, i would be either a partial winner or a full winner by bidding $\left(b_i, v_i^+\right)$. Then, we could see that bidding $\left(b_i, v_i^+\right)$ gives a higher utility than bidding $\left(b_i^-, v_i^+\right)$. The argument is the same as in the proof for Proposition 4: by bidding an extra budget of $b_i - b_i^-$ bidder i can get extra items at a per-unit price lower than her value, leading to a nonzero increase in her utility. ■

In other words, we should not expect to see $\left(b_i^-, v_i^+\right)$ to be played, since it is worse than either (b_i, v_i) or $\left(b_i, v_i^+\right)$.

Proposition 7 *For any bidder i with type (b_i, v_i), for $b_i^+ > b_i$ and $v_i^- < v_i$, whenever bidding $\left(b_i^+, v_i^-\right)$ brings a higher utility to i than bidding (b_i, v_i), the auctioneer's revenue with $\left(b_i^+, v_i^-\right)$ is not lower than the revenue with (b_i, v_i).*

Proof. Given $(\mathbf{b}_{-i}, \mathbf{v}_{-i})$, for some $b_i^+ > b_i$ and $v_i^- < v_i$, suppose that $u_i\left(\left(\mathbf{b}_{-i}, b_i^+\right), \left(\mathbf{v}_{-i}, v_i^-\right)\right) > u_i\left(\left(\mathbf{b}_{-i}, b_i\right), \left(\mathbf{v}_{-i}, v_i\right)\right)$. Since bidder i is budget constrained, she will have to be a partial winner by bidding $\left(b_i^+, v_i^-\right)$ (if she is a full winner her utility would be $-C$, and if she is a loser her utility would be 0).

- If she is a loser by bidding (b_i, v_i), the auctioneer's revenue clearly increases with $\left(b_i^+, v_i^-\right)$. This is because i's ranking with v_i^- is not higher than with v_i, and so by deviating from (b_i, v_i) to $\left(b_i^+, v_i^-\right)$, all full winners remain full winners and i becomes a partial winner.

- If she is a full winner by bidding (b_i, v_i), the partial winner with (b_i, v_i) has to become a full winner after i deviates to $\left(b_i^+, v_i^-\right)$. Otherwise, i would be worse off by bidding $\left(b_i^+, v_i^-\right)$ as she will have a worse pricing function. In this case the revenue has to increase. The argument is that, for this deviation to be beneficial, i has to get lower priced items after the deviation. For this to be the case, the partial winner's unused budget before the deviation, plus i's used budget after the deviation, has to be greater than i's budget b_i. But in this case, the revenue increases, since the new cut point is greater than the old one.

- If she is a partial winner by bidding (b_i, v_i), we need to analyze two cases: (i) i's ranking among the bidders is the same, or (ii) i's ranking is different. For (i), the pricing for

(b_i, v_i) and $\left(b_i^+, v_i^-\right)$ are the same. Since utility with $\left(b_i^+, v_i^-\right)$ is more than utility with (b_i, v_i), this means i is using more of her budget with $\left(b_i^+, v_i^-\right)$. Therefore the revenue increases. For (ii), i's ranking has to be worse with $\left(b_i^+, v_i^-\right)$. Now, similar to the previous case, we argue that total budget of "new full winners" after the deviation plus the used budget of i after deviation has to be greater than b_i. If that is not the case, i cannot get to lower prices.

■

In the above propositions we argued that playing $\left(b_i^-, v_i\right)$, $\left(b_i, v_i^-\right)$ and $\left(b_i^-, v_i^-\right)$ are not reasonable (they are dominated by (b_i, v_i)); playing $\left(b_i^-, v_i^+\right)$ is not reasonable in a weaker sense (it is dominated by a combination of (b_i, v_i) and $\left(b_i, v_i^+\right)$); also, playing $\left(b_i^+, v_i^-\right)$ is reasonable only when it is done by a winner, who becomes a partial winner after deviation and increases the overall revenue. We call the equilibria in which the strategies satisfy these conditions a *refined equilibrium.*

Definition 6 *A* refined equilibrium *is an equilibrium of Sort-Cut where for all bidders i, bidder i does not play $\left(b_i^-, v_i\right)$, $\left(b_i, v_i^-\right)$, $\left(b_i^-, v_i^-\right)$, or $\left(b_i^-, v_i^+\right)$. Moreover, a bidder i plays $\left(b_i^+, v_i^-\right)$ only when $u_i\left(\left(\mathbf{b}_{-i}, b_i^+\right), \left(\mathbf{v}_{-i}, v_i^-\right)\right) > u_i\left(\left(\mathbf{b}_{-i}, b_i\right), \left(\mathbf{v}_{-i}, v_i\right)\right)$.*

