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A near Pareto optimal auction with budget constraints[☆]Isa E. Hafalir^{*}, R. Ravi, Amin Sayedi

Tepper School of Business, Carnegie Mellon University, 5000 Forbes Ave., Pittsburgh, PA 15213, USA

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ABSTRACT

In a setup where a divisible good is to be allocated to a set of bidders with budget constraints, we introduce a mechanism in the spirit of the Vickrey auction. In the mechanism we propose, understating budgets or values is weakly dominated. Since the revenue is increasing in budgets and values, all kinds of equilibrium deviations from true valuations turn out to be beneficial to the auctioneer. We also show that ex-post Nash equilibrium of our mechanism is near Pareto optimal in the sense that all full winners' values are above all full losers' values.

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1. Introduction

The auction theory literature has paid relatively little attention to the case of budget-constrained bidders. However, budget constraints become quite relevant in multi-unit auctions especially when the auctioned items are of significantly high value for a single bidder to buy all units. Online advertisement auctions are arguably one such setup. Perhaps that is why the advertisers have to specify “a value per click” and “a daily maximum budget” for the advertisement auctions of companies like Google or Yahoo! The presence of budget constraints introduces important differences into traditional auction theory.

Prior literature on budget constraints in auctions have mainly focused on Bayesian Nash equilibria. The main motivation in this line of work was the constrained efficiency and/or revenue optimality in the presence of budget constraints.¹ Motivated by online advertisement auctions, we consider ex-post rather than Bayesian Nash equilibria, where the key efficiency criterion is that of Pareto optimality. The auctioneer's revenue is also an important additional consideration in the online auction setting. In these settings, budgets are usually private information. In a recent paper, Dobzinski et al. (2008) prove an impossibility result in a setup where the budgets are private information. They first consider the case in which the budgets of all players are common knowledge, and show that the unique mechanism which is truthful and efficient is a variation of the Ausubel (2004) auction. Then by showing that the unique mechanism is not truthful if budgets are private information, they conclude that there exists no mechanism that is individually rational, truthful, and Pareto optimal.²

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^{*} Corresponding author. Fax: +1 412 268 7930.

E-mail addresses: isaemin@cmu.edu (I.E. Hafalir), ravi@cmu.edu (R. Ravi), ssayedir@cmu.edu (A. Sayedi).

¹ Two of the earliest papers regarding budget-constrained bidders are Che and Gale (1998) and Benoit and Krishna (2001). Che and Gale (1998) show that in the presence of budget constraints, first-price auctions may yield higher expected revenue than second-price auctions; Benoit and Krishna (2001) show that for multiple objects, the sequential auction may yield more revenue than the simultaneous ascending auction. Maskin (2000) considers the problem of finding the efficient auction subject to the buyers' budget constraints. On the other hand, Laffont and Robert (1996), Malakhov and Vohra (2008), and Pai and Vohra (2010) consider the problem of optimal auctions when bidders have budget constraints.

² There is a quite large literature in the computer science community that tries to approximate optimal revenue in the presence of budget constraints. For instance, see Borgs et al. (2005), Abrams (2006), Ashlagi et al. (2010), Bhattacharya et al. (2010), Feldman et al. (2008).

In this paper, we model an environment where bidders have private constant marginal values and private budget constraints. Based on the above discussion, we are interested in designing a mechanism that has good Pareto optimality and revenue properties. As a starting point, consider selling a divisible item via a Vickrey auction when bidders have constant marginal values up to some limit (to keep the model simple, the limits are expressed not in terms of budgets, but as quantities). As an example, consider 6 bidders where bidders 1, 2, and 3 get positive marginal value up to 0.3 units and bidders 4, 5, and 6 get positive marginal value up to 0.4 units. The Vickrey auction distributes the objects according to the most efficient allocation and charges each winner the externality she imposes on the losing bidders. If bidders 1, 2 and 4 have the highest three marginal values and bidders 3 and 5 have the next two highest marginal values, in the Vickrey auction bidders 1, 2 and 4 will be awarded 0.3, 0.3, 0.4 units respectively. Moreover, bidders 1 and 2 will be paying bidder 3's value per unit, and bidder 4 will be paying bidder 3's value per unit up to 0.3 units, then would be paying bidder 5's value for the remaining 0.1 units. This example assumes common knowledge of the limits, and if that is the case, the Vickrey auction allocates the objects efficiently in dominant strategies.

We propose a natural generalization of the Vickrey auction—*Vickrey with Budgets*—and show that it yields good revenue and Pareto optimality properties. As in the Vickrey auction, we prove that understating budgets or values is weakly dominated in our mechanism. However, unlike Vickrey, overstating budget or value might be beneficial to the bidders. Nevertheless, we show that such deviations lead to higher revenue for the auctioneer than truthful revelations of values and budgets.

The idea of Vickrey with Budgets is the very idea of the Vickrey auction mentioned above. Taking budget constraints into account, we charge the winners, per item, the value of the highest-value loser, but only up to this loser's budget. After the highest-value loser's budget is exhausted, we start charging the winners the second-highest loser's value, up to her budget and so on. Given this pricing idea,³ the winners and losers are determined via a cut point to clear the market, i.e. all the available units are sold. The cut point may lead to the lowest-value winner being partially allocated, while the other bidders are all full winners or full losers.

Vickrey with Budgets has a number of desirable properties. First of all, bidders can only benefit by overstating their values or budgets, a deviation that is the most desirable for the auctioneer. Secondly, the allocation in the equilibrium of Vickrey with Budgets is nearly Pareto optimal in the sense that all full winners' values are above all full losers' values. Thirdly, Vickrey with Budgets reduces to a second-price auction when there are no budget constraints.

