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School choice with controlled choice constraints: Hard bounds versus soft bounds ☆

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Abstract

Controlled choice over public schools attempts giving parents selection options while maintaining diversity of different student types. In practice, diversity constraints are often enforced by setting *hard upper bounds* and *hard lower bounds* for each student type. We demonstrate that, with hard bounds, there might not exist assignments that satisfy standard fairness and non-wastefulness properties; and only *constrained non-wasteful* assignments that are *fair for same type students* can be guaranteed to exist. We introduce the student exchange algorithm that finds a constrained efficient assignment among such assignments. To achieve *fair* (across all types) and *non-wasteful* assignments, we propose control constraints to be interpreted as *soft bounds*–flexible limits that regulate school priorities dynamically. In this setting, (i) the student-proposing deferred acceptance algorithm produces an assignment that Pareto dominates all other

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fair assignments while eliciting true preferences and (ii) the school-proposing deferred acceptance algorithm finds an assignment that minimizes violations of controlled choice constraints among fair assignments.

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1. Introduction

School choice policies are implemented to grant parents the opportunity to choose the school their children will attend. In order to create a diverse environment for students, school districts often implement *controlled school choice* programs providing parental choice while maintaining the racial, ethnic or socioeconomic balance at schools. Before school choice policies were in effect, children were assigned to a public school in their immediate neighborhood. However, neighborhood-based assignment eventually led to socioeconomically segregated neighborhoods, as wealthy parents moved to the neighborhoods of their school of choice. Parents without such means had to continue to send their children to their neighborhood schools, regardless of the quality or appropriateness of those schools for their children. To overcome these shortcomings, controlled school choice programs have become increasingly more popular.

There are many examples of controlled public school admission policies in the United States. To name just a few, the Jefferson County School District has an assignment plan that requires elementary schools to allocate between 15 and 50 percent of their students coming from a particular geographic area inside the district that harbors the highest concentration of designated beneficiaries of the affirmative action policy.[1] Similarly, in New York City, "Educational Option" (EdOpt) schools have to accept students across different ability ranges. In particular, 16 percent of students that attend an EdOpt school must score above the grade level on the standardized English Language Arts test, 68 percent must score at the grade level, and the remaining 16 percent must score below the grade level (Abdulkadiroğlu et al. [3]).[2]

As it is evident from the two examples above, in practice, controlled school choice programs are often enforced by setting feasibility constraints with hard upper bounds and hard lower bounds for different student types.[3] In the first part of our paper, we analyze controlled

[1] More details on this policy are present on the "No Retreat" brochure on Jefferson Country School District's website (http://www.jefferson.k12.ky.us/Pubs/NoRetreatBro.pdf).

[2] There are similar constraints in other countries as well. For example in England, City Technology Colleges are required to admit a group of students across the ability range and their student body should be representative of the community in the catchment area (Donald Hirch [36, page 120]).

[3] There are many other examples of controlled school choice. A Racial Imbalance Law that was passed in 1965 in Massachusetts, prohibits racial imbalance and discourages schools from having student enrollments that are more than 50 percent minority. After a series of legal decisions, the Boston Public Schools (BPS) was ordered to implement a controlled choice plan in 1975. Although BPS has been relieved of legal monitoring, it still tries to achieve diversity across ethnic and socioeconomic lines at city schools (Abdulkadiroğlu et al. [4,5]). Likewise, St. Louis and Kansas City, Missouri must observe court-ordered racial desegregation guidelines for the placement of students in city schools. In contrast, the White Plains Board of Education employ their nationally recognized Controlled Parents' Choice Program voluntarily. Miami-Dade County Public Schools control for the socioeconomic status of students in order to diminish concentrations of low-income students at certain schools. Similarly, Chicago Public Schools diversify their student bodies by enrolling students in choice options at schools that are not the students' designated neighborhood schools.

school choice with hard bounds and demonstrate serious limitations as a consequence of this approach. In the second part, we provide a novel interpretation of controlled choice constraints as soft bounds and show that this new perspective has many advantages over its hard-bounds counterpart.

In general, a crucial feature of most school choice programs (not only controlled choice programs) is to give some students priority at certain schools. For example, some state and local laws require that students who live in the attendance area of a school must be given priority for that school over students who do not live in the school's attendance area; siblings of students already attending a school must be given priority; and students requiring a bilingual program must be given priority in schools that offer such programs. All these priority altering decisions, including controlled choice, should be implemented while preserving the notion of *fairness*.[4]

In order to provide a foundation for controlled school choice programs, a thorough analysis of fairness and controlled choice requires a substantial generalization of the standard matching models of school choice. In such an attempt, Abdulkadiroğlu and Sönmez [6] and Abdulkadiroğlu [1] consider a relaxed controlled choice problem by employing type-specific quotas (upper bounds). Control is imposed on the maximum number of students from each group that a school can enroll. This extension does not capture *controlled choice* to the fullest extent because they do not exclude segregated schools in fair assignments. For example, consider a school that can enroll 100 students with hard upper bound of 50 Caucasian students. In this case, a student body consisting of 50 Caucasian students would not violate the maximum quota, yet the school is fully segregated. Such a segregated assignment would violate the spirit of controlled choice for school districts.

Based on the laws of a state or the policies of a school choice program (or of the school district), an assignment is legally *feasible* (or politically acceptable) if both (i) every student is assigned to a public school and (ii) at each school the desegregation guidelines are respected. We incorporate these constraints in the definition of fairness. The nature of controlled choice imposes that a student–school pair can cause a *justified envy* (or blocks) only if matching this pair neither results in any unassigned student nor violates controlled choice constraints at any school. Later, we consider the case when (i) is relaxed.

Given the definition of fairness and justified envy in the controlled school choice context, we then explore the question of existence of fair and legally feasible assignments. In Section 3, we study this problem for the hard-bounds interpretation of the legal constraints. First, we show that feasible student assignments which are fair may not exist (Theorem 1). Due to this impossibility result, either fairness needs to be weakened to respect legal constraints, or the interpretation of legal constraints must be changed. We initially focus on the case where we relax the notion of fairness. In this setting, a natural route is to allow envy only among students of the same type. Then, for example, Caucasian students can justifiably envy other Caucasian students (but not any other student of a different type). It turns out that legally feasible assignments, which are fair for same types, may not exist if we also require non-wastefulness (Theorem 1).[5] With these two results, we demonstrate the difficulties associated with the implementation of hard bounds.

[4] Following Abdulkadiroğlu and Sönmez [6], we can define an assignment to be *fair* if there is no unmatched student–school pair where the student prefers the school to her assignment and she has higher priority than some other student who is assigned a seat at the school (together with non-wastefulness, this is the same notion as stability in the two-sided matching context).

[5] Non-wastefulness is a mild efficiency criterion (Balinski and Sönmez [11]). In our context, this condition requires that empty seats should not be wasted if students claim them while maintaining the legal constraints.

Since hard bounds are implemented in practice, it is important to investigate what the best mechanism in this context is. In this regard, we show that a positive result emerges if non-wastefulness is weakened: students can claim empty seats only if the resulting assignment does not cause any envy among students of the same type. In particular, we introduce a new algorithm called *the student exchange algorithm* (*SEA*) that finds a legally feasible assignment, which is both fair for same types and constrained non-wasteful. This assignment is also *constrained efficient* (Theorem 2), i.e., there does not exist another assignment that is fair for same types which is weakly better for all students. A significant advantage of SEA is that, as an input, it can take any feasible assignment that is fair for same types. Therefore, it can easily be adapted by school districts that already implement a version of controlled school choice. Furthermore, adapting SEA improves the welfare of students without violating feasibility or fairness for same types. Unfortunately, SEA is not strategy-proof, i.e., students may find it preferable to misreport their preferences. Indeed, we show that it is impossible to elicit true preferences in dominant strategies while maintaining fairness for same types and the hard-bounds interpretation of legal constraints (Theorem 3).

In the second part of the paper, instead of relaxing the notion of fairness, we reinterpret the legal constraints, which are implemented as upper and lower bounds (*floors* and *ceilings*, respectively) for each student type. Most school districts administer floors and ceilings as hard bounds, so a theoretical analysis of such policies is inarguably important. However, applications of these hard bounds are quite paternalistic in the sense that they are enforced *independently* of student preferences. With this specification, school districts may end up not allowing students to take some available seats, even if there are no physical limitations. For example, again consider a school with a capacity of 100 students and a hard upper bound of 50 for Caucasian students. Suppose only Caucasian students prefer this school. Then, after first 50 Caucasian students are admitted, the rest of the Caucasian students would not be allowed to enter this school, even though there are some available seats.

To circumvent these shortcomings of controlled school choice with hard bounds, we provide an alternative interpretation of legal constraints as *soft bounds*. To be more explicit, schools may adapt a dynamic priority structure, giving highest priority to student types who have not filled their floors, medium priority to student types who have filled their floors and not their ceilings, and lowest priority to student types who have filled their ceilings. Equivalently, school priorities, floors and ceilings are used in this fashion to construct (*school*) *choice rules* that specify which students to admit from any given set of applicants. With these choice rules, schools can still admit fewer students of the same race/ethnicity than their floor or more than their ceiling as long as students with higher priorities do not veto this match. In Section 4, we consider this soft bounds view by which control policies promote the desired balancing at schools, only when student preferences allow them to do so. In other words, soft bounds policies give schools an opportunity to establish desired balancing, but do not force students to accept an undesired balance. We also justify the soft bounds choice rule by showing that there is no acceptant choice rule[6] that is *closer to controlled choice constraints* than our constructed choice rule (Proposition 1).

With hard bounds, an assignment that is fair and non-wasteful might not exist even if fairness is restricted to students of the same type. On the other hand, with soft bounds the existence of an assignment that is fair across types and non-wasteful is guaranteed. In particular, the *student-proposing deferred acceptance algorithm* (*student-proposing DA*), in which schools tentatively

[6] A choice rule is *acceptant* for a school if it does not reject any student when there is an empty seat.

admit the set of students using the choice rules with the dynamic priority structure, yields such an assignment. With the student-proposing DA, all desirable properties of the deferred acceptance algorithm are restored in the controlled school choice context: the student-proposing DA produces the student-optimal, fair and non-wasteful assignment. Furthermore, the student-proposing DA is group strategy-proof (Theorems 4 and 5). In the proof leading to these results, we show that the choice rules for schools satisfy two axioms that are crucial for the success of the deferred acceptance algorithm: *substitutability* (Kelso and Crawford [38], Roth [48]) and the *law of aggregate demand* (Alkan and Gale [9], Hatfield and Milgrom [35]).[7]

We also consider the *school-proposing deferred acceptance algorithm* (*school-proposing DA*), which produces a fair and non-wasteful assignment as well (Theorem 6). We show that the assignment produced by the school-proposing DA minimizes violations of controlled choice constraints among fair and non-wasteful assignments (Theorem 7). Both the student-proposing DA and the school-proposing DA produce fair and non-wasteful assignments. Even though the student-proposing DA produces the student-optimal fair and non-wasteful assignment, the school-proposing DA maximizes diversity by minimizing violations of controlled choice constraints among fair and non-wasteful assignments.

In Section 5, we relax the assumption that all students have to be assigned to a school in the hard bounds model. For this case, the notion of justified envy becomes weaker, since the envied student can be expelled from a school without being assigned to another school. Compensating for the weaker envy property, the resulting fairness notion becomes stronger. We show that any assignment, which is strongly fair and non-wasteful, is Pareto dominated by the outcome of the student-proposing DA (Theorem 8). Hence, we demonstrate that the alternative approach of allowing students to be left unmatched in the hard bounds model is still inferior to the soft bounds approach.

Although we focus on controlled school choice, our results apply to various matching markets where diversity constraints are to be implemented. For instance, a college or MBA admissions office, which wants to avoid completely segregated student bodies, may use controlled policies. Other examples are entry-level labor markets where one may wish to create more balanced work forces in terms of race, gender or other socioeconomic attributes.

The paper is organized as follows. In Section 2, we formalize controlled school choice problems. In Section 3, we consider the controlled school choice with hard bounds. In Section 4, we consider the controlled school choice problem with soft bounds. In Section 5, we discuss a variant of the controlled school choice problem with hard bounds and obtain a Pareto comparison between hard bounds and soft bounds. In Section 6, we discuss the related literature. In Section 7, we conclude. In Appendix A, we propose an algorithm that can be used as an input assignment of the student exchange algorithm (SEA) and we give an example to demonstrate how SEA works. In Appendix B, we consider the stability notion for the corresponding many-to-one matching market and show that stability is equivalent to fairness under soft bounds and non-wastefulness under soft bounds. All proofs of the results in the main text are relegated to Appendix C.

[7] Substitutability requires that if a student is chosen from a set, then she should also be chosen from any subset of this set containing the student. Whereas, the law of aggregate demand states that the number of students chosen from a set should not be smaller than the number of students chosen from any subset of this set. These two notions are, by now, standard in the literature. Conveniently, the dynamic priority structure defined by soft bounds yields choice rules satisfying these two properties.