The refined equilibrium refinement of Sort-Cut's ex-post Nash equilibria is in the spirit of refinement of considering weakly undominated strategies: (b^-, v), (b, v^-), and (b^-, v^-) are weakly dominated; (b^-, v^+) is dominated by a combination of two strategies; and our refinement requires (b^+, v^-) to be played when it is better than (b, v). In a refined equilibrium, bidders never understate their budgets, and they understate their values only when they also simultaneously overstate their budgets, making them better off than their truthful announcements. Recall that when $u_i\left(\left(\mathbf{b}_{-i}, b_i^+\right), \left(\mathbf{v}_{-i}, v_i^-\right)\right) > u_i\left(\left(\mathbf{b}_{-i}, b_i\right), \left(\mathbf{v}_{-i}, v_i\right)\right)$, $\left(b_i^+, v_i^-\right)$ makes i a partial winner after the deviation and revenue is higher with $\left(b_i^+, v_i^-\right)$ than with (b_i, v_i).

3.2 Revenue

There are eight possible kinds of deviations from truthful revelation, (b_i, v_i). Five of them are discussed in the definition of refined equilibrium, and the remaining three of them, namely (b_i, v_i^+), (b_i^+, v_i), and (b_i^+, v_i^+), can only increase the revenue by Proposition 2. Hence we have the following result.

Theorem 1 *In a refined equilibrium of Sort-Cut, revenue is bounded below by the revenue of Sort-Cut with truthful revelations.*

Proof. Consider any refined equilibrium of Sort-Cut. Let b_i^- and v_i^- denote understating the types, and b_i^+ and v_i^+ denote overstating the types (with respect to true types). We know that (b_i^-, v_i), (b_i, v_i^-), (b_i^-, v_i^-), or (b_i^-, v_i^+) do not occur. Additionally, (b_i^+, v_i^-) could only occur for the current cut-point bidder, and by Proposition 7, if we change it back to (b_i, v_i), revenue cannot increase. Finally, the rest of the bidders are either bidding truthfully or using (b_i, v_i^+), (b_i^+, v_i), or (b_i^+, v_i^+). In any case, changing their bid to their truthful values cannot increase the revenue. Therefore revenue in a refined equilibrium of Sort-Cut is not smaller than revenue of Sort-Cut with truthful revelations. ■

3.3 Near Pareto Optimality

Among different efficiency concepts that could be considered, we consider that of Pareto optimality: we say that an allocation is Pareto optimal if there is no other allocation in which all players (including the auctioneer) are better off and at least one player is strictly better off.[8] In this setup, Dobzinski et al. (2008) has shown that Pareto optimality is equivalent to a "no trade" condition: an allocation is Pareto efficient if (a) all units are sold and (b) a player get a non-zero allocation only if all higher-value players exhaust their budgets. In other words, an allocation is Pareto optimal when, given the *true value* of the

[8] Maximizing social welfare dictates all items to be allocated to the bidder with the highest value, even if this bidder has a very small budget. We follow Dobzinski et al. (2008) and consider Pareto optimality as the appropriate efficiency concept.

partial winner, winners and losers are ordered in the right way: all winners have higher values and all losers have lower values.

The previous subsection demonstrated that Sort-Cut has good revenue properties. The following result pertains to the efficiency (near Pareto optimality) of the equilibria of Sort-Cut. It shows that in any ex-post Nash equilibrium of Sort-Cut, the full winners and losers are ordered in the right way given the *announced value* of the partial winner.

Theorem 2 *Consider any ex-post Nash equilibrium of Sort-Cut where v_j is the announced value of the partial winner j. Every bidder $i \neq j$ who has a true value $v_i^T > v_j$ is a full winner, and every bidder $i \neq j$ who has a true value $v_i^T < v_j$ is a loser in this equilibrium of Sort-Cut.*

Proof. First, consider a bidder i whose value is $v_i^T > v_j$. We prove that she must be a full winner in equilibrium. Assume for the sake of contradiction that bidder i is a loser, so her utility is zero. If she deviates and bids $v_j + \varepsilon$ (for $0 < \varepsilon < v_i^T - v_j$) and her true budget, she will become either a full winner or the cut-point bidder (otherwise revenue of Sort-cut will decrease with this deviation, which is not possible because of Proposition 2). Obviously her utility becomes strictly positive with this deviation (her price per unit is at most v_j). We thus reach the necessary contradiction to her individual rationality.