2. The model and Vickrey with Budgets

There is a single unit of a divisible good for sale. There are n bidders with a linear demand up to their budget limits. Specifically, each bidder $i \in N = \{1, \dots, n\}$ has a two-dimensional type (b_i, v_i) where b_i denotes her budget limit and v_i denotes her private value. Bidder i 's utility by getting q fractional units of the good and paying p is given by

$$u_i(q, p) = \begin{cases} qv_i - p & \text{if } p \leq b_i, \\ -C & \text{if } p > b_i \end{cases}$$

where $\infty \geq C > 0$.⁴

We are interested in mechanisms with good incentive, efficiency, and revenue properties to sell this good to n bidders. The equilibrium concept we use is that of an *ex-post Nash equilibrium*. In an ex-post Nash equilibrium, no bidder wants to deviate after she observes all other players' strategies.

We focus on direct mechanisms in which bidders announce their types (values and budgets). A mechanism consists of an allocation rule (how many units to allocate to each bidder) and a pricing rule (how much to charge each bidder). It takes the announcements as inputs and produces an allocation and a pricing scheme as the output. We consider pricing rules such that bidders who are not allocated any units (losers) do not pay anything. Bidders who are allocated nonzero units (winners), however, will be charged a positive price. Let us first introduce a general abstract *pricing rule*.

Definition 1. The price is set according to a pricing function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, if the marginal price of the next unit is $\alpha(y)$ dollars for a buyer who has already spent y dollars in the market. In other words, if the pricing of an item is set according to α , a buyer with b dollars can afford

$$x(\alpha, b) = \int_0^b \frac{1}{\alpha(y)} dy$$

units of the item. We are interested in pricing rules $\alpha(\cdot)$ that are nonincreasing and positive. Hence, we assume $\alpha(y) \leq \alpha(y')$ for all $y \geq y'$ and also $\alpha(y) > 0$ for all y .

³ There is a caveat here, which is that the lowest-value winner might not be able to exhaust all of her budget. Then all higher-value bidders are charged first at the lowest-value winner's value up to her unused budget. The pricing for the lowest-value winner starts from the highest-value loser.

⁴ C can also be a function of p . Our results are not affected as long as bidders get a negative utility once they exceed their budgets.

The following definition is also convenient for later discussions.

Definition 2 (*Shifted pricing*). For a given pricing function α and a positive real number z , we define the pricing function $\alpha^z(y)$ as:

$$\alpha^z(y) = \alpha(z + y).$$

Less formally, $\alpha^z(y)$ is the pricing function obtained by shifting $\alpha(y)$, z units to right. Note that we have, for any $z \in [0, b]$,

$$x(\alpha, b) = x(\alpha, z) + x(\alpha^z, b - z).$$

Throughout the proofs of our results, we sometimes make use of the terms “better (or worse) pricing function” and “getting to lower prices.” We say that α is a *better pricing function* than α' for a bidder if $\alpha(y) \leq \alpha'(y)$ for all y . We say that α *gets to lower prices* than α' for a bidder with budget b if marginal payment at b is lower with α than with α' .

We are now ready to introduce our mechanism, which we name *Vickrey with Budgets*.

Definition 3. A c -procedure cut algorithm takes budgets and values of the bidders $(\mathbf{b}, \mathbf{v}) \in \mathbb{R}_{++}^n \times \mathbb{R}_{++}^n$, a pricing rule $\alpha(\cdot)$, and a real number $c \in (0, \sum_{i=1}^n b_i]$ as input. First, it sorts bid and value vectors (\mathbf{b}, \mathbf{v}) in nonascending⁵ order of values and reindexes them so that $v_1 \geq v_2 \geq \dots \geq v_n$.⁶ Then, it picks j such that $c \leq \sum_{i=1}^j b_i$ and $c > \sum_{i=1}^{j-1} b_i$. Let $s = \sum_{i=1}^j b_i - c$. Procedure Cut sets the pricing function of bidders $1, \dots, j-1$ to α^c and the pricing function of bidder j to α^{c+s} , where $\alpha(\cdot)$ is a step function defined by (reindexed) (\mathbf{b}, \mathbf{v}) : $\alpha(y) = v_i$ for $y \in (\sum_{k=1}^{i-1} b_k, \sum_{k=1}^i b_k]$. The allocation of each bidder $1, \dots, j-1$ is such that she spends all her budget, i.e. $x_i = x(\alpha^c, b_i)$ for $i = 1, \dots, j-1$. The allocation of bidder j is such that she spends $b_j - s$ of her budget, i.e. $x_j = x(\alpha^{c+s}, b_j - s)$. Bidder j 's unused budget is denoted by s , where $s \in [0, b_j)$. All bidders $j+1, \dots, n$ get no allocation and pay nothing.

Define $X(c, (\mathbf{b}, \mathbf{v}))$ to be the total number of units allocated to all bidders, i.e. $X(c, (\mathbf{b}, \mathbf{v})) = \sum_{i=1}^j x_i$. We show in Proposition 1 below that there is a unique c^* that satisfies $X(c^*, (\mathbf{b}, \mathbf{v})) = 1$. *Vickrey with Budgets* is defined to be the c^* -procedure cut algorithm.

Bidders $1, \dots, j$ are called full winners, bidder j is called a partial winner (or a cut-point bidder), and bidders $j+1, \dots, n$ are called losers.

In other words, *Vickrey with Budgets* takes the vectors (\mathbf{b}, \mathbf{v}) and sorts them in nonascending order of values, calculates the unique cut point c^* which sells one unit according to the following pricing function: Each full winner (bidders $1, \dots, j-1$) pays v_j per unit up to a budget of s , then pays v_{j+1} per unit up to a budget of b_{j+1} , then pays v_{j+2} per unit up to a budget of b_{j+2} , and so on, until their budgets are exhausted; the partial winner (bidder j) pays v_{j+1} per unit up to a budget of b_{j+1} , then pays v_{j+2} per unit up to a budget of b_{j+2} , and so on, until she spends $b_j - s$; full losers pay nothing.