2. Controlled school choice

A **controlled school choice problem** or simply a **problem** consists of the following:

1. a finite set of students $S = \{s_1, \dots, s_n\}$;
2. a finite set of schools $C = \{c_1, \dots, c_m\}$;
3. a capacity vector $q = (q_{c_1}, \dots, q_{c_m})$, where q_c is the capacity of school $c \in C$ or the number of seats in $c \in C$;
4. a students' preference profile $P_S = (P_{s_1}, \dots, P_{s_n})$, where P_s is the strict preference relation of student $s \in S$ over C, i.e., $c P_s c'$ means that student s strictly prefers school c to school c';
5. a schools' priority profile $\succ_C = (\succ_{c_1}, \dots, \succ_{c_m})$, where $\succ_c$ is the strict priority ranking of school $c \in C$ over S; $s \succ_c s'$ means that student s has higher priority than student s' to be enrolled at school c;
6. a type space $T = \{t_1, \dots, t_k\}$;
7. a type function $\tau : S \to T$, where $\tau(s)$ is the type of student s;
8. for each school c, two vectors of type specific constraints $\underline{q}_c^T = (\underline{q}_c^{t_1}, \dots, \underline{q}_c^{t_k})$ and $\overline{q}_c^T = (\overline{q}_c^{t_1}, \dots, \overline{q}_c^{t_k})$ such that $\underline{q}_c^t \leqslant \overline{q}_c^t \leqslant q_c$ for all $t \in T$, and $\sum_{t \in T} \underline{q}_c^t \leqslant q_c \leqslant \sum_{t \in T} \overline{q}_c^t$.

Here, $\underline{q}_c^t$ is the minimal number of slots that school c must by law allocate to type t students, called **the floor for type t at school c**, whereas $\overline{q}_c^t$ is the maximal number of slots that school c is allowed by law to allocate to type t students, called **the ceiling for type t at school c**.

In summary, a **controlled school choice problem** is given by

$$\big(S, C, q, P_S, \succ_C, T, \tau, \big(\underline{q}_c^T, \overline{q}_c^T\big)_{c \in C}\big).$$

When all other parameters except P_S remain fixed, we simply refer to P_S as a controlled school choice problem.

The set of types may represent different characteristics of students such as: (i) race; (ii) socioeconomic status (determined by free or reduced-price lunch eligibility); or (iii) the district where the student lives. Controlled choice constraints are either enforced by law or state policies (via desegregation orders) or voluntarily adopted by the school district to increase diversity at schools.[8]

An **assignment** μ is a function from the set $C \cup S$ to the set of all subsets of $C \cup S$ such that

i. $\mu(s) \in C$ for every student s;
ii. $|\mu(c)| \leqslant q_c$ and $\mu(c) \subseteq S$ for every school c;
iii. $\mu(s) = c$ if and only if $s \in \mu(c)$.

In words, $\mu(s)$ denotes the school that student s is assigned to; $\mu(c)$ denotes the set of students that are assigned to school c; and student s is assigned to school c if and only if school c's assignment contains student s. Because any student is assigned to a school in an assignment (and in practice, any student is entitled to a seat at a public school), we will assume throughout

[8] Ehlers [22] also considers the case when students have multi-dimensional types and when control constraints are imposed in terms of percentages. He demonstrates that these extensions can be accommodated to the controlled school choice problems.

$|S| \leqslant \sum_{c \in C} q_c$. Furthermore, let $S^t \equiv \{s \in S:\ \tau(s) = t\}$ denote the set of type t students and $\mu^t(c) \equiv \mu(c) \cap S^t$ denote the students of type t that are assigned to school c.

Given two assignments μ and μ', we say that μ **Pareto dominates** μ' if all students weakly prefer μ to μ' and $\mu \neq \mu'$.

3. Controlled school choice with hard bounds

In this section, controlled choice constraints are taken as feasibility constraints that have to be implemented in practice. To be more explicit, a set of students $S' \subseteq S$ **respects (capacity and controlled choice) constraints at school** c if $|S'| \leqslant q_c$ and for every type $t \in T$, $\underline{q}_c^t \leqslant |\{s \in S':\ \tau(s) = t\}| \leqslant \overline{q}_c^t$. An assignment μ **respects constraints** if for every school c, $\mu(c)$ respects constraints at c, i.e., for every type t we have

$$\underline{q}_c^t \leqslant \left|\mu^t(c)\right| \leqslant \overline{q}_c^t.$$

As outlined before, many state laws in the US require students to be assigned to schools such that (i) at each school constraints are respected and (ii) each student is enrolled at a public school. An assignment μ is **(legally or politically) feasible (under hard bounds)** if μ respects the constraints and every student is assigned to a school.[9] According to the law, every student has a right to attend a public school. Hence, we assume that all students are acceptable to every school. Moreover, we consider the case when students have to give a full ranking of all schools. This is because if students are allowed to give shorter lists and the admissions process requires them to be assigned to a school in their lists, students could simply include only their favorite schools. This clearly may result in non-existence of feasible assignments.[10] We would like to emphasize that students can still prefer their outside options (going to a private school, or being home-schooled) to their assigned schools. Nonetheless, they are required to rank all schools.[11]

Obviously, a controlled school choice problem does not have a feasible solution if there are not enough students of a certain type to fill the minimal number of slots required by law for that type of students in all schools. There are other cases in which the controlled choice constraints cannot be satisfied and no feasible assignment exists.[12] In that case, either the laws are not compatible with each other and they need to be modified, which is out of this paper's scope, or controlled choice constraints need to be reconsidered (which we discuss in Section 4). In the rest of the paper, we assume that the legal constraints at schools are such that a feasible assignment exists.

What are desirable properties of feasible assignments in controlled school choice problems? The following notions are the natural adaptations of their counterparts in standard two-sided matching literature (without type constraints).

[9] Later in Section 4, we are going to reinterpret assumption (i) and study controlled choice with soft bounds. In Section 5, we are going to eliminate the assumption (ii), and obtain a welfare comparison between hard bounds and soft bounds.

[10] Indeed, in many school districts some students are always left unassigned at the end of the main assignment and ad-hoc methods are adapted to assign these students to schools with unfilled capacity. For example, the New York City student assignment system has such a second stage (Abdulkadiroğlu et al. [3]).

[11] Boston school district states the following in their website (http://www.bostonpublicschools.org/assignment) "If you don't receive one of your school choices, ... we will assign the student to the school closest to home that has a seat."

[12] It turns out that, given any controlled choice constraints, existence of a feasible assignment can be checked in polynomial time via the "transportation algorithm," which is well-known in the operations research literature. Some details about the transportation algorithm are given in Appendix A.

The first requirement is that a student cannot claim an empty slot in another school without violating the legal constraints.[13] Formally, we say that **student s justifiably claims an empty slot at school c under the feasible assignment μ** if

(nw1) $c P_s \mu(s)$ and $|\mu(c)| < q_c$,
(nw2) $\underline{q}_{\mu(s)}^{\tau(s)} < |\mu^{\tau(s)}(\mu(s))|$, and
(nw3) $|\mu^{\tau(s)}(c)| < \overline{q}_c^{\tau(s)}$.

Here (nw1) means student s prefers an empty slot at school c to the school assigned to her; (nw2) means that the floor of student s's type is not binding at school $\mu(s)$; and (nw3) means that the ceiling of student s's type is not binding at school c. Hence, under (nw1)–(nw3) student s can be assigned to the better school c without changing the assignments of the other students and without violating constraints at any school. A feasible assignment μ is **non-wasteful** if no student justifiably claims an empty slot at any school.

Another well-studied requirement in the literature is fairness or no-envy (Foley [27]).[14] In school choice, student s envies student s' when s prefers the school at which s' is enrolled, say school c, to her school. However, the nature of controlled school choice imposes the following (legal) constraints: Envy is justified only when

(i) student s has higher priority at school c than student s',
(ii) student s can be enrolled at school c without violating controlled choice constraints (at all schools) by removing s' from c, and
(iii) student s' can be enrolled at another school without violating constraints.

Formally, we say that **student s justifiably envies student s' at school c under the feasible assignment μ** if there exists another feasible assignment μ' such that

(f1) $\mu(s') = c$, $c P_s \mu(s)$ and $s \succ_c s'$,
(f2) $\mu'(s) = c$, $\mu'(s') \neq c$, and $\mu'(\hat{s}) = \mu(\hat{s})$ for all $\hat{s} \in S \backslash \{s, s'\}$.

Because μ' is feasible, (f2) simply says that $(\mu(c) \backslash \{s'\}) \cup \{s\}$ respects controlled choice constraints at school c and student s' can be enrolled at school $c' = \mu'(s')$ such that $(\mu(c') \backslash \{s\}) \cup \{s'\}$ respects controlled choice constraints at c'. In other words, assigning s a slot at c, s' a slot at c', and keeping all the other assignments intact do not violate any controlled choice constraint at any school. A feasible assignment μ is **fair across types (or fair)** if no student justifiably envies any other student.[15]

In what follows, we also consider a weaker version of envy (and fairness) where envy is justified only if both the envying student and the envied student are of the same type. If this is the case, then (ii) and (iii) are always true since then the envying student and the envied student

[13] This requirement is in the spirit of the "non-wastefulness" property introduced by Balinski and Sönmez [11].

[14] See for example Tadenuma and Thomson [51], for an excellent survey as well as Thomson [52,53] and Young [55].

[15] Note that if we do not require student s' to be assigned to another school, it will be easier to envy, hence the negative result we have in Theorem 1 would continue to hold. In Section 5, we analyze a "weak envy" condition.

can simply exchange schools. Formally, we say that **student s justifiably envies student s' of the same type at school c under the feasible assignment μ** if

(f1∗) $\mu(s') = c$, $c P_s \mu(s)$ and $s \succ_c s'$, and
(f2∗) $\tau(s) = \tau(s')$.

In (f1∗), student s' is enrolled at school c and both student s prefers school c to his assigned school $\mu(s)$ and student s has higher priority to be enrolled at school c than student s'. By (f2∗), student s and student s' are of the same type. Then we obtain a feasible assignment when students s and s' exchange their slots, i.e., choose μ' as follows: $\mu'(s) = \mu(s')$, $\mu'(s') = \mu(s)$, and $\mu'(\hat{s}) = \mu(\hat{s})$ for all $\hat{s} \in S \backslash \{s, s'\}$. The assignment μ' is feasible because s and s' are of the same type and μ is feasible. A feasible assignment μ is **fair for same types** if no student justifiably envies any other student of the same type. We say that an assignment μ that is fair for same types is **constrained efficient** if there is no assignment that satisfies fairness for same types and Pareto dominates μ.

Our first result shows the difficulty in finding assignments that satisfy the legal constraints together with other desirable properties such as fairness and non-wastefulness by establishing two benchmark incompatibility results (even under the assumption that a feasible assignment exists).

Theorem 1.

(i) *The set of feasible assignments that are fair across types may be empty in a controlled school choice problem.*
(ii) *The set of feasible assignments that are both fair for same types and non-wasteful may be empty in a controlled school choice problem.*

The proof of Theorem 1 is provided in Appendix C, and is by means of examples. Even though the upper bounds do not seem to cause any problems (Abdulkadiroğlu and Sönmez [6]), the lower bounds yield the negative results. Intuitively, the lower bounds create *complementarities* between students for schools, which is problematic. For example, if school c has a lower bound of 2 for students of type t, then student s_1 of type t or student s_2 of type t alone cannot be assigned to school c without any other type t student, even though students s_1 and s_2 together can be assigned to school c.

Theorem 1 is in the same spirit with the failure of the existence of competitive equilibrium when there are complementarities (Kelso and Crawford [38]). Complementarities are also known to cause problems in other environments such as auctions (Cramton et al. [16]), course allocation (Sönmez and Ünver [50]), and assignment problems (Budish [13]). In the context of matching with contracts, Hatfield and Milgrom [35] show that substitutability is needed for the existence of stable matchings in a maximal domain sense (see also Hatfield and Kojima [33]). These impossibility results do not immediately extend to our special setting because each school has linear priorities over individual students and complementarities are caused by lower bounds and, more importantly, the solution concepts are different.

Even though non-wastefulness is incompatible with fairness for same types, it can be replaced with a slightly weaker property. We say that a feasible assignment μ is **constrained non-wasteful** if a student s claims an empty slot at school c under μ, then the assignment μ' (where $\mu'(s) = c$ and $\mu'(s') = \mu(s')$ for all $s' \in S \backslash \{s\}$) is *not* fair for same types. This property is desirable because if a school district has fairness for same types as its main goal but also wants efficiency as a secondary goal, then constrained non-wastefulness naturally arises as an appropriate property. With this view, the school district may allocate empty seats to students who would like to take them as long as it does not cause justified envy between students of the same type.

If the feasible assignment μ is fair for same types and also constrained non-wasteful, then the above definition is equivalent to the requirement that whenever a student s of type t justifiably claims an empty slot at school c under μ, then some other student s' of same type t justifiably envies student s at school c under the assignment μ' (where μ' is defined above).

The idea of feasible assignments that are both fair for same types and constrained non-wasteful is similar to the one of "bargaining sets": if a type t student s has an objection to μ because s claims an empty slot at c, then there will be a counter-objection once s is assigned to c since some other type t student will then justifiably envy s at c. Roughly speaking, an outcome belongs to the "bargaining set" if and only if for any objection to the outcome there exists a counter-objection.[16]

We now show that there exists a feasible assignment that is both fair for same types and constrained non-wasteful in a controlled school choice problem. To show this, we introduce a new mechanism that we call the *student exchange algorithm* (*SEA*). The algorithm takes any feasible assignment that is fair for same types as input (we introduce a simple mechanism that finds a feasible assignment that is fair for same types in Appendix A), and produces a feasible assignment that is not only fair for same types but also constrained efficient. Therefore, the outcome is also constrained non-wasteful (Lemma 5). In the algorithm, schools exchange students to improve the welfare of students while the feasibility and fairness for same types are preserved. Hence, the new assignment is better for all exchanged students. The main difficulty in the algorithm is to find exchange cycles such that the new assignment satisfies feasibility and fairness for same types. To overcome this issue, for each assignment μ, we introduce the associated *application graph* $G(\mu)$.

For any assignment μ, define the directed application graph $G(\mu) = (V(\mu), E(\mu))$, where $V(\mu)$ is the set of nodes and $E(\mu) \subseteq V(\mu) \times V(\mu)$ is the set of directed edges, as follows. For ease of notation, we sometimes suppress the dependence on assignment μ and denote the application graph by $G = (V, E)$.