Now consider a bidder i whose value is $v_i^T < v_j$. Assume for the sake of contradiction that bidder i is a full winner. If b_i is smaller than the unused budget of the cut-point bidder (s), then she gets all items at a per-unit price v_j, and hence she obtains a negative utility. If this is the case, she would be better off announcing her true valuations to guarantee a nonnegative payoff. If $b_i > s$, then we argue that i would be better of by deviating to $(v_j - \varepsilon, b_i)$ for small enough ε. Let us first look at the limiting case in which i deviates to (v_j, b_i) and becomes the cut-point bidder. After this deviation, the unused budget of i would be exactly s. The allocation of original full winners will not change; bidder j will be getting $\frac{s}{v_j}$ more items by paying s more and bidder i will be getting $\frac{s}{v_j}$ less items by paying s less. Therefore, bidder

i's utility increases by $\frac{s}{v_j}\left(v_j - v_i^T\right) > 0$ (in a sense by this deviation, bidder i is selling $\frac{s}{v_j}$ units of the items to bidder j at the per-unit price of v_j). By deviating to $(v_j - \varepsilon, b_i)$, the original full winners' allocations would slightly increase; therefore bidder i's utility increase will be slightly smaller than $\frac{s}{v_j}\left(v_j - v_i^T\right)$.[9] But for small enough ε, it will always be positive, leading again to a contradiction. ■

This theorem establishes that given equilibrium cut-point value, all winners and losers will be rightly placed. But since the cut-point bidder may be misplaced, this does not imply full Pareto optimality. Consider the following example.

Example 1 *There are 2 units of the item to be sold, and there are four bidders with budget-value pairs* $(18, 19), (1, 9), \left(\frac{17}{9}, 8\right)$, *and* $(10, 1)$. *For this setup, it can be confirmed that bidders announcing their types (budget,value) as* $(18, 19), (1, 9), (36, 18)$, *and* $(10, 1)$ *constitute an ex-post equilibrium of Sort-Cut. In this equilibrium, bidder 3 overstates her value and budget and becomes the partial winner. Although the full winners and the losers are rightly ranked according to the announced value of the partial winner, the allocation is not Pareto optimal. Bidder 3 gets a positive allocation even though bidder 2 has a higher value and zero allocation.*

As an aside, note that the revenue of Sort-Cut in this ex-post equilibrium is 18 *while the revenue with truthful types is* $18 + \frac{17}{9}$.

4 Market Clearing Price Mechanism and Sort-Cut

In this section we compare Sort-Cut with the well-known *Market Clearing Price Mechanism (MCPM).* MCPM is a mechanism that sells m items to all interested bidders at a fixed price. That is, in MCPM all items are sold p dollars per unit and all bidders whose values are strictly greater than p spend all their budgets to buy these items (the bidders with values equal to p could be partially spending their budgets to clear the supply).

[9] There is an implicit continuity assumption here. However, it is not difficult to show that utilities of the bidders are continuous in type announcements.

Definition 7 *The Market Clearing Price Mechanism is an m-Procedure cut mechanism with fixed pricing rule, $\alpha(y) = p^*$ for all $y \geq 0$ where p^* satisfies $v_j \geq p^* > v_{j-1}$.*

One can easily argue that there will be a unique p^*. Consider a fixed pricing rule $\alpha(y) = p^*$ that satisfies the above definition. Then for any fixed pricing rule $\alpha(y) = p$ with $p > p^*$, we have $p > v_j$ (for v_j defined by $\alpha(y) = p$); and for any $p < p^*$, we have $p \leq v_{j-1}$ (for v_{j-1} defined by $\alpha(y) = p$).

Although it is a natural mechanism, as we demonstrate below, MCPM lacks good incentive properties.

Proposition 8 *Under the MCPM, overstating budget or value is weakly dominated by bidding true types, i.e, for bidder i with type (b_i, v_i), announcing $\left(b_i^+, v_i\right), \left(b_i, v_i^+\right)$, and $\left(b_i^+, v_i^+\right)$ are all weakly dominated by (b_i, v_i).*

Proof. Consider bidder i with type (b_i, v_i) who announces her type truthfully.