Proposition 1. $X(c, (\mathbf{b}, \mathbf{v}))$ is strictly increasing and continuous in c . Therefore, there is a unique c^* that satisfies $X(c^*, (\mathbf{b}, \mathbf{v})) = 1$.

Proof. In Appendix A. \square

Let us denote the revenue of *Vickrey with Budgets* by $R^B(\mathbf{b}, \mathbf{v})$. Note that $R^B(\mathbf{b}, \mathbf{v}) = c^*$ where $X(c^*, (\mathbf{b}, \mathbf{v})) = 1$.

Proposition 2. $R^B(\mathbf{b}, \mathbf{v})$ is nondecreasing in \mathbf{b} and \mathbf{v} .

Proof. In Appendix A. \square

3. Incentives, revenue, and near Pareto optimality

3.1. Incentives

In this section, we show that *Vickrey with Budgets* has good incentive properties. Specifically, we show that no bidder benefits from understating her value or budget. First we argue that three deviations that understate value or budget or both are weakly dominated in ex-post equilibria. Then we consider two other deviations that might potentially decrease revenue and argue that either they are unreasonable or they result in higher revenue.

⁵ It breaks ties among equal valued bidders arbitrarily.

⁶ Note that after reindexing, budgets are not necessarily sorted in a descending way. A bidder with a high valuation could have a small budget.

Proposition 3. For any bidder i with types (b_i, v_i) , bidding (b_i, v_i) weakly dominates bidding (b_i, v_i^-) for $v_i^- < v_i$.

Proof. Consider any $(\mathbf{b}_{-i}, \mathbf{v}_{-i})$. First of all, if i becomes a loser by bidding (b_i, v_i^-) , her utility cannot increase with this deviation. This is because losers' utilities are zero, and by construction, a bidder with type (b_i, v_i) achieves a nonnegative utility by bidding (b_i, v_i) . We will look at the possible cases one by one.

- If i loses by bidding (b_i, v_i) , then she will lose by bidding (b_i, v_i^-) (since the pricing function gets better for the winners). Hence her utility cannot increase by this deviation.
- If i is a partial winner by bidding (b_i, v_i) and by bidding (b_i, v_i^-) she is still a partial winner, then she will have the same pricing function but she will be able to use less of her budget (since the pricing function for full winners becomes better); hence her utility cannot increase. Bidder i cannot become a full winner by bidding (b_i, v_i^-) when she is a partial winner by bidding (b_i, v_i) .
- If i wins by bidding (b_i, v_i) and by bidding (b_i, v_i^-) she is still a winner, her utility does not change. This is because Vickrey with Budgets ignores the value of winners in the pricing calculation. If i wins by bidding (b_i, v_i) and bidding (b_i, v_i^-) makes her a partial winner, then the original partial winner j (with an unused budget s) has to be a winner after the deviation. We argue that i 's utility decreases. It is true that i would get the items at a lower per-unit price after the deviation, but at the same time she is using less of her budget. The argument is that, by this deviation, i cannot get to lower-priced items, and this follows from the fact that revenue of Vickrey with Budgets cannot decrease after the deviation. More formally, let us denote the unused budget of i after the deviation by s' . We know that $s' \geq s$ (because revenue cannot increase). Bidder i 's utility difference with the deviation can be shown to be nonpositive (where α and c are defined with respect to (\mathbf{b}, \mathbf{v})) as follows:

$$\begin{aligned}
& (x(\alpha^{c+s}, b_i - s')v_i - (b_i - s')) - (x(\alpha^c, b_i)v_i - b_i) \\
&= (x(\alpha^{c+s}, b_i - s') - x(\alpha^c, b_i))v_i + s' \\
&\leq (x(\alpha^{c+s'}, b_i - s') - x(\alpha^c, b_i))v_i + s' \\
&= (x(\alpha^{c+s'}, b_i - s') - (x(\alpha^c, s') + x(\alpha^{c+s'}, b_i - s'))v_i + s' \\
&= s' - x(\alpha^c, s')v_i \\
&\leq s' - \frac{s'}{v_i}v_i \\
&= 0
\end{aligned}$$

where the first inequality follows from $s' \geq s$ and the second inequality follows from $\alpha^c(y) \leq v_i$. \square

Proposition 4. For any bidder i with type (b_i, v_i) , bidding (b_i, v_i) weakly dominates bidding (b_i^-, v_i) for $b_i^- < b_i$.

Proof. Consider any $(\mathbf{b}_{-i}, \mathbf{v}_{-i})$: First of all, as in the previous proof, if i becomes a loser by bidding (b_i^-, v_i) , her utility cannot increase with this deviation. We look at the possible cases one by one.

- If i loses by bidding (b_i, v_i) , then she will lose by bidding (b_i^-, v_i) (since the pricing function gets better for the winners).
- If i is a partial winner by bidding (b_i, v_i) and by bidding (b_i^-, v_i) she is still a partial winner, then she will have the same pricing function but will be able to use less of her budget (since the pricing function for winners becomes better); hence her utility cannot increase. Bidder i cannot become a full winner by bidding (b_i, v_i^-) when she is a partial winner by bidding (b_i, v_i) .
- If i wins by bidding (b_i, v_i) and bidding (b_i^-, v_i) makes her a partial winner, then i would be worse off with this deviation. This is because she is using less of her budget, and her pricing got worse. If i is a full winner by bidding (b_i, v_i) and bidding (b_i^-, v_i) leaves her as a full winner, we can argue that her utility decreases. It is true that i may get the items at a lower per-unit price after the deviation, but at the same time she is using less of her budget. The argument is that by this deviation i cannot get to lower-priced items, which follows from the fact that the revenue of Vickrey with Budgets cannot increase after the deviation. More formally, bidder i 's utility difference with the deviation can be shown to be nonpositive as follows. Here α and c are defined with respect to (\mathbf{b}, \mathbf{v}) and $c' (\leq c)$ is the revenue of Vickrey with Budgets after the deviation:

$$\begin{aligned}
& (x(\alpha^{c'+b_i-b_i^-}, b_i^-)v_i - b_i^-) - (x(\alpha^c, b_i)v_i - b_i) \\
&= (x(\alpha^{c'+b_i-b_i^-}, b_i^-) - x(\alpha^c, b_i))v_i + b_i - b_i^- \\
&\leq (x(\alpha^{c'+b_i-b_i^-}, b_i^-) - x(\alpha^c, b_i))v_i + b_i - b_i^-
\end{aligned}$$

$$\begin{aligned}
&= (x(\alpha^{c+b_i-b_i^-}, b_i^-) - (x(\alpha^c, b_i - b_i^-) + x(\alpha^{c+b_i-b_i^-}, b_i^-)))v_i + b_i - b_i^- \\
&= b_i - b_i^- - x(\alpha^c, b_i - b_i^-)v_i \\
&\leq b_i - b_i^- - \frac{b_i - b_i^-}{v_i}v_i \\
&= 0
\end{aligned}$$

where the first inequality follows from $c \geq c'$ and the second inequality follows from $\alpha^c(y) \leq v_i$ for all y .⁷ \square

Now we can argue that bidding (b_i^-, v_i^-) for $b_i^- < b_i$ and $v_i^- < v_i$ is weakly dominated by bidding (b_i, v_i) . This follows from the proofs above. The two previous propositions imply that both (b_i^-, v_i) and (b_i, v_i^-) dominate (b_i^-, v_i^-) when $b_i^- < b_i$ and $v_i^- < v_i$. Applying either of them one more time, we have the following result.

Proposition 5. For any bidder i with type (b_i, v_i) , bidding (b_i, v_i) weakly dominates bidding (b_i^-, v_i^-) for $b_i^- < b_i$ and $v_i^- < v_i$.

Propositions 3, 4, and 5 establish that these revenue-decreasing deviations should not occur in equilibrium (they are weakly dominated). There are two deviations, however, that may increase or decrease the revenue. These deviations are “understating budget and overstating value” and “overstating budget and understating value.” We now show that the former deviation is not reasonable in the sense that it could be a best response only when the utility with that strategy is zero. Then we show that the latter deviation could happen in an equilibrium, yet whenever it is a (strict) profitable deviation from truthful revelation, the revenue increases with the deviation.

Proposition 6. For any bidder i with type (b_i, v_i) , for $b_i^- < b_i$ and $v_i^+ > v_i$, bidding (b_i^-, v_i^+) can never be in the set of best responses unless bidder i 's utility in her best response is 0.

Proof. Given $(\mathbf{b}_{-i}, \mathbf{v}_{-i})$, suppose that (b_i^-, v_i^+) is a best response for i where $b_i^- < b_i$ and $v_i^+ > v_i$. Since bidding (b_i, v_i) would give nonnegative utility to bidder i , the utility by bidding (b_i^-, v_i^+) has to be nonnegative. We claim that bidding (b_i, v_i^+) is a better response than (b_i^-, v_i^+) , and it is strictly better when the utility by bidding (b_i, v_i^+) is strictly positive. This implies (b_i^-, v_i^+) could be a best response only when bidder i 's utility in her best response is 0.

Suppose bidder i 's utility by bidding (b_i^-, v_i^+) is nonnegative, and consider the utility difference between bidding (b_i^-, v_i^+) and bidding (b_i, v_i^+) . The utility difference is clearly zero if i is a loser in both cases. For all other cases, i would be either a partial winner or a full winner by bidding (b_i, v_i^+) . Then we can see that bidding (b_i, v_i^+) gives a higher utility than bidding (b_i^-, v_i^+) . The argument is the same as in the proof for Proposition 4: by bidding an extra budget of $b_i - b_i^-$, bidder i can get extra items at a per-unit price lower than her value, leading to a nonzero increase in her utility. \square

In other words, we should not expect to see (b_i^-, v_i^+) to be reported in an ex-post equilibrium, since it is worse than either (b_i, v_i) or (b_i, v_i^+) .

Proposition 7. For any bidder i with type (b_i, v_i) , for $b_i^+ > b_i$ and $v_i^- < v_i$, whenever bidding (b_i^+, v_i^-) brings a higher utility to i than bidding (b_i, v_i) , the auctioneer's revenue with (b_i^+, v_i^-) is not lower than the revenue with bidder i bidding (b_i, v_i) .

Proof. Given $(\mathbf{b}_{-i}, \mathbf{v}_{-i})$, for some $b_i^+ > b_i$ and $v_i^- < v_i$, suppose that $u_i((\mathbf{b}_{-i}, b_i^+), (\mathbf{v}_{-i}, v_i^-)) > u_i((\mathbf{b}_{-i}, b_i), (\mathbf{v}_{-i}, v_i))$. Since bidder i is budget constrained, she will have to be a partial winner by bidding (b_i^+, v_i^-) (if she is a full winner her utility would be $-C$, and if she is a loser her utility would be 0).

- If she loses by bidding (b_i, v_i) , the auctioneer's revenue clearly increases with (b_i^+, v_i^-) . This is because i 's ranking with v_i^- is not higher than that with v_i , and so by deviating from (b_i, v_i) to (b_i^+, v_i^-) , all full winners remain full winners and i becomes a partial winner.
- If she is a full winner by bidding (b_i, v_i) , the partial winner with (b_i, v_i) has to become a full winner after i deviates to (b_i^+, v_i^-) . Otherwise, i would be worse off by bidding (b_i^+, v_i^-) as she will have a worse pricing function. In this case the revenue has to increase. The argument is that, for this deviation to be beneficial, i has to get lower-priced items after the deviation. For this to be the case, the partial winner's unused budget before the deviation plus i 's used budget after the deviation, has to be greater than i 's budget b_i . But in this case, the revenue increases, since the new cut point is greater than the old one.