First, for each school c, we create a node for each student type that is present in $\mu(c)$. Moreover, if there is an empty seat in school c, then we create an additional node $c(t_0)$ to represent the empty seats. Formally,

$$V(\mu) \equiv \bigcup_{c \in C} \left(\left\{ c(t)\colon\ t \in T \ \&\ \mu^t(c) \neq \emptyset \right\} \right) \cup \left\{ c(t_0)\colon\ c \in C \ \&\ \left|\mu(c)\right| < q_c \right\}.$$

Second, $E(\mu)$ consists of the following directed edges. For each student type t and school c, we consider all type t students who would prefer to be matched with c rather than their current assignments, i.e., $\{s \in S^t\colon\ c P_s \mu(s)\}$. If this set is empty, then we do nothing. Otherwise, if this set is non-empty, then we consider the student with the highest priority according to $\succ_c$ in this set. Let $\hat{s}$ be this student.[17] Assume $\hat{s}$ is assigned to school $\hat{c} \equiv \mu(\hat{s})$. Then,

[16] In a paper subsequent to Ehlers [22], Alcalde and Romero-Medina [7] weakened stability in a similar fashion in school choice problems without constraints in order to improve efficiency of stable assignments. Kesten [39] also proposes a method for the latter.

[17] The reason that we only consider $\hat{s}$, and not other students with lower priorities according to $\succ_c$ is to be able to satisfy fairness for same types throughout the mechanism.

(i) $(\hat{c}(t) \to c(t)) \in E(\mu)$ if $c(t) \in V(\mu)$,
(ii) $(\hat{c}(t) \to c(t')) \in E(\mu)$ if $t' \neq t$, $|\mu^{t'}(c)| > \underline{q}_c^{t'}$ and $|\mu^t(c)| < \overline{q}_c^t$,
(iii) $(\hat{c}(t) \to c(t_0)) \in E(\mu)$ if $c(t_0) \in V(\mu)$ and $|\mu^t(c)| < \overline{q}_c^t$,
(iv) $(c(t_0) \to c'(t')) \in E(\mu)$ if $t' \in T$, $c' \neq c$, $|\mu^{t'}(c')| > \underline{q}_{c'}^{t'}$.

In (i), $\hat{c}(t)$ points to $c(t)$, if there is a type t student in $\mu(c)$. In (ii), $\hat{c}(t)$ points to $c(t')$ if $\hat{s}$ can be admitted to c by replacing a student of type t' without violating the feasibility constraints in c (type t' can leave c and type t can join c). In (iii), $\hat{c}(t)$ points to $c(t_0)$ if c has an empty seat in μ and $\hat{s}$ can take that seat without violating the feasibility constraints in c (type t can join c). Finally, in (iv), $c(t_0)$ points to $c'(t')$ where $c' \neq c$ if a student of type t' can be expelled from c' without violating feasibility constraints in c' (type t' can leave c').

Intuitively, in the student exchange algorithm, we search for "trading cycles" in which we improve the assignments of the students included in cycles while making sure that controlled choice constraints are not violated and fairness for same types is preserved. Conditions (i)–(iv) ensure that trading cycles preserve feasibility according to controlled choice constraints.

A more formal definition of the algorithm is in order.

Student Exchange Algorithm (SEA).

Step 0. Consider an assignment μ_0 that is fair for same types.

Step ℓ. Construct $G(\mu_{\ell-1})$. If there are no cycles in $G(\mu_{\ell-1})$, then stop. Otherwise, consider a cycle in the graph, $c'_1(t'_1) \to c'_2(t'_2) \to \cdots \to c'_p(t'_p) \to c'_1(t'_1)$. Rematch students associated with each node in the cycle as follows. For each $c'_i(t'_i)$ with $t'_i \neq t_0$, $i \in \{1, \dots, p\}$, there exists $s_i \in \mu_{\ell-1}^{t'_i}(c'_i)$ such that $s_i \succ_{c'_{i+1}} s'$ for all $s' \in \{s \in S^{t'_i}: c'_{i+1} P_s \mu_{\ell-1}(s)\}$ where $c'_{p+1} \equiv c'_1$. Let $\mu_\ell(s_i) = c'_{i+1}$ for all such students, otherwise let $\mu_\ell(s) = \mu_{\ell-1}(s)$. Go to Step $\ell + 1$.

In the student exchange algorithm, we start with a feasible assignment that is fair for same types. Then we improve the assignments of students in each step such that this property is preserved.[18] The algorithm has to end in a finite number of steps since in each step at least one student's assignment is improved. We are now ready to state our main result for this section.

Theorem 2. *For any controlled school choice problem, the student exchange algorithm yields a feasible assignment μ that is fair for same types and constrained non-wasteful. Moreover, μ is constrained efficient.*

The proof of Theorem 2 is provided in Appendix C. A key lemma in the proof, Lemma 6, shows that if an assignment is fair for same types but not constrained efficient, then there exists a cycle in its application graph. Therefore, such an assignment cannot be the termination point of the algorithm. In other words, the algorithm produces a feasible assignment that is fair for same types, which is also constrained efficient. Moreover, any such assignment also has to be constrained non-wasteful (Lemma 5).

[18] Note that by construction of $G(\mu_{\ell-1})$, any cycle contains a type in T (and cannot consist only of nodes $c(t_0)$ where $c \in C$).

Erdil and Ergin [24] also deploy cycles to improve the welfare of students while preserving fairness. However, their model does not have controlled choice constraints. In contrast, the main difficulty in our setup is finding the exchange cycles to improve the welfare of students in a *feasible* way. To this end, we use the application graph to find the exchange cycles, which is the main crux of our proof.

The input to the student exchange algorithm is any feasible assignment that is fair for same types. Such an assignment can be found in a number of ways, for example, by using the controlled deferred acceptance algorithm of Ehlers [22]. In Appendix A, we provide a simpler and more intuitive algorithm for finding such an assignment that we call *the deferred acceptance algorithm with fixed type allotments* (*DAAFTA*). We also show how DAAFTA followed by SEA works in Appendix A.

3.1. Incentives

Apart from student preferences, all other components of a controlled school choice problem are exogenously determined (like the capacities of the schools) or set by law (like the priority rankings and the controlled choice constraints). On the other hand, each student's preference is her private information. Therefore, student preferences should be revealed by students to the school choice program. Since students must be assigned to schools for any possible preference profile, the assignment mechanism of the program should be incentive compatible while respecting the constraints. We define a mechanism to be **(legally) feasible** if it selects a feasible assignment for *any* reported students' preference profile. A feasible mechanism is **fair for same types** if it selects an assignment that is fair for same types for any controlled school choice problem. Analogously we define **non-wastefulness** and **constrained non-wastefulness**, respectively, for a mechanism.

Any program would like to elicit the true preferences from students. If students choose to misreport, then the assignment chosen by the program is based on *false* preferences and may be highly unfair given the *true* preferences. Avoiding this problem means constructing a mechanism where no student has ever an incentive to strategically misrepresent her true preference. Any mechanism which makes truthful revelation of preferences a dominant strategy for each student is called **strategy-proof**. Formally, an assignment mechanism ϕ is strategy-proof if for any $s \in S$, and for any profile P_S, there exists no P'_s such that $\phi_s(P'_s, P_{S\setminus\{s\}}) P_s \phi_s(P_S)$.

It turns out that it is impossible to construct a mechanism that is strategy-proof, fair for same types and constrained non-wasteful while respecting the diversity constraints.

Theorem 3. *In controlled school choice problem, there is no feasible mechanism that is strategy-proof, fair for same types and constrained non-wasteful.*

The proof of Theorem 3 is in Appendix C where we provide an example to prove the non-existence of such mechanisms.[19]

[19] In school choice problems without control and legal constraints, Ergin and Sönmez [26] consider revelation games induced by the Boston school choice mechanism and the deferred acceptance algorithm.

4. Controlled school choice with soft bounds

Some school districts administer floors and ceilings as hard bounds, so a theoretical analysis of such policies is inarguably important. In the previous sections, we accommodate this constraint by considering an assignment infeasible if it assigns less than the floor or more than the ceiling for some type at some school. However, applications of these hard bounds are quite paternalistic in the sense that assignments are enforced independently of student preferences: feasibility may require to admit a number of Caucasian students to a particular school even when that school is each Caucasian student's least preferred one. In contrast, in this section we view these constraints as *soft bounds*. In controlled school choice with soft bounds, school districts adopt a dynamic priority structure: giving highest priority to student types who have not filled their floors; medium priority to student types who have filled their floors, but not filled their ceilings; and lowest priority to student types who have filled their ceilings. Yet, schools can still admit fewer students than their floors or more than their ceilings as long as students with higher priorities do not veto this assignment. With this view, there are no feasibility constraints as long as school capacities are not exceeded (and our approach below can be used for situations where no feasible assignment exists). All controlled choice concerns are embedded in the choice rules of the schools that we describe below. A **choice rule** $\mathcal{C}$ is a function on the power set of students such that for all $\emptyset \neq S' \subseteq S$, $\mathcal{C}(S')$ is a subset of S'.

The choice rule for school c depends on capacity q_c, floors $\underline{q}_c^T$, and ceilings $\overline{q}_c^T$ as described above. However, we are going to take these parameters as given and simplify the notation by omitting them. To define the choice rule more formally, given $S' \subseteq S$, let $H_c(S', \tilde{q}_c, (\tilde{q}_c^t)_{t\in T})$ be the largest subset of S' (with respect to set inclusion) that includes the highest ranked students in S' according to $\succ_c$ such that there are no more than $\tilde{q}_c$ students in total and $\tilde{q}_c^t$ students of type t. In addition, let

$$\mathcal{C}_c^{SB(1)}(S') \equiv H_c\big(S', q_c, (\underline{q}_c^t)_{t\in T}\big),$$
$$\mathcal{C}_c^{SB(2)}(S') \equiv H_c\big(S' \setminus \mathcal{C}_c^{SB(1)}(S'), q_c - \big|\mathcal{C}_c^{SB(1)}(S')\big|, (\overline{q}_c^t - \underline{q}_c^t)_{t\in T}\big), \quad \text{and}$$
$$\mathcal{C}_c^{SB(3)}(S') \equiv H_c\big(S' \setminus \big(\mathcal{C}_c^{SB(1)}(S') \cup \mathcal{C}_c^{SB(2)}(S')\big), q_c - \big|\mathcal{C}_c^{SB(1)}(S') \cup \mathcal{C}_c^{SB(2)}(S')\big|, (q_c - \overline{q}_c^t)_{t\in T}\big).$$

Intuitively, $\mathcal{C}_c^{SB(1)}(S')$ is the set of students chosen with the highest priorities among S' without exceeding the floor of each student type, $\mathcal{C}_c^{SB(2)}(S')$ is the set of remaining students chosen from S' with the highest priorities without exceeding the ceilings, and $\mathcal{C}_c^{SB(3)}(S')$ is the set of students chosen above the ceilings. Finally,

$$\mathcal{C}_c^{SB}(S') \equiv \mathcal{C}_c^{SB(1)}(S') \cup \mathcal{C}_c^{SB(2)}(S') \cup \mathcal{C}_c^{SB(3)}(S')$$

is the set of students chosen from S'. It is apparent from this formulation that schools dynamically give highest priority to student types who have not filled their floors; medium priority to student types who have filled their floors, but not filled their ceilings; and lowest priority to student types who have filled their ceilings.

Now we define the desirable properties under the soft bounds approach. In controlled school choice with soft bounds, any assignment is feasible. An assignment μ is **non-wasteful under soft bounds** if for any student s and any school c, $cP_s\mu(s)$ implies $|\mu(c)| = q_c$. Previously, non-wastefulness required student s to be matched to school c without violating ceilings and floors, which is not required anymore. Furthermore, an assignment μ **removes justifiable envy**

under soft bounds if for any student s and any school c such that $c P_s \mu(s)$ with $\tau(s) = t$, we have both $|\mu^t(c)| \geqslant \underline{q}_c^t$ and $s' \succ_c s$ for all $s' \in \mu^t(c)$, and either

(i) $|\mu^t(c)| \geqslant \overline{q}_c^t$ and $s' \succ_c s$ for all $s' \in \mu(c)$ such that $|\mu^{\tau(s')}(c)| > \overline{q}_c^{\tau(s')}$, or
(ii) $\overline{q}_c^t > |\mu^t(c)| \geqslant \underline{q}_c^t$, and
  (a) $|\mu^{t'}(c)| \leqslant \overline{q}_c^{t'}$ for all $t' \in T \setminus \{t\}$, and
  (b) $s' \succ_c s$ for all $s' \in \mu(c)$ such that $\overline{q}_c^{\tau(s')} \geqslant |\mu^{\tau(s')}(c)| > \underline{q}_c^{\tau(s')}$.

In other words, an assignment removes justifiable envy under soft bounds if a student s of type t cannot attend a favorable school c, then type t students fill their floor in c and c prefers all type t students that it has been assigned to s. In addition, either (i) c has admitted more type t students than its ceiling, and all students with types exceeding their ceilings are preferred to s; or (ii) c has admitted more type t students than its floor, but not more than its ceiling, yet there are no students with types exceeding their ceilings, and all students with types exceeding their floors are preferred to s. An assignment μ is **fair under soft bounds** if it removes justifiable envy under soft bounds.

Next, we establish that soft bounds choice function satisfies important properties for choice rules that are crucial for the existence of stable assignments and strategy-proofness.

Definition 1. Choice rule $\mathcal{C}$ satisfies **substitutability** if for any $S' \subseteq S$ that contains students s and s' ($s \neq s'$), $s \in \mathcal{C}(S')$ implies $s \in \mathcal{C}(S' \setminus \{s'\})$.