- If she is a full winner, she is indifferent to announcing v_i^+ and would be strictly worse off by announcing b_i^+ (she would either get negative payoff by staying a full winner or will get zero utility by becoming a partial winner or a loser).
- If she is a partial winner, since overstating her value or budget can only increase the market clearing price p^*, she never can obtain a strictly positive payoff by deviating to v_i^+ or b_i^+.
- If she is a loser, by overstating her value or budget, she may become a winner, but the market clearing price after deviation is going to be greater than the previous market clearing price and hence greater than her value.

■

However, understating her value or budget in general can be beneficial. Consider the following example.

Example 2 *Consider two bidders with (budget,value) pairs* $(16, 10)$ *and* $(8, 9)$ *where the supply is* $m = 3$. *Under truthful report of types, the market clearing price is* $p^* = 8$. *However, if bidder 1 understates her value as* 7, *the new market clearing price will be* 7 *with the first bidder spending* 13 *of her budget for an allocation of* $\frac{13}{7}$ *units. Her new payoff is* $(10 - 7)\frac{13}{7} \cong 5.57$ *versus* $(10 - 8)\,2 = 4$, *which shows that her understatement of value is a profitable deviation. Similarly, if bidder* 1 *understates her budget as* 10, *the new market clearing price will be* 6 *with the first bidder spending* 10 *of her budget for an allocation of* $\frac{10}{6}$ *units. Her new payoff is* $(10 - 6)\frac{10}{6} \cong 6.67$, *which shows that her understatement of budget is a profitable deviation. In fact, for this example an ex-post equilibrium is when bidders announce their types as* $\left(\frac{2700}{361}, 10\right)$, $\left(\frac{2430}{361}, 9\right)$, *which brings only a revenue of* 14.21 *(in comparison with truthful revenue of* 24.*)*

The above discussion illustrates that the revenue from an (undominated) ex-post equilibrium of MCPM is bounded above by the revenue of MCPM with truthful revelations. Next, we obtain a lower bound for the revenue of Sort-Cut. For any announcements $(\mathbf{b}, \mathbf{v})$, we show that the revenue difference between MCPM and Sort-Cut is at most equal to the maximum budget of the players. For the same announcement of the types, since Sort-Cut's pricing function for the winners decreases in winners' budgets, whereas MCPM's pricing is constant, MCPM's revenue would be higher than the revenue of Sort-Cut. However, the following proposition shows that the difference in revenues is bounded above by the maximum of the winners' budgets. Let $R^M(\mathbf{b}, \mathbf{v})$ denote MCPM's revenue and $b_{\max}$ denote the maximum budget of the bidders.[10]

Proposition 9 *For any announcements* $(\mathbf{b}, \mathbf{v})$, $R^M(\mathbf{b}, \mathbf{v}) - R^S(\mathbf{b}, \mathbf{v}) \leq b_{\max}$.

Proof. Given $(\mathbf{b}, \mathbf{v})$, let Sort-Cut's cut point be denoted by c^*, and let MCPM's cut point (fixed price) be denoted by c^{**}. We argue that $c^{**} - c^* \leq b_{\max}$. By the definition of MCPM, $c^{**} = m \times p^*$ where p^* satisfies $v_j \geq p^* > v_{j-1}$ and v_j is the partial winner in MCPM.

[10] $b_{\max}$ can also be taken as the maximum budget of the full winners.

Since $c^* \leq c^{**}$, j cannot be a full winner in Sort-Cut. If she is a partial winner, then $c^{**} - c^* \leq b_j \leq b_{\max}$ holds since the difference between c^{**} and c^* is smaller than b_j. If j is a loser in Sort-Cut, then we argue as follows. At least one of the winners of Sort-cut has to pay at most p^* per unit (otherwise the revenue of Sort-Cut has to be greater than c^{**}). Now, this bidder's budget has to be greater than $c^{**} - c^*$, because otherwise her price per unit cannot be smaller than p^*. Hence, $c^{**} - c^* \leq b_{\max}$. ■

Proposition 9, together with proposition 1, establishes the following result.

Theorem 3 *Let us denote the revenue of MCPM with the truthful revelation of types by R^*. Then the revenue of any refined equilibrium of Sort-Cut is not lower than $R^* - b_{\max}$.*

Unlike Sort-Cut, we next show an example where MCPM obtains a revenue that is an order of magnitude (as the number of bidders) lower than R^*.