⁷ To see why bidder i 's pricing function after the deviation is $\alpha^{c'+b_i-b_i^-}$, note that her pricing function is $\alpha^{c'}$ according to types after the deviation, and this translates to $\alpha^{c'+b_i-b_i^-}$ with the original types.

- If she is a partial winner by bidding (b_i, v_i) , we need to analyze two cases: (i) i 's ranking among the bidders is the same, or (ii) i 's ranking is different. For (i), the pricing for (b_i, v_i) and (b_i^+, v_i^-) are the same. Since bidder i 's utility by bidding (b_i^+, v_i^-) is more than that by bidding (b_i, v_i) , this means i is using more of her budget with (b_i^+, v_i^-) . Therefore the revenue increases. For (ii), i 's ranking has to be worse with (b_i^+, v_i^-) . Now, similar to the previous case, we argue that total budget of "new full winners" after the deviation plus the used budget of i after deviation has to be greater than b_i . If that is not the case, i cannot get to lower prices. \square

In the above propositions we argued that playing (b_i^-, v_i) , (b_i, v_i^-) , or (b_i^-, v_i^-) is not reasonable (they are dominated by (b_i, v_i)); playing (b_i^-, v_i^+) is not reasonable in a weaker sense (it is dominated by a combination of (b_i, v_i) and (b_i, v_i^+)); also, playing (b_i^+, v_i^-) is reasonable only when it is done by a winner, who becomes a partial winner after deviation and increases the overall revenue. We call an equilibrium in which the strategies satisfy these conditions a *refined equilibrium*.

Definition 4. A *refined equilibrium* is an equilibrium of Vickrey with Budgets where for all bidders i , bidder i does not play (b_i^-, v_i) , (b_i, v_i^-) , (b_i^-, v_i^-) , or (b_i^-, v_i^+) . Moreover, a bidder i plays (b_i^+, v_i^-) only when $u_i((\mathbf{b}_{-i}, b_i^+), (\mathbf{v}_{-i}, v_i^-)) > u_i((\mathbf{b}_{-i}, b_i), (\mathbf{v}_{-i}, v_i))$.

In other words, in a refined equilibrium, bidders never understate their budgets, and they understate their values only when they also simultaneously overstate their budgets, making them strictly better off than their truthful announcements. Recall that when $u_i((\mathbf{b}_{-i}, b_i^+), (\mathbf{v}_{-i}, v_i^-)) > u_i((\mathbf{b}_{-i}, b_i), (\mathbf{v}_{-i}, v_i))$, bidding (b_i^+, v_i^-) makes i a partial winner after the deviation and the revenue is higher with (b_i^+, v_i^-) than with (b_i, v_i) .

3.2. Revenue

There are eight possible kinds of deviations from the truthful revelation (b_i, v_i) . Five of them are discussed in the definition of a refined equilibrium. The remaining three of them, namely (b_i, v_i^+) , (b_i^+, v_i) , and (b_i^+, v_i^+) , can only increase the revenue by Proposition 2. Hence we have the following result.

Theorem 1. In a refined equilibrium of Vickrey with Budgets, revenue is bounded below by the revenue of Vickrey with Budgets with truthful revelations.

Proof. Consider any refined equilibrium of Vickrey with Budgets. Let b_i^- and v_i^- denote understating the types, and b_i^+ and v_i^+ denote overstating the types (with respect to true types). We know that (b_i^-, v_i) , (b_i, v_i^-) , (b_i^-, v_i^-) , and (b_i^-, v_i^+) do not occur. Additionally, (b_i^+, v_i^-) could occur only for the current cut-point bidder, and by Proposition 7, if we change it back to (b_i, v_i) , revenue cannot increase. Finally, the rest of the bidders are either bidding truthfully or using (b_i, v_i^+) , (b_i^+, v_i) , or (b_i^+, v_i^+) . In any case, changing their bid to their truthful values cannot increase the revenue. Therefore, revenue in a refined equilibrium of Vickrey with Budgets is not smaller than revenue of Vickrey with Budgets with truthful revelations. \square

3.3. Near Pareto optimality

We say that an allocation is Pareto optimal if there is no other allocation in which all players (including the auctioneer) are better off and at least one player is strictly better off.⁸ In this setup, Dobzinski et al. (2008) has shown that Pareto optimality is equivalent to a "no trade" condition: an allocation is Pareto efficient if (a) all units are sold and (b) a player get a nonzero allocation only if all higher-value players exhaust their budgets. In other words, an allocation is Pareto optimal when, given the *true value* of the partial winner, winners and losers are ordered in the right way: all winners have higher values and all losers have lower values.

The following shows that in any ex-post Nash equilibrium of Vickrey with Budgets, the full winners and losers are ordered in the right way given the *announced value* of the partial winner.

Theorem 2. Consider any ex-post Nash equilibrium of Vickrey with Budgets where v_j is the announced value of the partial winner j . Every bidder $i \neq j$ who has a true value $v_i^T > v_j$ is a full winner, and every bidder $i \neq j$ who has a true value $v_i^T < v_j$ is a loser in this equilibrium of Vickrey with Budgets.

Proof. First, consider a bidder i whose value is $v_i^T > v_j$. We prove that she must be a full winner in equilibrium. Assume for the sake of contradiction that bidder i is a loser, so her utility is zero. If she deviates and bids $v_j + \varepsilon$ (for $0 < \varepsilon < v_i^T - v_j$) and her true budget, she will become either a full winner or the cut-point bidder (otherwise revenue of Vickrey with

⁸ Maximizing social welfare dictates all items to be allocated to the bidder with the highest value, even if this bidder has a very small budget. We follow Dobzinski et al. (2008) and consider Pareto optimality as the appropriate efficiency concept.