Substitutability was introduced by Kelso and Crawford [38] for matching with transfers and adapted to matching without transfers by Roth [48].

In our setup, the choice rules satisfy substitutability even though school priorities are determined dynamically using floors and ceilings.

Definition 2. Choice rule $\mathcal{C}$ satisfies **the law of aggregate demand (LAD)** if for any $S'' \subseteq S' \subseteq S$, we have $|\mathcal{C}(S'')| \leqslant |\mathcal{C}(S')|$.

The law of aggregate demand was first introduced in Alkan [8] and Alkan and Gale [9] for matching without transfers as size monotonicity. Later it was used in matching with contracts by Hatfield and Milgrom [35].

Consider a school c with capacity q_c. We say that school c's choice rule is q_c-acceptant if the school selects q_c students when there are at least q_c applicants, and otherwise selects all students.

Definition 3. Choice rule $\mathcal{C}$ is q_c**-acceptant** if for any $S' \subseteq S$, we have $|\mathcal{C}(S')| = \min\{q_c, |S'|\}$.

Note that being q_c-acceptant is a stronger requirement than the law of aggregate demand. Kojima and Manea [41] and Ehlers and Klaus [23] use acceptant priorities.

Next we show that $\mathcal{C}_c^{SB}$ satisfies all of the properties above.

Lemma 1. *For every school c, $\mathcal{C}_c^{SB}$ is substitutable and q_c-acceptant.*

Since q_c-acceptance implies LAD, this lemma implies that $\mathcal{C}_c^{SB}$ satisfies LAD.

Lastly, we motivate the construction of $\mathcal{C}_c^{SB}$ by showing that there cannot be any other choice rule that is q_c-*acceptant* and *closer to constraints* than $\mathcal{C}_c^{SB}$. Before we state our result, some terminology is in order.

Define the following functions on sets of students measuring the violations of controlled choice constraints for each type t at a given school c:

$$\underline{d}_c^t(S') \equiv \max\{\underline{q}_c^t - |S' \cap S^t|, 0\},$$
$$\overline{d}_c^t(S') \equiv \max\{|S' \cap S^t| - \overline{q}_c^t, 0\}.$$

Similarly, we can measure the violations for lower and upper bounds for all types, let $\underline{d}_c(S') \equiv \sum_t \underline{d}_c^t(S')$ and $\overline{d}_c(S') \equiv \sum_t \overline{d}_c^t(S')$. This definition can be expanded to assignments μ by setting $\underline{d}_c(\mu) \equiv \sum_t \underline{d}_c^t(\mu(c))$ and $\overline{d}_c(\mu) \equiv \sum_t \overline{d}_c^t(\mu(c))$.

Definition 4. Let $\mathcal{C}$ and $\mathcal{C}'$ be two choice rules. We say that $\mathcal{C}$ is closer to controlled choice constraints than $\mathcal{C}'$ for school c if for each type $t \in T$ and for each $S' \subseteq S$, we have $\max\{\underline{d}_c^t(\mathcal{C}'(S')), \overline{d}_c^t(\mathcal{C}'(S'))\} \geqslant \max\{\underline{d}_c^t(\mathcal{C}(S')), \overline{d}_c^t(\mathcal{C}(S'))\}$, and for some $t \in T$ and some $S' \subseteq S$, $\max\{\underline{d}_c^t(\mathcal{C}'(S')), \overline{d}_c^t(\mathcal{C}'(S'))\} > \max\{\underline{d}_c^t(\mathcal{C}(S')), \overline{d}_c^t(\mathcal{C}(S'))\}$.

In words, if $\mathcal{C}$ is closer to controlled choice constraints than $\mathcal{C}'$ for school c, then whenever $\mathcal{C}'$ violates constraints at school c, $\mathcal{C}$ is closer to satisfy constraints at school c (where we do not differentiate between floors and ceilings); and if $\mathcal{C}'$ satisfies constraints at school c, then so does $\mathcal{C}$.

Proposition 1. *There is no q_c-acceptant choice rule that is closer to controlled choice constraints than $\mathcal{C}_c^{SB}$ for school c.*[20]

The proof of Proposition 1 is provided in Appendix C and is by contradiction.

In the next two subsections we consider the student-proposing deferred acceptance algorithm and the school-proposing deferred acceptance algorithm.

4.1. Student-proposing DA

With hard bounds, no assignment that is fair and non-wasteful exists even if fairness is restricted to students within the same types (Theorem 1). However, with soft bounds we guarantee the existence of an assignment that is non-wasteful under soft bounds and fair under soft bounds. To show this, we consider the student-proposing deferred acceptance algorithm with soft bounds.

Student-Proposing DA.

Step 1 Start with the assignment in which no student is matched. Each student s applies to her first-choice school. Let $S_{c,1}$ denote the set of students who applied to school c. School c tentatively accepts the students in $\mathcal{C}_c^{SB}(S_{c,1})$ and permanently rejects the rest.

[20] Note that there are q_c-acceptant choice rules $\mathcal{C} \neq \mathcal{C}_c^{SB}$ such that no other q_c-acceptant choice rule is closer to controlled constraints than $\mathcal{C}$: let all floors and all ceilings be equal to zero, and for any $S' \subseteq S$, first choose all type t_1 students in S', second choose all type t_2 students in S', and so on (without violating capacity q_c).

Step k Start with the tentative assignment obtained at the end of Step $k-1$. Each student s who got rejected at Step $k-1$ applies to her next-choice school. Let $S_{c,k}$ denote the set of students who either were tentatively matched to c at the end of Step $k-1$, or applied to school c in this step. Each school tentatively accepts the students in $\mathcal{C}_c^{SB}(S_{c,k})$ and permanently rejects the rest. If there are no rejections, then stop.

The student-proposing DA terminates when there are no rejections. At each step of the algorithm, there is at least one student who gets rejected. Hence, the algorithm ends in finite time.

Next, we establish the well-known properties of the student-proposing DA in the current setting.

Theorem 4. *For any controlled school choice problem, the student-proposing DA yields an assignment that is fair under soft bounds and non-wasteful under soft bounds. Moreover, each student prefers this assignment to any other assignment that is fair under soft bounds and non-wasteful under soft bounds.*

In Appendix B, we show that stability with respect to choice rule profile $(\mathcal{C}_c^{SB})_{c\in C}$ is equivalent to fairness under soft bounds and non-wastefulness under soft bounds (Lemma 2). Consequently, the student-proposing DA continues to work well because the choice rules satisfy substitutability.[21] Thus, the proof of Theorem 4 follows immediately from Theorem 6.8 in Roth and Sotomayor [49] since choice rules are substitutable.

Even though the student exchange algorithm fails to satisfy strategy-proofness, the student-proposing DA satisfies a stronger property: An assignment mechanism ϕ is **group strategy-proof** if for any group of students $\hat{S} \subseteq S$, for any profile P_S there exists no $P'_{\hat{S}}$ such that $\phi_s(P'_{\hat{S}}, P_{S\setminus\hat{S}}) P_s \phi_s(P_S)$ for all $s \in \hat{S}$. If a mechanism is group strategy-proof, then there exists no group of students who can jointly change their preference profiles to make each student in the group better off.

Theorem 5. *The student-proposing DA is group strategy-proof.*

To prove Theorem 5 we rely on a result of Hatfield and Kojima [34]. They show that, in a many-to-one matching model with contracts, if schools' choice rules satisfy the law of aggregate demand and substitutability, then the student-proposing DA is group strategy-proof. Our setup can be trivially embedded in the many-to-one matching model with contracts of Hatfield and Kojima [34], so the conclusion follows.[22]

It is important to analyze the student-proposing DA because each student gets the best possible school among assignments that are fair under soft bounds and non-wasteful under soft bounds. On the other hand, we show that the assignment produced by the student-proposing DA may

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[21] In a recent paper Aygün and Sönmez [10] show that when choice rules are primitives of the model, rather than the preferences, an additional axiom called *irrelevance of rejected students* (IRS) is needed for the deferred acceptance algorithm to work well. $\mathcal{C}$ satisfies IRS if $\mathcal{C}(S'') \subseteq S' \subseteq S''$ then $\mathcal{C}(S') = \mathcal{C}(S'')$. In our setup, for every c, $\mathcal{C}_c^{SB}$ satisfies IRS because substitutability and LAD imply IRS (Proposition 1, Aygün and Sönmez [10]).

[22] See also Martínez et al. [44].

not respect constraints even when there exists an assignment that is fair under soft bounds and non-wasteful under soft bounds.[23]

Example 1. Assume that there are two students $\{s_1, s_2\}$, two schools $\{c_1, c_2\}$ and two student types $\{t_1, t_2\}$ such that $\tau(s_1) = t_1$ and $\tau(s_2) = t_2$. Both schools have capacity one. The only effective controlled choice constraint is $\underline{q}_{c_1}^{t_2} = 1$. Suppose that $c_1 P_{s_1} c_2$ and $c_2 P_{s_2} c_1$, and $s_2 \succ_{c_1} s_1$ and $s_1 \succ_{c_2} s_2$. Let μ be the assignment such that $\mu(s_1) = c_1$ and $\mu(s_2) = c_2$, and μ' be the assignment such that $\mu'(s_1) = c_2$ and $\mu'(s_2) = c_1$. Obviously, μ' is feasible with respect to hard bounds; moreover, it is fair under soft bounds and non-wasteful under soft bounds. Yet, the student-proposing DA produces μ, which does not satisfy controlled choice constraints.

Next, we introduce the school-proposing DA. In the previous example, the school-proposing DA produces μ'. In fact, we show that the school-proposing DA yields an assignment that minimizes violations of controlled choice constraints among assignments that are fair under soft bounds and non-wasteful under soft bounds.

4.2. School-proposing DA

In the school-proposing deferred acceptance algorithm, schools make the proposals instead of the students.

School-Proposing DA. For each school c, denote the set of students who have rejected school c in the first t steps of the algorithm defined below by $R_c(t)$. Let $R_c(0) \equiv \emptyset$.

Step 1. Each school c proposes to students in $\mathcal{C}_c^{SB}(S \setminus R_c(0))$. Each student s tentatively accepts the best school according to P_s and rejects the rest. For each school c, update the set of rejecting students, $R_c(1)$, accordingly. If there are no rejections, then stop.

Step ℓ. Each school c proposes to students in $\mathcal{C}_c^{SB}(S \setminus R_c(\ell - 1))$. Each student s considers all proposals, accepts the best school with respect to P_s and rejects the rest. For each school c, update the set of rejecting students, $R_c(\ell)$, accordingly. If there are no rejections, then stop.

The school-proposing DA ends in finite time since there is at least one rejection at every step of the algorithm except the last one.

Theorem 6. *For any controlled school choice problem, the school-proposing DA yields an assignment, say μ^*, that is fair under soft bounds and non-wasteful under soft bounds. Moreover, if μ is any assignment that is fair under soft bounds and non-wasteful under soft bounds, then for any school c, $\mathcal{C}_c^{SB}(\mu^*(c) \cup \mu(c)) = \mu^*(c)$.*

The proof of this result is immediate from Roth [48] since we know that $(\mathcal{C}_c^{SB})_{c \in C}$-stability is equivalent to fairness under soft bounds and non-wastefulness under soft bounds (Lemma 2). Next we show that the outcome of the school-proposing DA minimizes violations of controlled choice constraints.

23 We are grateful to an anonymous referee for raising this issue which allowed us to construct this example and consider the school-proposing DA.

Theorem 7. *Let μ^* be the outcome of the school-proposing DA and μ be an assignment that is fair under soft bounds and non-wasteful under soft bounds. Then for any school c, $\underline{d}_c(\mu^*) \leqslant \underline{d}_c(\mu)$ and $\overline{d}_c(\mu^*) \leqslant \overline{d}_c(\mu)$.*

The proof follows from an important lemma and the rural hospitals theorem and is relegated to Appendix C. An important corollary of this result is the following.

Corollary 1. *If there exists an assignment that respects constraints and that is fair under soft bounds and non-wasteful under soft bounds, then the outcome of the school-proposing DA respects constraints.*

5. A Pareto comparison between hard bounds and soft bounds

The assumption that all students have to be matched is critical in our hard bounds approach (Section 3). Most crucially, it is incorporated in the definition of justified-envy: A student can only justifiably-envy another student at school c if the latter can be expelled from school c and admitted at another school *feasibly*. On the other hand, this assumption is not needed in the soft bounds approach. Here, we discuss the implications of removing this assumption in the hard bounds model.

Suppose that students can be left unmatched in the hard bounds model. Then the definition of envy changes as follows. A student s envies student s' when s prefers the school at which s' is enrolled, say school c, to her school. A weak envy is justified only when

(i) student s has higher priority at school c than student s',
(ii) student s can be enrolled at school c by replacing s' without violating controlled choice constraints.

Recall that in Section 3 envy is justified with the additional requirement that s' can be enrolled at another school and s can be removed from $\mu(s)$ (possibly replacing with s') without violating controlled choice constraints. Therefore, envy is justified more easily now, which leads to a stronger fairness notion. An assignment is **strongly-fair across types** if no student justifiably weakly-envies any student.

Finally, we show that any assignment that is strongly-fair across types and non-wasteful[24] is Pareto dominated by the outcome of the student-proposing DA. This result shows us that if we took a different approach in Section 3 and allowed a feasible assignment to leave students unassigned, then any stable solution of this alternative approach would be inferior to the outcome of the student-proposing DA.