Example 3 *Consider two types of bidders with budget, value pairs $(b_0, v_0) = (16, 18)$ and $(b_1, v_1) = (8, 9)$; our basic example has one bidder of each type with a supply of $m = 3$ units. Under truthful reports of budgets and values, the market-clearing price is $p = 8$. Let us look for an ex-post equilibrium, in which the announcements are $(a_0, 18)$ and $(a_1, 9)$. The pair of values a_0 and a_1 solve the optimization problems of $\max(v_i - p)\frac{a_i}{p}$ for $i = 0, 1$ where p is the market clearing price for the given announcements and supply. In our case $p = \frac{a_0 + a_1}{3}$.*

Thus the optimization problem becomes $\max f(a_i) = \frac{3v_i a_i}{a_i + a_{1-i}} - a_i$. Taking derivatives, we get $f'(a_i) = \frac{3v_i a_{1-i}}{(a_1 + a_{1-i})^2} - 1$, with $f"(a_i) < 0$. Solving the pair of first-order equations by setting $f'(a_0) = f'(a_1) = 0$, we get $a_1 = 6$ and $a_2 = 12$ for a market clearing price of 6. The total revenue of this equilibrium is therefore 18 compared to $R^* = 24$.

If we now scale the example to have N bidders of each type and a supply of $3N$, we may assume that all the optimal budget announcements of each type of bidder are the same by symmetry. The clearing price stays unchanged at $p = \frac{N(a_0 + a_1)}{3N} = \frac{a_0 + a_1}{3}$. The optimization problem for determining each a_i remains identical, giving the same solutions as before.

However, the revenue now is $16N$ *compared to* $R^* = 24N$ *and is thus a whole third less than* R^**, while by Theorem 3, Sort-Cut's revenue in a refined equilibrium is not smaller than* $24N - 16$.

5 Conclusion and Discussion

In this paper, we have introduced a mechanism to sell m divisible units to a set of bidders with budget constraints. In this practically important setting, where a mechanism that is simultaneously truthful and Pareto optimal is precluded, our mechanism, Sort-Cut, achieves good incentive, revenue, and efficiency properties. Specifically, in Sort-Cut, (i) there are profitable deviations from truthful revelations of types, but these can only happen in a revenue-increasing way; (ii) in a refined ex-post equilibrium, the revenue of Sort-Cut is bounded below by $R^* - b_{\max}$, and (iii) the equilibrium allocation is *nearly Pareto efficient* in the sense that full winners and losers are ordered in the right way given the announced value of the partial winner. We then compare Sort-Cut to a well-known mechanism, Market Clearing Price Mechanism (MCPM). We show that in MCPM, (i) revenue increasing deviations are dominated, and (ii) the revenue can be much smaller than $R^* - b_{\max}$.

There are many ways our work can be generalized. In the context of online advertisement auctions, our model can be interpreted as "there is a *single* sponsored link that gets m clicks a day and there are n advertisers." However, in reality, there are many sponsored links. In generalized second-price auctions studied by Edelman et al. (2007), the winner of the best item (first sponsored link) is charged the bid of the second-best item, the winner of the second-best item is charged the bid of the third-best item, and so on. In this environment there are no budget constraints and the second-highest bid is always the competitor of the highest value. The idea of Sort-Cut can be applied in this setup with budget constraints. More specifically, it would be interesting to consider a model in which there are budget-constrained bidders and multiple slots available for a query (in which an advertiser cannot appear in more than one slot per query).

In our model, we consider a setting of hard budget constraints in which the bidders cannot spend more than their budgets. Extending our results to a soft-budget problem in which bidders are able to finance further budgets at some cost is a promising direction. One can model this kind of soft-budget constraint as specifying value per-click up to some budget, then specifying a smaller value per-click up to some other extra budget, and so on. By replicating a bidder into as many copies as the number of pieces in her value/budget function, and allowing them all to participate in our mechanism, it seems reasonable that we may preserve some of the desirable properties of Sort-Cut.