Budgets will decrease with this deviation, which is not possible because of Proposition 2). Obviously her utility becomes strictly positive with this deviation (her price per unit is at most v_j). We thus reach the necessary contradiction to her individual rationality.

Now consider a bidder i whose value is $v_i^T < v_j$. Assume for the sake of contradiction that bidder i is a full winner. If b_i is smaller than the unused budget of the cut-point bidder (s), then she gets all items at a per-unit price v_j , and hence she obtains a negative utility. If this is the case, she would be better off announcing her true valuations to guarantee a nonnegative payoff. If $b_i > s$, then we argue that i would be better off by deviating to $(v_j - \varepsilon, b_i)$ for small enough ε . Let us first look at the limiting case in which i deviates to (v_j, b_i) and becomes the cut-point bidder. After this deviation, the unused budget of i would be exactly s . The allocation of original full winners will not change; bidder j will be getting $\frac{s}{v_j}$ more items by paying s more and bidder i will be getting $\frac{s}{v_j}$ less items by paying s less. Therefore, bidder i 's utility increases by $\frac{s}{v_j}(v_j - v_i^T) > 0$ (in a sense by this deviation, bidder i is selling $\frac{s}{v_j}$ units of the items to bidder j at the per-unit price of v_j). By deviating to $(v_j - \varepsilon, b_i)$, the original full winners' allocations would slightly increase; therefore bidder i 's utility increase will be slightly smaller than $\frac{s}{v_j}(v_j - v_i^T)$.⁹ But for small enough ε , it will always be positive, leading again to a contradiction. \square

This theorem establishes that given equilibrium cut-point value, all winners and losers will be correctly placed. But since the cut-point bidder may be misplaced, this does not imply full Pareto optimality. Consider the following example.

Example 1. There are 2 units of the item to be sold, and there are four bidders with budget-value pairs $(18, 19)$, $(1, 9)$, $(\frac{17}{9}, 8)$, and $(10, 1)$. For this setup, it can be confirmed that bidders announcing their types (budget, value) as $(18, 19)$, $(1, 9)$, $(36, 18)$, and $(10, 1)$ constitute an ex-post equilibrium of Vickrey with Budgets. In this equilibrium, bidder 3 overstates her value and budget and becomes the partial winner. Although the full winners and the losers are rightly ranked according to the announced value of the partial winner, the allocation is not Pareto optimal. Bidder 3 gets a positive allocation even though bidder 2 has a higher value and zero allocation.

As an aside, note that the revenue of Vickrey with Budgets in this ex-post equilibrium is 18 while the revenue with truthful types is $18 + \frac{17}{9}$.

4. Conclusion and discussion

In this paper, we have introduced a mechanism, Vickrey with Budgets, to sell a divisible good to a set of bidders with budget constraints. In this important setting, where a mechanism that is simultaneously truthful and Pareto optimal is precluded, our mechanism achieves good incentive, revenue, and efficiency properties. Specifically, in Vickrey with Budgets, (i) there are profitable deviations from truthful revelations of types, but these can only happen in a revenue-increasing way; and (ii) the equilibrium allocation is *nearly Pareto efficient* in the sense that full winners and losers are ordered in the right way given the announced value of the partial winner.

There are many ways our work can be generalized. In the context of online advertisement auctions, our model can be interpreted as "there is a *single* sponsored link that gets (normalized) 1 click a day and there are n advertisers." However, in reality, there are many sponsored links. In generalized second-price auctions, studied by Edelman et al. (2007), the winner of the best item (first sponsored link) is charged the bid of the second-best item, the winner of the second-best item is charged the bid of the third-best item, and so on. In this environment there are no budget constraints and the second-highest bid is always the competitor of the highest value. The idea of Vickrey with Budgets can be applied in this setup with budget constraints. More specifically, it would be interesting to consider a model in which there are budget-constrained bidders and multiple slots available for a query in which an advertiser cannot appear in more than one slot per query.

In our model, we consider a setting of hard budget constraints in which the bidders cannot spend more than their budgets. Extending our results to a soft-budget problem in which bidders are able to finance further budgets at some cost is a promising direction.¹⁰ One can model this kind of soft-budget constraint as specifying marginal value up to some budget, then specifying a smaller marginal value up to some other extra budget, and so on in a piecewise linear fashion. By replicating a bidder into as many copies as the number of pieces in her value/budget function, and allowing them all to participate in our mechanism, it seems reasonable that we may preserve some of the desirable properties of Vickrey with Budgets.

Appendix A

A.1. Proof of Proposition 1

First, note that $x(\alpha^c, b)$ is weakly increasing in c : since α is nonincreasing, for $c' \geq c \geq 0$, we have $\alpha^{c'}(y) = \alpha(y + c') \leq \alpha(y + c) = \alpha^c(y)$, and hence

⁹ There is an implicit continuity assumption here. However, it is not difficult to show that utilities of the bidders are continuous in type announcements.

¹⁰ See Gaviols et al. (2002) where the utility of an agent is declining in price via "bid costs."

$$x(\alpha^{c'}, b) = \int_0^b \frac{1}{\alpha^{c'}(y)} dy \geq \int_0^b \frac{1}{\alpha^c(y)} dy = x(\alpha^c, b).$$

Also, obviously $x(\alpha^c, b)$ is strictly increasing in b .

Now we can show that $X(c, (\mathbf{b}, \mathbf{v}))$ is strictly increasing in c . Consider $c' > c \geq 0$; we have

$$X(c, (\mathbf{b}, \mathbf{v})) = \left(\sum_{i=1}^{j-1} x(\alpha^c, b_i) \right) + x(\alpha^{c+s}, b_j - s)$$

where j satisfies $c \leq \sum_{i=1}^j b_i$ and $c > \sum_{i=1}^{j-1} b_i$ (and $s = \sum_{i=1}^j b_i - c$). For $c' > c$, we can have one of two cases: either the index j stays the same or j is larger.