Theorem 8. *Suppose that μ is a feasible assignment that is strongly-fair across types and non-wasteful. Then all students weakly prefer the outcome of the student-proposing DA to μ.*

The proof of Theorem 8 is provided in Appendix C, and the proof strategy closely follows that of Theorem 1 in Hafalir et al. [31]. Since μ is a feasible assignment that is strongly-fair across types, it is also fair under soft bounds. In addition, if μ is also non-wasteful under soft bounds,

[24] Recall that Theorem 1 implies such an assignment may fail to exist.

then the result follows from Theorem 4. Otherwise, if μ is wasteful under soft bounds, then there exist a school c and a student s such that $c P_s \mu(s)$ and $|\mu(c)| < q_c$. Whenever there exists such a pair, we can improve the matches of students by reassigning them to the empty seats. In the proof, we show that this improvement process delivers an assignment that is fair under soft bounds and non-wasteful under soft bounds. The conclusion is then reached by applying Theorem 4. Therefore, if feasible assignments that are strongly-fair across types and non-wasteful exist, then the outcome of the student-proposing DA (weakly) Pareto dominates *all* such assignments. In such situations all students are weakly better off under soft bounds than under hard bounds.[25]

6. Related literature

In a seminal paper, Abdulkadiroğlu and Sönmez [6] propose *the student-proposing deferred acceptance algorithm* (also known as Gale–Shapley student optimal stable mechanism) as an alternative to some popular school choice mechanisms. This mechanism finds the fair assignment which is preferred by every student to any other fair assignment. Moreover, revealing preferences truthfully is a weakly dominant strategy for every student in the preference revelation game in which students submit their preferences over schools first, and then the assignment is determined via the student-proposing deferred acceptance algorithm using the submitted preferences (Dubins and Freedman [18]; Roth [47]).[26] Abdulkadiroğlu and Sönmez also study control constraints only with upper bounds and extend the Gale and Shapley algorithm to this setting.

In a recent paper, Kojima [40] considers a model where there are two kinds of students (minority and majority) and only a quota for majority students. He investigates the consequences of such affirmative action policies and shows that these policies may hurt minority students, the purported beneficiaries. To overcome this shortcoming, Hafalir et al. [31] propose *affirmative action with minority reserves* in which schools give higher priority to minority students up to the point that the minorities fill the reserves. They establish that minorities are on average better off with minority reserves while adverse effects on majorities are mitigated. There are only two types in Hafalir et al. [31] and their affirmative action policy can be interpreted as either a soft upper bound on majorities or a soft lower bound on minorities. These two interpretations are equivalent since they only have two types of students. In the current manuscript, there is an arbitrary number of student types, and more importantly, schools may have both ceilings and floors for each type at every school.[27] Even though our soft bounds model can be viewed as a generalization of Hafalir et al. [31], the generalization is very substantial and not straightforward because in Hafalir et al. [31] each school can be represented by two copies where one copy is "minority favoring" (minority students have higher priority than majority students) and the other one (with the number of seats reduced by the minority reserve) uses the "original" priorities. Such a representation is not possible for an arbitrary number of student types.

[25] The corresponding result for assignments that are fair across types and non-wasteful does not hold. An example showing the contrary is available from the authors.

[26] Although for schools it is not a weakly dominant strategy to truthfully reveal their preferences in the student-proposing DA, Kojima and Pathak [42] have recently shown under some regularity conditions that in the student-proposing DA the fraction of participants that can gain from misreporting approaches zero as the market becomes large.

[27] In a recent paper, Budish et al. [14] consider expected assignments satisfying both lower and upper bounds and determine when such expected assignments can be implemented by a lottery over deterministic assignments.

In a related paper, Abdulkadiroğlu [2] has the same controlled school choice environment as in this paper but proposes different feasibility and fairness concepts. In particular, due to the non-existence of feasible and fair student assignments, he relaxes feasibility by not requiring that all students are enrolled at a school and then looks for fair assignments that are not dominated by any other fair assignment.

Westkamp [54] studies a model of controlled choice in which choice rules are constructed sequentially by considering different groups of students at each step. Our soft bound choice rules cannot be constructed in this fashion because at each step we have to consider not only the number of remaining seats but also the type composition of the previously admitted students. However, Westkamp's model is neither a generalization nor a special case of the soft bounds model that we consider. In another related paper, Fragiadakis et al. [28] study a model where schools have minimum quotas on the number of students and come up with strategy-proof mechanisms that satisfy weaker fairness or non-wastefulness notions. There are no student types in Fragiadakis et al. [28], which is the main difference with the current model. More recently, Erdil and Kumano [25] introduce "substitutable priorities with ties" that capture various notions of diversity and ensure existence of stable matchings.

Kamada and Kojima [37] consider entry-level medical markets with regional caps: hospitals (or schools) are partitioned into regions and each region is controlled by a cap (or ceiling) determining the maximal number of students that can be assigned to the hospitals in that region. Similar to our context, they propose different stability notions like "strong stability" and "stability". Some of their results have a similar flavor like ours: (i) strongly stable assignments do not exist (like fairness (for same types) and non-wastefulness are incompatible under hard bounds for school choice with control) and (ii) stable assignments exist (like fairness and non-wastefulness are compatible under soft bounds) and (iii) their "flexible deferred acceptance algorithm" finds a stable assignment and is incentive compatible (like the student-proposing DA finds a fair and non-wasteful assignment under soft bounds and is incentive compatible). In addition, Kamada and Kojima [37] construct choice rules for regions by accommodating hospital preferences with a weak ordering on the vector of number of doctors that hospitals can admit. Since each doctor can only apply to at most one hospital at any step, their choice rule works as follows. First using the order, the number of doctors that each hospital is going to admit are determined. Then for each hospital the best doctors are chosen using the preference of the hospital. In our paper, students have preferences over schools for which choice rules are constructed whereas in Kamada and Kojima [37] students (or doctors) do not have preferences over regions for which the choice rules are constructed but they have preferences over the hospitals.

In a subsequent paper, Kominers and Sönmez [43] examine diversity in a two-sided many-to-one matching model. Each school has a preference associated with every available seat, and the school's choice rule is constructed by taking the union of the best available student with respect to the preference relation in a predetermined sequence. Even though this construction generalizes the affirmative action with minority reserves of Hafalir et al. [31], it is neither a generalization nor a special case of the soft bounds approach that we use.

Finally, Echenique and Yenmez [21] provide an axiomatic study of choice rules by imposing substitutability and some diversity axioms in addition to other normative axioms. They characterize choice rules including 1) choice rules in which the school has a hard upper bound for each student type and 2) choice rules in which the school has a soft lower bound for each student type (which is a special case of our soft bounds choice rules since schools may also have

Table 1
Properties of SEA and student-proposing DA.

	SEA	Student-proposing DA
Fairness	Fair for same types	Fair across types
Non-wastefulness	Constrained	Full
Strategy-proofness	Manipulable	Group strategy-proof
Diversity	Guaranteed	Contingent on student preferences

soft upper bounds). They also consider the implications of these choice rules on matching markets.

7. Conclusion

Although there is a large literature in education evaluating and estimating the effects of segregation across schools on students' achievements (Hanushek et al. [32], Guryan [30], Card and Rothstein [15], and others),[28] on how to measure segregation and determine optimal desegregation guidelines,[29] none of these papers discusses the problem of *how in practice* to assign students to schools while complying with these desegregation guidelines. This is exactly what our paper does.

To analyze controlled school choice problems, we have taken two different approaches. In the first approach controlled choice constraints define feasibility of assignments, i.e., they are hard bounds. In this case, we have shown that it may be impossible to eliminate justified envy across types. However, justified envy can be eliminated among students of the same type by the student exchange algorithm (SEA).

In the second approach, we have provided a new interpretation of controlled choice constraints as soft bounds. With this view, schools' preferences can be described through choice rules that satisfy two critical properties: substitutability and the law of aggregate demand. Therefore, the deferred acceptance algorithm can be applied to achieve attractive fairness, efficiency, and incentive properties. In addition, the assumption that students submit full rankings of all schools can be relaxed for the soft bounds model.

Soft bounds approach has clear benefits over hard bounds approach as it restores fairness, non-wastefulness and truthfulness. On the other hand, there is a potential shortcoming of the soft bounds approach: with soft bounds, the desired diversity in schools is achieved only when student preferences are also in line with them. In some peculiar cases, schools may have segregated student bodies. In this case, the school-proposing DA does better compared to student-proposing DA in terms of diversity even though the problem is not completely overcame.

This information is summarized in Table 1.

To sum up, we provide two different approaches to handle controlled choice problems that are found in many school districts. If the school districts' objectives are not very paternalistic, soft bound policies is the clear winner. Otherwise, the choice is hard bounds, even though it has less attractive properties.

[28] We refer the interested reader to Echenique et al. [19] for an illuminating account of this literature.

[29] School segregation can be purely racial or, as in Echenique et al. [19], school segregation is measured according to the spectral segregation index of Echenique and Fryer [20] which uses the intensity of social interactions among the members of a group (see also Cutler and Glaeser [17]).

Appendix A. DAAFTA and an example

In Appendix A, we describe the deferred acceptance algorithm with fixed type allotments (DAAFTA) and an example that demonstrates the workings of DAAFTA and SEA.

Deferred Acceptance Algorithm with Fixed Type Allotments (DAAFTA). First we find a feasible assignment using linear programming. Note that it is assumed that there exists at least one feasible assignment.

We want to check whether there exists a feasible assignment such that the number of type t students in school c is x_c^t. Let $|S^t| = y^t$. Given a type allocation vector $y = (y^t)_{t\in T}$, a capacity vector $q = (q_c)_{c\in C}$, and floor and ceiling matrices $\underline{q} = \{\underline{q}_c^t\}_{t\in T, c\in C}$, and $\overline{q} = \{\overline{q}_c^t\}_{t\in T, c\in C}$, a type assignment matrix $\{x_c^t\}_{t\in T, c\in C}$ is feasible if,

(i) for all $t \in T$, we have $\sum_{c\in C} x_c^t = y^t$,
(ii) for all $c \in C$, we have $\sum_{t\in T} x_c^t \leqslant q_c$, and
(iii) for all $t \in T$ and $c \in C$, we have, $\underline{q}_c^t \leqslant x_c^t \leqslant \overline{q}_c^t$.

First, the floors can be reduced to zero by defining a new variable $\hat{x}_c^t \equiv x_c^t - \underline{q}_c^t$. The rest of the constraints then can be written in terms of $\hat{x}_c^t$. The reduced set of constraints corresponds to the so-called *transportation problem*, which is well-known in the operations research literature. It is a network flow on a bipartite graph, and the linear programming relaxation (allowing x to be non-integer) provides a feasible *integer* solution in polynomial time (note that y^t, q_c, $\underline{q}_c^t$, and $\overline{q}_c^t$ are integers).[30] Let x_c^{*t} be a solution of the problem above.

For each school c and type t, we create a pseudo school $c(t)$ with a capacity of x_c^{*t}. For each student type t, we create a school choice problem without controlled-choice constraints in which only type t students and schools $c(t)$ participate. Each school $c(t)$ ranks type t students according to $\succ_c$ and each student s ranks schools according to $\succ_s$.

That is, we consider $K = |T|$ standard school choice problems, one for each $t \in T$. In school choice problem for t, there are m schools, denoted by $c(t)$ for $c \in C$, and S^t is the set of students participating in this problem. School capacities and priorities, and student preferences are defined as follows: $c(t)$ has a capacity of x_c^{*t} (which can be 0). For $s \in S^t$, preferences are defined by $c(t) P_s^t c'(t)$ if and only if $c P_s c'$. For school $c(t)$ priorities are the same as original priorities: for $s, s' \in S^t$, we have $s \succ_{c(t)} s'$ if and only if $s \succ_c s'$.

In each different school choice problem, we run the student-proposing deferred acceptance algorithm (Gale and Shapley [29]). Let us denote the assignment obtained by DA algorithm by μ^{*t}. Then we aggregate all of the outcomes these problems. That is, we define μ^* as follows: $\mu^*(c) = \bigcup_{t\in T} \mu^{*t}(c(t))$ and for $s \in S^t$, $\mu^*(s) = \mu^{*t}(s)$.

Assignment μ^* is the output of DAAFTA. Now, we can easily argue that μ^* is feasible and fair for same types. It is feasible since the total number of seats and number of students in problem t are exactly equal to each other, hence we have $|\mu^{*t}(c)| = x_c^{*t}$. It is fair for same types because outcome of the DA algorithm for type t cannot have justifiable envy.

DAAFTA, however, can be wasteful. Consider a student s of type t and her top school c. Suppose that the initial feasible assignment μ is such that $|\mu^t(c)| = 0$ and $|\mu(c)| < q_c$, then an

[30] For more information, see Nemhauser and Wolsey [45] or Reed and Leavengood [46]. We thank Ersin Korpeoglu, R. Ravi and Willem-Jan van Hoeve for discussions.

Table 2
Preferences, capacities, floors and ceilings.

	$P_{s_1} = P_{s_4} = P_{s_5} = P_{s_6}$	$P_{s_2} = P_{s_3}$	$\succ_{c_1} = \succ_{c_2} = \succ_{c_3} = \succ_{c_4}$
	c_1	c_2	s_3
	c_2	c_1	s_5
	c_3	c_3	s_1
	c_4	c_4	s_2
			s_4
			s_6
Capacities	$q_{c_1} = 2$	$q_{c_2} = 1, q_{c_3} = 2$	$q_{c_4} = 2$
Effective ceilings	$\overline{q}_{c_3}^{t_2} = 0$	$\overline{q}_{c_3}^{t_3} = 0$	
Effective floors	$\underline{q}_{c_1}^{t_3} = 1$		

empty seat in c is wasted which can be happily taken by s. Hence, DAAFTA is "rigid": for each type t; the slots, which will be filled with type-t students, are exogenously given by the feasible assignment μ. This rigidity is also a source of inefficiency. To see this consider a problem with no control constraints, with two schools: c_1 and c_2 with capacities of 1 each, and two students: s_1 of type t_1 and s_2 of type t_2. Suppose s_i prefers c_i more than c_j, and s_i has top priority for c_i ($i \in \{1, 2\}$ and $j = 3 - i$). Consider μ which matches s_i with c_j and another assignment $\hat{\mu}$ which matches s_i with c_i. It is easy to see that applying DAAFTA on μ will not change the assignment; yet μ is dominated by $\hat{\mu}$.