One very important extension is to consider the environment of multi-item auctions with budget constraints. Again consider advertisement department of a computer manufacturer, who this time wants to appear in a search engine's queries for "laptops" and "desktops." This advertisement department might have a total budget to allocate between all online ads and their per-click values for different items might be different. For instance one firm might have higher per-click values for desktops, but lower per-click values of laptops, as compared to a second firm. Designing an allocation and pricing rule that would have good efficiency and revenue properties for this setup is very challenging. Devanur et al. (2002) provided an algorithm for finding "market clearing prices." This mechanism, however, lacks good incentive properties. Bidders would have an incentive to understate their budgets, thereby decreasing the prices. The extension of Sort-Cut to this setting is not straightforward, since how bidders will want to split the budgets between different items would depend on the pricing rule of each of these items, which in turn depends on the budget splits. Bidders' effective valuations for different goods are given by the ratios of "per-click values and the average prices" of different items. This multi-item extension seems to be the most important yet also the most challenging extension of our model.

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6 Appendix

6.1 Proof of Proposition 1

First, note that $x(\alpha^c, b)$ is weakly increasing in c : since α is nonincreasing, for $c' \geq c \geq 0$, we have $\alpha^{c'}(y) = \alpha(y + c') \leq \alpha(y + c) = \alpha^c(y)$, and hence

$$x\left(\alpha^{c'}, b\right) = \int_0^b \frac{1}{\alpha^{c'}(y)} dy \geq \int_0^b \frac{1}{\alpha^c(y)} dy = x(\alpha^c, b).$$

Also, obviously $x(\alpha^c, b)$ is strictly increasing in b.

Now, we can show that $X(c, (\mathbf{b}, \mathbf{v}))$ is strictly increasing in c. Consider $c' > c \geq 0$; we have

$$X(c, (\mathbf{b}, \mathbf{v})) = \left(\sum_{i=1}^{j-1} x(\alpha^c, b_i)\right) + x\left(\alpha^{c+s}, b_j - s\right)$$

where j satisfies $c \leq \sum_{i=1}^{j} b_i$ and $c > \sum_{i=1}^{j-1} b_i$ (and $s = \sum_{i=1}^{j} b_i - c$). For $c' > c$, we can have one of the two cases: either j is the same or j is bigger.

If j is bigger, then we have

$$\begin{aligned} X(c', (\mathbf{b}, \mathbf{v})) &> \sum_{i=1}^{j} x\left(\alpha^{c'}, b_i\right) \\ &\geq \left(\sum_{i=1}^{j-1} x(\alpha^c, b_i)\right) + x\left(\alpha^{c'}, b_j\right) > X(c, (\mathbf{b}, \mathbf{v})) \end{aligned}$$

This is because $x\left(\alpha^{c'}, b_i\right) \geq x(\alpha^c, b_i)$ for all $i = 1, .., j-1$ and $x\left(\alpha^{c'}, b_j\right) > x\left(\alpha^{c+s}, b_j - s\right)$ since $c' > c + s$.

If j is the same (if $c' < c + s$), then we have

$$\begin{aligned} X(c', (\mathbf{b}, \mathbf{v})) &= \left(\sum_{i=1}^{j-1} x\left(\alpha^{c'}, b_i\right)\right) + x\left(\alpha^{c'+s'}, b_j - s'\right) \\ &> \left(\sum_{i=1}^{j-1} x(\alpha^c, b_i)\right) + x\left(\alpha^{c+s}, b_j - s\right) = X(c, (\mathbf{b}, \mathbf{v})) \end{aligned}$$

where $s' = \sum_{i=1}^{j} b_i - c' < s$. This is because $x\left(\alpha^{c'}, b_i\right) \geq x(\alpha^c, b_i)$ for all $i = 1, .., j - 1$ and

$x\left(\alpha^{c'+s'}, b_j - s'\right) > x\left(\alpha^{c+s}, b_j - s\right)$ since $c' + s' = c + s$ and $b_j - s' > b_j - s$.

Next, we show that $X\left(c, (\mathbf{b}, \mathbf{v})\right)$ is continuous in c. By definition, $x\left(\alpha^c, b\right)$ is continuous in c and b (this is because $x\left(\alpha^c, b\right) = \int_0^b \frac{1}{\alpha(y+c)} dy$ and is continuous in c and b even when α is not a continuous function). Moreover,

$$X\left(c, (\mathbf{b}, \mathbf{v})\right) = \left(\sum_{i=1}^{j-1} x\left(\alpha^c, b_i\right)\right) + x\left(\alpha^{c+s}, b_j - s\right).$$