If j is larger, then we have

$$X(c', (\mathbf{b}, \mathbf{v})) > \sum_{i=1}^j x(\alpha^{c'}, b_i) \geq \left(\sum_{i=1}^{j-1} x(\alpha^c, b_i) \right) + x(\alpha^{c'}, b_j) > X(c, (\mathbf{b}, \mathbf{v})).$$

This is because $x(\alpha^{c'}, b_i) \geq x(\alpha^c, b_i)$ for all $i = 1, \dots, j-1$ and $x(\alpha^{c'}, b_j) > x(\alpha^{c+s}, b_j - s)$ since $c' > c + s$.

If j stays the same (i.e., if $c' < c + s$), then we have

$$X(c', (\mathbf{b}, \mathbf{v})) = \left(\sum_{i=1}^{j-1} x(\alpha^{c'}, b_i) \right) + x(\alpha^{c'+s'}, b_j - s') > \left(\sum_{i=1}^{j-1} x(\alpha^c, b_i) \right) + x(\alpha^{c+s}, b_j - s) = X(c, (\mathbf{b}, \mathbf{v}))$$

where $s' = \sum_{i=1}^j b_i - c' < s$. This is because $x(\alpha^{c'}, b_i) \geq x(\alpha^c, b_i)$ for all $i = 1, \dots, j-1$ and $x(\alpha^{c'+s'}, b_j - s') > x(\alpha^{c+s}, b_j - s)$ since $c' + s' = c + s$ and $b_j - s' > b_j - s$.

Next we show that $X(c, (\mathbf{b}, \mathbf{v}))$ is continuous in c . By definition, $x(\alpha^c, b)$ is continuous in c and b (this is because $x(\alpha^c, b) = \int_0^b \frac{1}{\alpha(y+c)} dy$ and is continuous in c and b even when α is not a continuous function). Moreover

$$X(c, (\mathbf{b}, \mathbf{v})) = \left(\sum_{i=1}^{j-1} x(\alpha^c, b_i) \right) + x(\alpha^{c+s}, b_j - s).$$

If c increases from c to $c + \varepsilon$, j changes only when $s = 0$. If $s \neq 0$, then $X(c, (\mathbf{b}, \mathbf{v}))$ is obviously continuous in c as all of the terms in the summation are continuous in c . If $s = 0$, then

$$X(c + \varepsilon, (\mathbf{b}, \mathbf{v})) = \left(\sum_{i=1}^j x(\alpha^{c+\varepsilon}, b_i) \right) + x(\alpha^{c+\varepsilon+s'}, b_{j+1} - s')$$

and this goes to $X(c, (\mathbf{b}, \mathbf{v}))$ as ε goes to zero. This is because $\sum_{i=1}^j x(\alpha^{c+\varepsilon}, b_i) \rightarrow \sum_{i=1}^j x(\alpha^c, b_i) = X(c, (\mathbf{b}, \mathbf{v}))$ and $x(\alpha^{c+\varepsilon+s'}, b_{j+1} - s') \rightarrow 0$ since $s' \rightarrow b_{j+1}$.

Hence we conclude that $X(c, (\mathbf{b}, \mathbf{v}))$ is strictly increasing and continuous in c .

We consider pricing rules that are not too high, in the sense that they will be able to sell all the items if all budgets are exhausted. Hence we assume that for $B \equiv \sum_{i=1}^n b_i$, we have

$$\alpha(B) \leq B.$$

With this assumption, we can easily conclude that $X(B, (\mathbf{b}, \mathbf{v})) \geq 1$. This is because when $c = 1$, all bidders are full winners and their allocations satisfy

$$x(\alpha^B, b_i) \geq \frac{b_i}{\sum_{i=1}^n b_i}$$

and hence

$$X(B, (\mathbf{b}, \mathbf{v})) = \sum_{i=1}^n x(\alpha^B, b_i) \geq 1.$$

Thus we conclude that there is a unique c^* such that $X(c^*, (\mathbf{b}, \mathbf{v})) = 1$.

A.2. Proof of Proposition 2

Consider bidder i with announced type (b_i, v_i) .

- First we show that revenue is nondecreasing in budgets. Consider bidder i who decreases her budget to $b_i^- < b_i$. We show that revenue cannot increase with this deviation.
 - If bidder i was originally a loser by announcing (b_i, v_i) , then she cannot become a winner or partial winner by deviating to $b_i^- < b_i$. This is because by this deviation, the pricing function for everybody becomes better and winners pay less per unit. Therefore the revenue cannot increase.
 - Next, consider bidder i who is a partial winner by bidding b_i . If bidder i deviates to b_i^- and becomes a loser, then the revenue has to decrease since the set of losers becomes larger with this deviation. If she deviates to b_i^- and remains a partial winner, since all winners' pricing gets better, the revenue has to decrease. If she deviates to b_i^- , she cannot become a full winner. If this were the case, the pricing function for every (full or partial) winner gets better, and then the total number of units allocated will be greater than one.
 - Lastly, consider bidder i who is originally a winner by announcing (b_i, v_i) . If she deviates to b_i^- and if she becomes a loser or a partial winner after the deviation, then the revenue clearly decreases. This is because the set of full winners before the deviation is a strict superset of the set of full winners after the deviation. Now consider the case where bidder i deviates to b_i^- and remains a winner. Let us denote $b_i - b_i^-$ by Δ . Suppose that the initial cut point is c and the new cut point after the deviation is c' . Let α be the n -piece step function defined by (\mathbf{b}, \mathbf{v}) . Note that the initial revenue is c and the new revenue is c' . We will show that $c \geq c'$.