An Example. To demonstrate how DAAFTA followed by SEA works, we provide the following example.

Example 2 *(An illustration of SEA).* Assume that there are six students $\{s_1, s_2, s_3, s_4, s_5, s_6\}$, four schools $\{c_1, c_2, c_3, c_4\}$ and three student types $\{t_1, t_2, t_3\}$ such that $\tau(s_1) = \tau(s_3) = t_1$, $\tau(s_2) = \tau(s_5) = t_2$, and $\tau(s_4) = \tau(s_6) = t_3$. Schools c_1, c_3, c_4 have capacities of two and c_2 has a capacity of one. The only effective control constraints are $\underline{q}_{c_1}^{t_3} = 1$ and $\overline{q}_{c_3}^{t_3} = \overline{q}_{c_3}^{t_2} = 0$ (all other floors are zero and all other ceilings are equal to capacities). For all schools, student priorities are the same and given as follows; for all $c \in C$,

$$s_3 \succ_c s_5 \succ_c s_1 \succ_c s_2 \succ_c s_4 \succ_c s_6.$$

For students $s \in \{s_1, s_4, s_5, s_6\}$ the preferences are $c_1 P_s c_2 P_s c_3 P_s c_4$; whereas for students $s \in \{s_2, s_3\}$ the preferences are $c_2 P_s c_1 P_s c_3 P_s c_4$. This information is summarized in Table 2.

Consider a feasible assignment in which c_1 admits one type t_1 student and one type t_3 student; c_2 admits one type t_2 student; c_3 admits one type t_1 student; and c_4 admits one type t_2 student and one type t_3 student. For instance, consider μ as:

$$\mu = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ \{s_1, s_4\} & s_2 & s_3 & \{s_5, s_6\} \end{pmatrix}.$$

We first apply DAAFTA to μ to get a feasible assignment that is fair for same types: s_1 and s_3 exchange seats, and s_2 and s_5 exchange seats. Hence, the following assignment μ_0 is the input to SEA:

$$\mu_0 = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ \{s_3, s_4\} & s_5 & s_1 & \{s_2, s_6\} \end{pmatrix}.$$

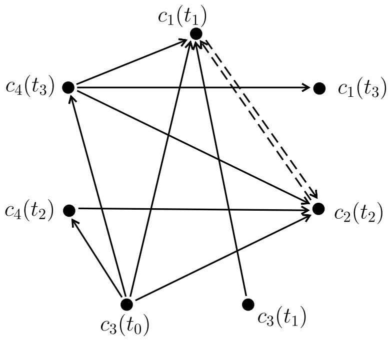

Fig. 1. Application graph $G(\mu_0)$.

In Step 1 of SEA, we first construct $G(\mu_0) = (V(\mu_0), E(\mu_0))$. School c_1 is assigned s_3 and s_4, whose types are t_1 and t_3, respectively: $c_1(t_1), c_1(t_3) \in V(\mu_0)$. School c_2 is assigned s_5, whose type is t_2: $c_2(t_2) \in V(\mu_0)$. School c_3 is assigned s_1, whose type is t_1, and has an empty seat: $c_3(t_1), c_3(t_0) \in V(\mu_0)$. Finally, school c_4 is assigned s_2 and s_6 whose types are t_2 and t_3, respectively: $c_4(t_2), c_4(t_3) \in V(\mu_0)$.

Similarly the set of edges $E(\mu_0)$ is constructed. For example, consider type t_1 and school c_2. The set of type t_1 students who would like to switch to c_2 is $\{s_1, s_3\}$. Since $s_3 \succ_{c_2} s_1$, we get $(c_1(t_1) \to c_2(t_2)) \in E(\mu_0)$ and $(c_3(t_1) \to c_2(t_2)) \notin E(\mu_0)$. We construct all of the edges in this way to get $G(\mu_0)$ depicted in Fig. 1.

The only cycle in this graph is $c_1(t_1) \to c_2(t_2) \to c_1(t_1)$. Hence, we rematch students associated with each node in the cycle, so s_3 is matched to c_2 and s_5 is matched with c_1. Note that both s_3 and s_5 prefer their new schools to old schools. The new assignment μ_1 is given by:

$$\mu_1 = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ \{s_4, s_5\} & s_3 & s_1 & \{s_2, s_6\} \end{pmatrix}.$$

In Step 2 of SEA, we construct $G(\mu_1)$. It can be checked that there are no cycles in $G(\mu_1)$, so the algorithm outputs μ_1. By Theorem 2, μ_1 is fair for same types, constrained non-wasteful, and constrained efficient.

Appendix B. Stability

We now show that in two sided matching markets stability is equivalent to fairness under soft bounds and non-wastefulness under soft bounds. For any student s, let R_s be defined as cR_sc' if and only if $c = c'$ or cP_sc' where $c, c' \in C \cup \{s\}$. Note that cR_ss by assumption (where s stands for student s being unassigned).

An assignment μ is $(\mathcal{C}_c)_{c\in C}$-**stable** for choice rule profile $(\mathcal{C}_c)_{c\in C}$ if

1. (**individual rationality**) $\mathcal{C}_c(\mu(c)) = \mu(c)$ for all $c \in C$, and $\mu(s)R_ss$ for all $s \in S$; and
2. (**no blocking pair**) there exists no $(c, s) \in C \times S$ such that $s \notin \mu(c)$, $s \in \mathcal{C}_c(\mu(c) \cup \{s\})$ and $cP_s\mu(s)$.

Individual rationality means that each school wants to keep all of its assigned students and each student prefers her assignment to being unassigned. No blocking pair rules out the existence of school-student pairs who would like to get matched with each other. We show

$(\mathcal{C}_c^{SB})_{c\in C}$-*stability* is equivalent to non-wastefulness under soft bounds and fairness under soft bounds. Note that by Lemma 1 for each school c, $\mathcal{C}_c^{SB}$ is substitutable and q_c-acceptant. In addition, $\mathcal{C}_c^{SB}$ satisfies *irrelevance of rejected students* (IRS): if $\mathcal{C}_c^{SB}(S'') \subseteq S' \subseteq S''$, then $\mathcal{C}_c^{SB}(S') = \mathcal{C}_c^{SB}(S'')$ (because q_c-acceptance implies LAD, and substitutability and LAD imply IRS (Proposition 1, Aygün and Sönmez [10])).

Lemma 2. *An assignment μ is $(\mathcal{C}_c^{SB})_{c\in C}$*-stable *if and only if μ is non-wasteful under soft bounds and fair under soft bounds.*

Proof. To show the only if-direction, suppose that μ is $(\mathcal{C}_c^{SB})_{c\in C}$-*stable*. We show that μ is non-wasteful under soft bounds and fair under soft bounds.

Non-wastefulness under soft bounds: Let $s \in S$ and $c \in C$ be such that $cP_s\mu(s)$. Since μ is $(\mathcal{C}_c^{SB})_{c\in C}$-*stable*, (c,s) is not a blocking pair. Thus, $s \notin \mathcal{C}_c^{SB}(\mu(c)\cup\{s\})$. Since $\mathcal{C}_c^{SB}$ is q_c-acceptant, it cannot be that $|\mu(c)| < q_c$. Therefore, $|\mu(c)| = q_c$ and μ is non-wasteful under soft bounds.

Fairness under soft bounds: Suppose that $s \in S$ and $c \in C$ are such that $cP_s\mu(s)$. Let $\tau(s) = t$. Since (c,s) is not a blocking pair, we have $s \notin \mathcal{C}_c^{SB}(\mu(c)\cup\{s\})$. Thus, $\mathcal{C}_c^{SB}(\mu(c)\cup\{s\}) \subseteq \mu(c) \subseteq \mu(c)\cup\{s\}$ and by IRS and non-wastefulness under soft bounds, $\mathcal{C}_c^{SB}(\mu(c)\cup\{s\}) = \mathcal{C}_c^{SB}(\mu(c)) = \mu(c)$. This implies $|\mu^t(c)| \geqslant \underline{q}_c^t$ and $s' \succ_c s$ for any $s' \in \mu^t(c)$. The rest of the analysis depends on the number of type t students in $\mu(c)$.

- First, suppose that $\overline{q}_c^t > |\mu^t(c)| \geqslant \underline{q}_c^t$. By $s \notin \mathcal{C}_c^{SB}(\mu(c)\cup\{s\})$, we have $s \notin \mathcal{C}_c^{SB(2)}(\mu(c)\cup\{s\})$ and $s \notin \mathcal{C}_c^{SB(1)}(\mu(c)\cup\{s\})$. Therefore, there cannot be any student chosen at the third step. Hence, $\overline{q}_c^{t'} \geqslant |\mu^{t'}(c)|$ for every $t' \neq t$. Therefore, for any $s' \in \mu(c)$ with $\overline{q}_c^{\tau(s')} \geqslant |\mu^{\tau(s')}(c)| > \underline{q}_c^{\tau(s')}$, we have either (i) $s' \in \mathcal{C}_c^{SB(2)}(\mu(c)\cup\{s\})$ and $s' \succ_c s$ or (ii) $s' \in \mathcal{C}_c^{SB(1)}(\mu(c)\cup\{s\})$ and for some $s'' \in \mathcal{C}_c^{SB(2)}(\mu(c)\cup\{s\})$, $s' \succ_c s'' \succ_c s$ and by transitivity of $\succ_c$, $s' \succ_c s$, the desired conclusion.
- Second, suppose that $|\mu^t(c)| \geqslant \overline{q}_c^t$. By $s \notin \mathcal{C}_c^{SB}(\mu(c)\cup\{s\})$, we have $s \notin \mathcal{C}_c^{SB(2)}(\mu(c)\cup\{s\})$ and $s \notin \mathcal{C}_c^{SB(1)}(\mu(c)\cup\{s\})$, and s is still available in the third step of the construction of $\mathcal{C}_c^{SB}(\mu(c)\cup\{s\})$. Therefore, for any $s' \in \mu(c)$ with $|\mu^{\tau(s')}(c)| > \overline{q}_c^{\tau(s')}$, we have either (i) $s' \in \mathcal{C}_c^{SB(3)}(\mu(c)\cup\{s\})$ and $s' \succ_c s$ or (ii) $s' \in \mathcal{C}_c^{SB(1)}(\mu(c)\cup\{s\}) \cup \mathcal{C}_c^{SB(2)}(\mu(c)\cup\{s\})$ and for some $s'' \in \mathcal{C}_c^{SB(3)}(\mu(c)\cup\{s\})$, $s' \succ_c s'' \succ_c s$ and by transitivity of $\succ_c$, $s' \succ_c s$, the desired conclusion.

To show the if-direction, suppose that μ is non-wasteful under soft bounds and fair under soft bounds. We show that μ is $(\mathcal{C}_c^{SB})_{c\in C}$-*stable*.

Individual rationality: Since μ is an assignment, we have $|\mu(c)| \leqslant q_c$. Because $\mathcal{C}_c^{SB}$ is q_c-acceptant, we have $\mathcal{C}_c^{SB}(\mu(c)) = \mu(c)$. In addition, for any student s, $\mu(s)R_s s$ since $\mu(s) \in C\cup\{s\}$.

No blocking pair: Suppose that for some $(c,s) \in C\times S$, we have both $s \notin \mu(c)$ and $cP_s\mu(s)$. By non-wastefulness under soft bounds, $|\mu(c)| = q_c$. By fairness under soft bounds, $|\mu^t(c)| \geqslant \underline{q}_c^t$ and $s' \succ_c s$ for all $s' \in \mu^t(c)$. Thus, $s \notin \mathcal{C}_c^{SB(1)}(\mu(c)\cup\{s\})$. The rest of the analysis depends on the number of type t students in $\mu(c)$.

Table 3
Preferences, capacities, floors and ceilings.