If c increases from c to $c + \varepsilon$, j changes only when $s = 0$. If $s \neq 0$, then $X\left(c, (\mathbf{b}, \mathbf{v})\right)$ is obviously continuous in c as all of the terms in the summation are continuous in c. If $s = 0$, then

$$X\left(c + \varepsilon, (\mathbf{b}, \mathbf{v})\right) = \left(\sum_{i=1}^{j} x\left(\alpha^{c+\varepsilon}, b_i\right)\right) + x\left(\alpha^{c+\varepsilon+s'}, b_{j+1} - s'\right)$$

and this goes to $X\left(c, (\mathbf{b}, \mathbf{v})\right)$ as ε goes to zero. This is because $\sum_{i=1}^{j} x\left(\alpha^{c+\varepsilon}, b_i\right) \to \sum_{i=1}^{j} x\left(\alpha^c, b_i\right) = X\left(c, (\mathbf{b}, \mathbf{v})\right)$ and $x\left(\alpha^{c+\varepsilon+s'}, b_{j+1} - s'\right) \to 0$ since $s' \to b_{j+1}$.

6.2 Proof of Proposition 2

Consider bidder i with announced type (b_i, v_i).

- First, we show that revenue is nondecreasing in budgets. Consider bidder i who decreases her budget to $b_i^- < b_i$. We show that revenue cannot increase with this deviation.
  - If bidder i was originally a loser by announcing (b_i, v_i), then she cannot become a winner or partial winner by deviating to $b_i^- < b_i$. This is because by this deviation, the pricing function for everybody becomes better and winners pay less per unit. Therefore the revenue cannot increase.
  - Next, consider bidder i who is a partial winner by bidding b_i. If bidder i deviates to b_i^- and becomes a loser, then the revenue has to decrease since the set of losers becomes larger with this deviation. If she deviates to b_i^- and remains a partial

winner, since all winners' pricing gets better, the revenue has to decrease. If she deviates to b_i^-, she cannot become a full winner. If this were the case, the pricing function for every (full or partial) winner gets better, and then the total number of units allocated will be greater than m.

– Lastly, consider bidder i who is originally a winner by announcing (b_i, v_i). If she deviates to b_i^- and if she becomes a loser or a partial winner after the deviation, then the revenue clearly decreases. This is because the set of full winners before the deviation is a strict superset of the set of full winners after the deviation. Now consider the case where bidder i deviates to b_i^- and remains a winner. Let us denote $b_i - b_i^-$ by Δ. Suppose that the initial cut point is c and the new cut point after the deviation is c'. Let α be the n-piece step function defined by $(\mathbf{b}, \mathbf{v})$. Note that the initial revenue is c and the new revenue is c'. We will show that $c \geq c'$.

  Since i has understated her budget, there will be a shortage of demand and the pricing of all original winners will be better. Therefore, with this deviation, all original winners except i will be allocated (weakly) more units of the object. Assume for contradiction that $c' > c$. This means that there will be new winners who use an extra budget strictly greater than Δ, say Δ'. We now argue that the extra units allocated to these new winners have to be greater than the number of units i is giving up with the deviation. Extra units allocated to new winners are priced at the values starting from the new cut point $c + \Delta'$ (according to $(\mathbf{b}, \mathbf{v})$) and the total budget used is Δ'. The number of units i is giving up are priced at the values in the range of c to $c + \Delta < c + \Delta'$ and the total budget used is Δ. Since extra units are given with higher budget ($\Delta' > \Delta$) and lesser prices ($c + \Delta < c + \Delta'$) than the units given up, we conclude that with the assumption $c' > c$, the total number of units allocated has to be strictly greater than m, which is a contradiction.

We can present this argument more formally. Consider the case in which Δ is small enough so that the original partial winner j remains a partial winner. All full winners $k \neq i$ with $k < j$ will be allocated more items since j will be using more of her budget after the deviation. Let us consider the difference between the total amounts allocated to bidders i and j before and after the deviation. Bidder i's allocation is decreased by

$$A \equiv x\left(\alpha^{c}, b\right) - x\left(\alpha^{c+\Delta'}, b - \Delta\right)$$

since

$$x\left(\alpha^{c+\Delta'}, b - \Delta\right) > x\left(\alpha^{c+\Delta'}, b - \Delta'\right).$$

We have

$$\begin{aligned} A &< x\left(\alpha^{c}, b\right) - x\left(\alpha^{c+\Delta'}, b - \Delta'\right) \\ &= x\left(\alpha^{c}, \Delta'\right). \end{aligned}$$