Since i has understated her budget, there will be a shortage of demand and the pricing of all original winners will be better. Therefore, with this deviation, all original winners except i will be allocated (weakly) more units of the object. Assume for a contradiction that $c' > c$. This means that there will be new winners who use an extra budget strictly greater than Δ , say Δ' . We now argue that the extra units allocated to these new winners have to be greater than the number of units bidder i is giving up with the deviation. Extra units allocated to new winners are priced at the values starting from the new cut point $c + \Delta'$ (according to (\mathbf{b}, \mathbf{v})) and the total budget used is Δ' . The number of units bidder i is giving up are priced at the values in the range of c to $c + \Delta < c + \Delta'$ and the total budget used is Δ . Since extra units are given with higher budget ($\Delta' > \Delta$) and lesser prices ($c + \Delta < c + \Delta'$) than the units given up, we conclude that with the assumption $c' > c$, the total number of units allocated has to be strictly greater than one, which is a contradiction.

We can present this argument more formally. Consider the case in which Δ is small enough so that the original partial winner j remains a partial winner. All full winners $k \neq i$ with $k < j$ will be allocated more items since bidder j will be using more of her budget after the deviation. Let us consider the difference between the total amounts allocated to bidders i and j before and after the deviation. Bidder i 's allocation is decreased by

$$A \equiv x(\alpha^c, b) - x(\alpha^{c+\Delta}, b - \Delta)$$

since

$$x(\alpha^{c+\Delta'}, b - \Delta) > x(\alpha^{c+\Delta'}, b - \Delta').$$

We have

$$A < x(\alpha^c, b) - x(\alpha^{c+\Delta'}, b - \Delta') = x(\alpha^c, \Delta').$$

On the other hand, bidder j 's allocation is increased by

$$B \equiv x(\alpha^{c+s}, b_j - s + \Delta') - x(\alpha^{c+s}, b_j - s) = x(\alpha^{c+b_j}, \Delta').$$

Since

$$x(\alpha^{c+b_j}, \Delta') \geq x(\alpha^c, \Delta'),$$

we conclude that $B > A$. The argument for the case when the deviation results in a change of the partial winner is very similar but not illuminating. Thus, the total number of units allocated has to increase after the deviation.

- Now, we show that revenue is increasing in values. Consider bidder i who increases her value to $v_i^+ > v_i$. We show that revenue cannot decrease with this deviation.
 - First, if bidder i is a winner by bidding (b_i, v_i) and she deviates to $v_i^+ > v_i$, then she remains a winner after the deviation, and the revenue does not change. This is because the allocation and pricing rule of Vickrey with Budgets is invariant to full winners' values (so long as they remain full winners).
 - Second, consider a bidder i who is a loser by bidding (b_i, v_i) and deviates to $v_i^+ > v_i$. If she remains a loser after the deviation, since the pricing function for winners gets worse, the revenue has to increase. Let us now consider the deviation which makes bidder i a partial winner. If the original partial winner becomes a full winner after the

deviation ($v_i^+ < v_j$ where j is the original partial winner), the revenue obviously increases with the deviation, since the cut point has increased.

Let us consider the case in which $v_i^+ > v_j$; bidder i becomes a partial winner and bidder j becomes a loser after the deviation. Assume for contradiction that the revenue decreases with the deviation. If this is the case, it can be seen that the pricing function for all winners becomes worse after the deviation (the total unspent budget of price setters with $v_k \geq v_i$ becomes greater and some of the values increase). Hence all full winners will be allocated less units of items after the deviation. This implies that the number of units allocated to i after the deviation has to be greater than the number of units allocated to j before the deviation. But again, the pricing function for i after the deviation is worse than the pricing function for j before the deviation. For i to be allocated more, her budget spent after the deviation has to be greater than j 's budget spent before the deviation, which is a contradiction.

Suppose bidder i is currently a loser and deviates to v_i^+ and becomes a full winner. We can split this into two deviations. First, i deviates to $v_i^{++} > v_j$ and becomes a partial winner (which increases the revenue), then she deviates to v_i^+ and becomes a full winner which next be shown to increase the revenue.

- Lastly, consider bidder i who is a partial winner by bidding (b_i, v_i) . It is obvious that she cannot become a loser after deviating to v_i^+ . If she deviates to v_i^{++} and remains a partial winner, then the pricing function for all winners get worse, hence the revenue has to increase. If she deviates to v_i^+ and becomes a full winner, then we argue that revenue has to increase.

Consider the case where bidder i is currently the partial winner, and she deviates to $v_i^+ > v_{i-1}$ (where bidder $i-1$ has the next highest value after bidder i) so that $i-1$ is the new partial winner and i is a full winner. Denote the original unused budget of bidder i by s'_i and after deviation, the unused budget of bidder $i-1$ by s'_{i-1} . It suffices to show that $s'_i \geq s'_{i-1}$. Assume for contradiction that $s'_{i-1} > s'_i$. First it is easy to see that the pricing function for all winners other than i or $i-1$ gets worse, therefore they will be allocated (weakly) less number of items. As in the previous discussion, we show that the total number of units allocated to bidder i and $i-1$ has to (strictly) decrease after the deviation, which gives us the desired contradiction. Bidder $i-1$'s allocation is decreased by

$$x(\alpha^c, b_{i-1}) - x(\alpha^{c+s'_i}, b_{i-1} - s'_{i-1})$$

which is strictly greater than

$$x(\alpha^c, s'_i).$$

Bidder i 's allocation is increased by at most

$$x(\alpha^{c+s'_i-s'_{i-1}}, b_i) - x(\alpha^{c+s'_i}, b_i - s'_i)$$

which is smaller than

$$x(\alpha^{c+s'_i-s'_{i-1}}, s'_i).$$

Since $c + s'_i - s'_{i-1} < c$, we conclude that the total number of units allocated to players has to be strictly less than one, leading to a contradiction.

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