	$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$	P_{s_1}	P_{s_2}	P_{s_3}
	s_2	s_2	s_1	c_2	c_3	c_2
	s_1	s_1	s_2	c_3	c_2	c_3
	s_3	s_3	s_3	c_1	c_1	c_1
Capacities	$q_{c_1}=1$	$q_{c_2}=1$	$q_{c_3}=1$			
Ceiling for t_1	$\overline{q}_{c_1}^{t_1}=1$	$\overline{q}_{c_2}^{t_1}=1$	$\overline{q}_{c_3}^{t_1}=1$			
Floor for t_1	$\underline{q}_{c_1}^{t_1}=1$	$\underline{q}_{c_2}^{t_1}=0$	$\underline{q}_{c_3}^{t_1}=0$			
Ceiling for t_2	$\overline{q}_{c_1}^{t_2}=1$	$\overline{q}_{c_2}^{t_2}=1$	$\overline{q}_{c_3}^{t_2}=1$			
Floor for t_2	$\underline{q}_{c_1}^{t_2}=0$	$\underline{q}_{c_2}^{t_2}=0$	$\underline{q}_{c_3}^{t_2}=0$			

- First, suppose that $\overline{q}_c^t > |\mu^t(c)| \geqslant \underline{q}_c^t$. Since $s' \succ_c s$ for all $s' \in \mu^t(c)$ and $|\mu^t(c)| \geqslant \underline{q}_c^t$, by fairness under soft bounds we have $\overline{q}_c^{t'} \geqslant |\mu^{t'}(c)|$ for every $t' \neq t$. By fairness under soft bounds, for any $s' \in \mu(c)$ with $\overline{q}_c^{\tau(s')} \geqslant |\mu^{\tau(s')}(c)| > \underline{q}_c^{\tau(s')}$, we have $s' \succ_c s$. Because $|\mu(c)| = q_c$ and $\mathcal{C}_c^{SB}$ satisfies q_c-acceptance, we have $|\mathcal{C}_c^{SB}(\mu(c) \cup \{s\})| = q_c$. But now the previous facts imply $\mathcal{C}_c^{SB(1)}(\mu(c) \cup \{s\}) = \mathcal{C}_c^{SB(1)}(\mu(c))$ and $\mathcal{C}_c^{SB(2)}(\mu(c) \cup \{s\}) = \mathcal{C}_c^{SB(2)}(\mu(c))$. Because $|\mu(c)| = q_c$ and all students are chosen in the first two steps out of $\mu(c) \cup \{s\}$, we obtain $s \notin \mathcal{C}_c^{SB}(\mu(c) \cup \{s\})$, the desired conclusion.
- Second, suppose that $|\mu^t(c)| \geqslant \overline{q}_c^t$. Since $s' \succ_c s$ for all $s' \in \mu^t(c)$ and $|\mu^t(c)| \geqslant \overline{q}_c^t$, we have $s \notin \mathcal{C}_c^{SB(1)}(\mu(c) \cup \{s\}) \cup \mathcal{C}_c^{SB(2)}(\mu(c) \cup \{s\})$. This implies $\mathcal{C}_c^{SB(1)}(\mu(c) \cup \{s\}) = \mathcal{C}_c^{SB(1)}(\mu(c))$ and $\mathcal{C}_c^{SB(2)}(\mu(c) \cup \{s\}) = \mathcal{C}_c^{SB(2)}(\mu(c))$. By fairness under soft bounds, for any student $s' \in \mu(c)$ such that $|\mu^{\tau(s')}(c)| > \overline{q}_c^{\tau(s')}$, we have $s' \succ_c s$. Because $|\mu(c)| = q_c$ and $\mathcal{C}_c^{SB}$ satisfies q_c-acceptance, we have $|\mathcal{C}_c^{SB}(\mu(c) \cup \{s\})| = q_c$. By the fact that any student s' chosen in the third step has higher priority than student s, we now have $s \notin \mathcal{C}_c^{SB(3)}(\mu(c) \cup \{s\})$. This implies $s \notin \mathcal{C}_c^{SB}(\mu(c) \cup \{s\})$, the desired conclusion. □

Appendix C. Omitted proofs

In Appendix C, we provide the proofs omitted from the main text.

Proof of Theorem 1. The proof of both parts is by means of an example. For part (i) consider the following problem consisting of three schools $\{c_1, c_2, c_3\}$ and three students $\{s_1, s_2, s_3\}$. Each school has a capacity of one ($q_c = 1$ for all schools c). The type space consists of two types t_1 and t_2. Students s_1 and s_2 are of type t_1 whereas student s_3 is of type t_2. For all types the ceiling is equal to one at all schools ($\overline{q}_c^t = 1$ for all types t and all schools c). School c_1 has a floor for type t_1 of $\underline{q}_{c_1}^{t_1} = 1$. All other floors are equal to zero. The schools' priorities are given by $s_2 \succ_{c_1} s_1 \succ_{c_1} s_3$, $s_2 \succ_{c_2} s_1 \succ_{c_2} s_3$ and $s_1 \succ_{c_3} s_2 \succ_{c_3} s_3$. The students' preferences are given by $c_2 P_{s_1} c_3 P_{s_1} c_1$, $c_3 P_{s_2} c_2 P_{s_2} c_1$ and $c_2 P_{s_3} c_3 P_{s_3} c_1$. This information is summarized in Table 3.

Next we determine the set of assignments which are both feasible and fair across types for this problem. Feasibility requires that student s_1 or student s_2 is assigned school c_1 and all students are enrolled at a school. Therefore,

$$
\begin{tikzcd}[ampersand replacement=\&]
\mu_1 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & s_2 & s_3 \end{pmatrix} \arrow[r, "s_2 \text{ envies } s_3"] \& \mu_2 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & s_3 & s_2 \end{pmatrix}, \arrow[d, "s_1 \text{ envies } s_2"] \\
\mu_4 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & s_1 & s_3 \end{pmatrix} \arrow[u, "s_2 \text{ envies } s_1"] \& \mu_3 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & s_3 & s_1 \end{pmatrix} \arrow[l, "s_1 \text{ envies } s_3"']
\end{tikzcd}
$$

are the only assignments which are feasible. Now (as indicated above)

(i) μ_1 is not fair across types because s_2 justifiably envies s_3 at c_3,
(ii) μ_2 is not fair across types because s_1 justifiably envies s_2 at c_3,
(iii) μ_3 is not fair across types because s_1 justifiably envies s_3 at c_2, and
(iv) μ_4 is not fair across types because s_2 justifiably envies s_1 at c_2.

Hence there is no assignment which is both feasible and fair across types.

For part (ii), consider the same problem as in (i) and simply delete s_3 from the model.[31]

The following are all feasible assignments:

$$
\begin{tikzcd}[ampersand replacement=\&]
\mu_1 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & s_2 & \emptyset \end{pmatrix} \arrow[r, "s_2 \text{ claims } c_3"] \& \mu_2 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & \emptyset & s_2 \end{pmatrix}. \arrow[d, "s_1 \text{ envies } s_2"] \\
\mu_4 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & s_1 & \emptyset \end{pmatrix} \arrow[u, "s_2 \text{ envies } s_1"] \& \mu_3 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & \emptyset & s_1 \end{pmatrix} \arrow[l, "s_1 \text{ claims } c_2"']
\end{tikzcd}
$$

Now (as indicated above):

(i) μ_1 is wasteful because s_2 claims the empty seat at c_3,
(ii) μ_2 is not fair for same types because s_1 justifiably envies s_2 at c_3,
(iii) μ_3 is wasteful because s_1 claims the empty seat at c_2, and
(iv) μ_4 is not fair for same types because s_2 justifiably envies s_1 at c_2.

Hence there is no assignment which is both feasible and fair across types. □

Proof of Theorem 2. Let P_S be a controlled school choice problem. We prove the claim by using the following lemmas.

Lemma 3. *The assignment produced by the student exchange algorithm is feasible.*

Proof. For $t, t' \in T$, each node $c(t)$ only points to a node $c'(t')$ when a type t' student can be fired from school c' and a type t student can be admitted to c' without violating the feasibility conditions in school c'. Thus, when we execute a cycle consisting of such nodes we get a feasible assignment. On the other hand, suppose that we execute a cycle containing $c(t_0)$. Let the cycle include the following path $c'(t) \to c(t_0) \to c''(t')$. Since $c'(t)$ is pointing $c(t_0)$, then a type t student can take an empty seat in c without violating feasibility constraints. Similarly, since $c(t_0)$

[31] We thank an anonymous referee for suggesting this simpler example.

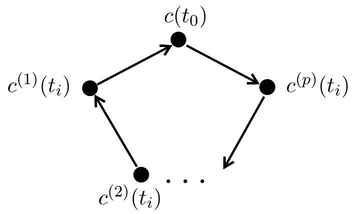

Fig. 2. Application graph $G(\mu)$ for Case 1.

is pointing $c''(t')$ a type t' student can be fired from school c'' without violating the feasibility constraints. Therefore, the assignment produced is feasible. □

Lemma 4. *The assignment produced by the student exchange algorithm is fair for same types.*

Proof. Let μ be assignment that is the input of the algorithm and μ' be the assignment produced by it. Suppose for contradiction that μ' is not fair for same types. Therefore, there exist students s and s' of the same type such that s' justifiably envies s at school c under μ': $\tau(s) = \tau(s')$, $c \equiv \mu'(s) P_{s'} \mu'(s')$, and $s' \succ_c s$. There are two cases depending on whether $\mu(s) = \mu'(s)$ or $\mu(s) \neq \mu'(s)$.

- $\mu(s) = \mu'(s)$: For student s', let $R_{s'}$ be the weak order associated with $P_{s'}$. Since μ is fair for same types and $s' \succ_c s$, we have $\mu(s') R_{s'} c$. Since the algorithm improves the match of every student or keeps it the same, we get $\mu'(s') R_{s'} \mu(s')$. This and $\mu(s') R_{s'} c$ imply $\mu'(s') R_{s'} c$.
- $\mu(s) \neq \mu'(s)$: In this case, s must have matched with c in one of the steps of the algorithm. To have a node for type $\tau(s)$ in $\mu(s)$ point to any node for school c, s must have the highest priority among type $\tau(s)$ students who prefer c to their current assignments. Since s' wants to switch to c at any step of the algorithm, we get $s \succ_c s'$.

In both cases, we get a contradiction. The conclusion follows. □

Lemma 5. *Suppose that μ is a feasible assignment that is fair for same types, which is also constrained efficient. Then μ is constrained non-wasteful.*

Proof. Suppose, otherwise, that μ violates constrained non-wastefulness. Then there exists a student s and school c with an empty seat such that the assignment in which school c admits student s without changing the matches of any other student is fair for same types. This gives a contradiction to constrained efficiency. □

Lemma 6. *Let μ be an assignment that is fair for same types, which is not constrained efficient. Then there exists a cycle in $G(\mu)$, and hence the assignment produced by the student exchange algorithm is different than μ.*

Proof. Let μ' be an assignment that is fair for same types, which Pareto dominates μ. We are going to show that there exists a cycle in $G(\mu)$. To do this, we split the analysis whether there exists a school c such that $|\mu(c)| \neq |\mu'(c)|$ or not.

Case 1: (There exists c such that $|\mu(c)| \neq |\mu'(c)|$.) (See Fig. 2.) Since the total number of assigned students is the same in both μ and μ', there exists c such that $|\mu'(c)| > |\mu(c)|$. Hence,

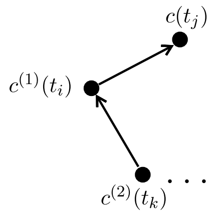

Fig. 3. Application graph $G(\mu)$ for Case 2.

there exists $t_i \in T$ such that $|\mu'^{t_i}(c)| > |\mu^{t_i}(c)|$. Thus, in $G(\mu)$ there exists a school $c^{(1)} \neq c$ such that $c^{(1)}(t_i)$ is pointing to $c(t_0)$. Choose $c^{(1)}$ such that some $s \in \mu^{t_i}(c^{(1)})$ has highest $\succ_c$-priority among $\{\hat{s} \in S^{t_i}\colon c P_{\hat{s}} \mu(\hat{s})\}$ and $c P_s c^{(1)}$. If the floor of type t_i in $c^{(1)}$ is not binding in μ, then $c(t_0)$ is also pointing to $c^{(1)}(t_i)$. Therefore, there exists a cycle and we are done. Suppose otherwise that the floor of type t_i in $c^{(1)}$ is binding at μ. By $|\mu'^{t_i}(c)| > |\mu^{t_i}(c)|$ and our choice of s, s cannot be assigned to $c^{(1)}$ in μ' since μ' is fair for same types and μ' Pareto dominates μ. This implies that there exists a student of type t_i who is in $\mu'(c^{(1)})$ but not in $\mu(c^{(1)})$ since μ' is feasible and $|\mu^{t_i}(c^{(1)})| = \underline{q}^{t_i}_{c^{(1)}}$. Because μ' Pareto dominates μ, now there exists a school $c^{(2)} \neq c^{(1)}$ such that $c^{(2)}(t_i)$ is pointing to $c^{(1)}(t_i)$ in $G(\mu)$. Again choose $c^{(2)}$ such that some $s \in \mu^{t_i}(c^{(2)})$ has highest $\succ_{c^{(1)}}$-priority among $\{\hat{s} \in S^{t_i}\colon c^{(1)} P_{\hat{s}} \mu(\hat{s})\}$ and $c^{(1)} P_s c^{(2)}$. By a similar argument, we see that either $c(t_0)$ is pointing to $c^{(2)}(t_i)$ or that there exists a school $c^{(3)}$ such that $c^{(3)}(t_i)$ is pointing to $c^{(2)}(t_i)$. Since there is a finite number of schools, by mathematical induction, we see that there exists a positive number p such that $c(t_0)$ is pointing to $c^{(p)}(t_i)$ and for every $l = 1, \ldots, p$, $c^{(l)}(t_i)$ is pointing to $c^{(l-1)}(t_i)$. Hence, there exists a cycle consisting of type t_i nodes and a node for an empty seat in $G(\mu)$.