On the other hand, bidder j's allocation is increased by

$$\begin{aligned} B &\equiv x\left(\alpha^{c+s}, b_j - s + \Delta'\right) - x\left(\alpha^{c+s}, b_j - s\right) \\ &= x\left(\alpha^{c+b_j}, \Delta'\right) \end{aligned}$$

since

$$x\left(\alpha^{c+b_j}, \Delta'\right) \geq x\left(\alpha^{c}, \Delta'\right)$$

we conclude $B > A$. The argument for the case when the deviation results in a change of the partial winner is very similar but not illuminating. Thus, the total number of units allocated has to increase after the deviation.

- Now, we show that revenue is increasing in values. Consider bidder i who increases her value to $v_i^+ > v_i$. We show that revenue cannot decrease with this deviation.

– First, if bidder i is a winner by bidding (b_i, v_i) and she deviates to $v_i^+ > v_i$, then she remains a winner after the deviation, and the revenue does not change. This is because Sort-Cut's allocation and pricing rule is invariant to full winners' values (so long as they remain full winners).

– Second, consider a bidder i who is a loser by bidding (b_i, v_i) and deviates to $v_i^+ > v_i$. If she remains a loser after the deviation, since the pricing function for winners gets worse, the revenue has to increase. Let us now consider the deviation which makes i a partial winner. If the partial winner becomes a full winner after the deviation ($v_i^+ < v_j$ where j is the original partial winner), the revenue obviously increases with the deviation, since the cut-point has increased.

  Let us consider the case in which $v_i^+ > v_j$; i becomes a partial winner and j becomes a loser after the deviation. Assume for contradiction that the revenue decreases with the deviation. If this is the case, it can be seen that the pricing function for all winners becomes worse after the deviation (total unspent budget of price setters with $v_k \geq v_i$ becomes greater and some of the values increase). Hence all full winners will be allocated less units of items after the deviation. This implies that the number of units allocated to i after the deviation has to be greater than the number of units allocated to j before the deviation. But again, the pricing function for i after the deviation is worse than the pricing function for j before the deviation. For i to be allocated more, her budget spent after the deviation has to be greater than j's budget spent before the deviation, which is a contradiction.

  Suppose i is currently a loser and deviates to v_i^+ and becomes a full winner. We can split this into two deviations. First, i deviates to $v_i^{+\prime} > v_j$ and becomes a partial winner (which increases the revenue), then she deviates to v_i^+ and becomes a full winner which will next be shown to increase the revenue.

– Lastly, consider bidder i who is a partial winner by bidding (b_i, v_i). It is obvious that she cannot become a loser after deviating to v_i^+. If she deviates to v_i^+ and remains a partial winner, then the pricing function for all winners get worse, hence the revenue has to increase. If she deviates to v_i^+ and becomes a full winner, then we argue that revenue has to increase.

  Consider the case where i is currently the partial winner, and she deviates to $v_i^+ > v_{i-1}$ (where bidder $i-1$ has the next highest value after bidder i) so that $i-1$ is the new partial winner and i is a full winner. Denote the original unused budget of bidder i by s_i' and after deviation, the unused budget of bidder $i-1$ by s_{i-1}'. It suffices to show that $s_i' \geq s_{i-1}'$. Assume for contradiction that $s_{i-1}' > s_i'$. First, it is easy to see that the pricing function for all winners other than i or $i-1$ gets worse, therefore they will be allocated (weakly) less number of items. Similarly to the previous discussion, we show that the total number of units allocated to bidder i and $i-1$ has to (strictly) decrease after the deviation, which gives us the desired contradiction. Bidder $i-1$'s allocation is decreased by

$$x\left(\alpha^c, b_{i-1}\right) - x\left(\alpha^{c+s_i'}, b_{i-1} - s_{i-1}'\right)$$

  which is strictly greater than

$$x\left(\alpha^c, s_i'\right).$$

  Bidder i's allocation is increased by at most

$$x\left(\alpha^{c+s_i'-s_{i-1}'}, b_i\right) - x\left(\alpha^{c+s_i'}, b_i - s_i'\right)$$

  which is smaller than

$$x\left(\alpha^{c+s_i'-s_{i-1}'}, s_i'\right).$$

Since $c + s'_i - s'_{i-1} < c$, we conclude that the total number of units allocated to players has to be strictly less than m, leading to a contradiction.