Case 2: (For all c, $|\mu(c)| = |\mu'(c)|$.) (See Fig. 3.) In this case, since $\mu \neq \mu'$ there exist $t_i \in T$, $s \in S^{t_i}$ and $c \in C$ such that $s \in \mu'(c) \setminus \mu(c)$ and $c P_s \mu(s)$. If $|\mu^{t_i}(c)| \leqslant |\mu'^{t_i}(c)|$, then (since μ' Pareto dominates μ) in $G(\mu)$, $c(t_i)$ is being pointed by $c^{(1)}(t_i)$ for some $c^{(1)} \neq c$. Otherwise, if $|\mu^{t_i}(c)| > |\mu'^{t_i}(c)|$, there exists $t_j \in T \backslash \{t_i\}$ such that $|\mu^{t_j}(c)| < |\mu'^{t_j}(c)|$. Since μ' Pareto dominates μ and both $\underline{q}^{t_i}_c < |\mu^{t_i}(c)|$ and $|\mu^{t_j}(c)| < \overline{q}^{t_j}_c$, in $G(\mu)$, $c(t_i)$ is being pointed by $c^{(1)}(t_j)$ for some $c^{(1)} \in C$. In either case choose $c^{(1)}$ such that (i) if $|\mu^{t_i}(c)| \leqslant |\mu'^{t_i}(c)|$, then some $s' \in \mu^{t_i}(c^{(1)})$ has highest $\succ_c$-priority among $\{\hat{s} \in S^{t_i}\colon c P_{\hat{s}} \mu(\hat{s})\}$ and $c P_{s'} c^{(1)}$ and (ii) if $|\mu^{t_i}(c)| > |\mu'^{t_i}(c)|$, then some $s' \in \mu^{t_j}(c^{(1)})$ has highest $\succ_c$-priority among $\{\hat{s} \in S^{t_j}\colon c P_{\hat{s}} \mu(\hat{s})\}$ and $c P_{s'} c^{(1)}$. By construction and the fact that both μ' Pareto dominates μ and μ' is fair for same types, s' cannot be assigned to $c^{(1)}$ in μ' and $s' \in \mu(c^{(1)}) \backslash \mu'(c^{(1)})$. Now in the same way we find $c^{(2)}$ and s'' (where $s'' \in \mu'^{t_j}(c^{(2)})$ or $s'' \in \mu'^{t_k}(c^{(2)})$ for $t_k \in T \backslash \{t_j\}$) and in $G(\mu)$, either $c^{(2)}(t_j)$ points to $c^{(1)}(t_j)$ or $c^{(2)}(t_k)$ points to $c^{(1)}(t_j)$. We continue in this way constructing a path in $G(\mu)$. Since there exists a finite number of nodes in $G(\mu)$, we conclude that this path must cycle at some point. This completes the argument. $\square$

Now we establish Theorem 2 using these lemmas. Let μ be the assignment produced by the student exchange algorithm. It is fair for same types (Lemma 4). Since the algorithm produces μ, there are no cycles in $G(\mu)$, so μ is also constrained efficient (Lemma 6). Therefore, it is also constrained non-wasteful (Lemma 5). $\square$

Proof of Theorem 3. The proof is by means of an example. Consider the following problem consisting of three schools $\{c_1, c_2, c_3\}$ and two students $\{s_1, s_2\}$. Each school has a capacity of

Table 4
Preferences, capacities, floors and ceilings.

	$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$	P_{s_1}	P_{s_2}
	s_2	s_2	s_1	c_2	c_3
	s_1	s_1	s_2	c_1	c_1
				c_3	c_2
Capacities	$q_{c_1}=2$	$q_{c_2}=2$	$q_{c_3}=2$		
Ceiling for t	$\overline{q}^t_{c_1}=2$	$\overline{q}^t_{c_2}=2$	$\overline{q}^t_{c_3}=2$		
Floor for t	$\underline{q}^t_{c_1}=1$	$\underline{q}^t_{c_2}=0$	$\underline{q}^t_{c_3}=0$		

two ($q_c=2$ for all schools c). The type space consists of a single type t, i.e., both students are of the same type t. The ceiling for type t is equal to two for each school ($\overline{q}^t_c=2$ for all schools c). School c_1 has a floor for type t of $\underline{q}^t_{c_1}=1$ and both other schools have a floor of 0 for type t. Schools c_1 and c_2 give higher priority to student s_2 whereas school c_3 gives higher priority student s_1. The students' preferences are given by $c_2 P_{s_1} c_1 P_{s_1} c_3$ and $c_3 P_{s_2} c_1 P_{s_2} c_2$. This information is summarized in Table 4.

Next we determine the set of feasible assignments. Feasibility requires that one of the students is assigned to c_1 and each student is assigned to a school. Then it is straightforward to verify that

$$\mu_1=\begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & \emptyset & s_2 \end{pmatrix},\qquad \mu_2=\begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & s_2 & \emptyset \end{pmatrix},$$
$$\mu_3=\begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & \emptyset & s_1 \end{pmatrix},\qquad \mu_4=\begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & s_1 & \emptyset \end{pmatrix},\qquad \mu_5=\begin{pmatrix} c_1 & c_2 & c_3 \\ \{s_1,s_2\} & \emptyset & \emptyset \end{pmatrix}$$

represent all feasible assignments.

It is easy to check that μ_1 and μ_4 are the only feasible assignments which are both fair for same types and constrained non-wasteful for this controlled school choice problem. Note that under P_S,

(i) μ_2 and μ_5 are not constrained non-wasteful since s_2 justifiably claims an empty slot at c_3 under both μ_2 and μ_5 and the resulting assignment μ_1 is fair for same types, and
(ii) μ_3 is not constrained non-wasteful since s_1 justifiably claims an empty slot at c_2 under μ_3 and the resulting assignment μ_4 is fair for same types.

Any feasible mechanism which is both fair for same types and constrained non-wasteful must select either the assignment μ_1 or the assignment μ_4. We will show that in each case there is a student who profitably manipulates the mechanism.

Case 1: *The mechanism selects* μ_1. Under μ_1 student s_1 is assigned to c_1. We will show that student s_1 gains by misreporting his true preference. Suppose that student s_1 states the (false) preference P'_{s_1} given by $c_2 P'_{s_1} c_3 P'_{s_1} c_1 P'_{s_1} s_1$, and student s_2 were to report his true preference P_{s_2}. Keeping all other components of the above problem fixed, in the new problem the students' preferences are $P'_S=(P'_{s_1},P_{s_2})$.

In the new problem under μ_1 student s_1 justifiably envies student s_2 at school c_3 since (f1) $\mu_1(s_1)=c_1$, $c_3 P'_{s_1} c_1$ and $s_1 \succ_{c_3} s_2$, and (f2) $\tau(s_1)=\tau(s_2)$. Note that under P'_S,

(i) μ_1 and μ_2 are not fair for same types, and
(ii) μ_3 and μ_5 are not constrained non-wasteful since s_1 justifiably claims an empty slot at c_2 under both μ_3 and μ_5 and the resulting assignment μ_4 is fair for same types.

Thus, the unique feasible assignment, which is both fair for same types and constrained non-wasteful for the new problem, is μ_4. Hence, any feasible mechanism, which is both fair for same types and constrained non-wasteful, must select the assignment μ_4 for the new problem. Under μ_4 student s_1 is assigned school c_2 which is strictly preferred to c_1 under the true preference P_{s_1}. Thus student s_1 does better by stating P'_{s_1} than by stating his true preference P_{s_1}, and the mechanism is not strategy-proof.

Case 2: *The mechanism selects* μ_4. Under μ_4 student s_2 is assigned school c_1. Similarly as in Case 1 we will show that student s_2 gains by misreporting his preference. Suppose that student s_2 states the (false) preference P'_{s_2} given by $c_3 P'_{s_2} c_2 P'_{s_2} c_1 P'_{s_2} s_2$, and student s_1 were to report his true preference P_{s_1}. Keeping all other components of the above problem fixed, in the new problem the students' preferences are $P'_S = (P_{s_1}, P'_{s_2})$.

In the new problem under μ_4 student s_2 justifiably envies student s_1 at school c_2 since (f1) $\mu_4(s_2) = c_1$, $c_2 P'_{s_2} c_1$ and $s_2 \succ_{c_2} s_1$, and (f2) $\tau(s_2) = \tau(s_1)$. Note that under P'_S,

(i) μ_4 is not fair for same types,
(ii) μ_2 and μ_5 are not constrained non-wasteful since s_2 justifiably claims an empty slot at c_3 under both μ_2 and μ_5 and the resulting assignment μ_1 is fair for same types, and
(iii) μ_3 is not constrained non-wasteful since s_1 justifiably claims an empty slot at c_1 under μ_3 and the resulting assignment μ_5 is fair for same types.

The unique feasible assignment, which is both fair for same types and constrained non-wasteful for the new problem, is μ_1. Hence, any feasible mechanism, which is both fair for same types and constrained non-wasteful, must select the assignment μ_1 for the new problem. Under μ_1 student s_2 is assigned school c_3 which is strictly preferred to c_1 under the true preference P_{s_2}. Thus student s_2 does better by stating P'_{s_2} than by stating his true preference P_{s_2}, and the mechanism is not strategy-proof. $\square$

Proof of Lemma 1.[32] We show that $\mathcal{C}_c^{SB}$ satisfies substitutability and q_c-acceptance. Note that for fixed quotas $\tilde{q}_c$ and $(\tilde{q}_c^t)_{t\in T}$, $H_c(\cdot, \tilde{q}_c, (\tilde{q}_c^t)_{t\in T})$ satisfies substitutability and the following monotonicity property: $H_c(S', \tilde{q}_c, (\tilde{q}_c^t)_{t\in T}) \subseteq H_c(S', \tilde{q}_c + 1, (\tilde{q}_c^t)_{t\in T})$ for all $S' \subseteq S$.

Substitutability: It is easy to see that $\mathcal{C}_c^{SB(1)}$ satisfies both substitutability and the law of aggregate demand (LAD). First, we show that $\mathcal{C}_c^{SB(1)} \cup \mathcal{C}_c^{SB(2)}$ also satisfies LAD.

Suppose that $S'' \subseteq S'$. Note that substitutability of $\mathcal{C}_c^{SB(1)}$ implies $S'' \setminus \mathcal{C}_c^{SB(1)}(S'') \subseteq S' \setminus \mathcal{C}_c^{SB(1)}(S')$, so a larger set is considered in the construction of $\mathcal{C}_c^{SB(2)}(S')$ compared to $\mathcal{C}_c^{SB(2)}(S'')$. If $|\mathcal{C}_c^{SB(2)}(S'')| \leqslant |\mathcal{C}_c^{SB(2)}(S')|$, then we get $|\mathcal{C}_c^{SB(1)}(S'') \cup \mathcal{C}_c^{SB(2)}(S'')| \leqslant |\mathcal{C}_c^{SB(1)}(S') \cup \mathcal{C}_c^{SB(2)}(S')|$ since $\mathcal{C}_c^{SB(1)}$ also satisfies LAD. Otherwise, if $|\mathcal{C}_c^{SB(2)}(S'')| > |\mathcal{C}_c^{SB(2)}(S')|$, then $|\mathcal{C}_c^{SB(2)}(S')| = q_c - |\mathcal{C}_c^{SB(1)}(S')|$ because from $S'' \setminus \mathcal{C}_c^{SB(1)}(S'') \subseteq S' \setminus \mathcal{C}_c^{SB(1)}(S')$ the second argument in the construction of $\mathcal{C}_c^{SB(2)}(S')$ must be binding. Therefore, $|\mathcal{C}_c^{SB(2)}(S'')| - |\mathcal{C}_c^{SB(2)}(S')| \leqslant (q_c - |\mathcal{C}_c^{SB(1)}(S'')|) - (q_c - |\mathcal{C}_c^{SB(1)}(S')|) = |\mathcal{C}_c^{SB(1)}(S')| - |\mathcal{C}_c^{SB(1)}(S'')|$ which is equivalent to $|\mathcal{C}_c^{SB(1)}(S'') \cup \mathcal{C}_c^{SB(2)}(S'')| \leqslant |\mathcal{C}_c^{SB(1)}(S') \cup \mathcal{C}_c^{SB(2)}(S')|$. Hence, $\mathcal{C}_c^{SB(1)} \cup \mathcal{C}_c^{SB(2)}$ satisfies LAD.

Now, suppose that $\mathcal{C}_c^k \equiv \mathcal{C}_c^{SB(1)} \cup \mathcal{C}_c^{SB(k)}$ satisfies substitutability and LAD for some $k \in \{1, 2\}$. We show that $\overline{\mathcal{C}}_c^k \equiv \mathcal{C}_c^{SB(1)} \cup \mathcal{C}_c^{SB(k)} \cup \mathcal{C}_c^{SB(k+1)}$ satisfies substitutability as well. Let $s, s' \in S'$ be such that $s \neq s'$ and $s \in \overline{\mathcal{C}}_c^k(S')$. If $s \in \mathcal{C}_c^k(S')$ then by substitutability of $\mathcal{C}_c^k$ we get $s \in \mathcal{C}_c^k(S' \setminus \{s'\})$,

[32] We would like to thank an anonymous referee for suggesting the following proof.

which implies $s \in \overline{\mathcal{C}}_c^k(S' \setminus \{s'\})$. Otherwise, if $s \notin \mathcal{C}_c^k(S')$ then $s \in \mathcal{C}_c^{SB(k+1)}(S')$. We now consider two cases. By LAD, either $|\mathcal{C}_c^k(S' \setminus \{s'\})| < |\mathcal{C}_c^k(S')|$ or $|\mathcal{C}_c^k(S' \setminus \{s'\})| = |\mathcal{C}_c^k(S')|$.

Case 1: $|\mathcal{C}_c^k(S' \setminus \{s'\})| < |\mathcal{C}_c^k(S')|$. By substitutability of $\mathcal{C}_c^k$, this implies $\mathcal{C}_c^k(S' \setminus \{s'\}) = \mathcal{C}_c^k(S') \setminus \{s'\}$. Since $\mathcal{C}_c^k$ also satisfies IRS, we have $(S' \setminus \{s'\}) \setminus \mathcal{C}_c^k(S' \setminus \{s'\}) = S' \setminus \mathcal{C}_c^k(S')$. Therefore, the same set of students is considered in $\mathcal{C}_c^{SB(k+1)}(S')$ and $\mathcal{C}_c^{SB(k+1)}(S' \setminus \{s'\})$. The monotonicity property of H_c implies that $s \in \mathcal{C}_c^{SB(k+1)}(S' \setminus \{s'\})$ since $s \in \mathcal{C}_c^{SB(k+1)}(S')$. This shows that $s \in \overline{\mathcal{C}}_c^k(S' \setminus \{s'\})$.

Case 2: $|\mathcal{C}_c^k(S' \setminus \{s'\})| = |\mathcal{C}_c^k(S')|$. If $s \in \mathcal{C}_c^k(S' \setminus \{s'\})$, then we are done. Otherwise, if $s \notin \mathcal{C}_c^k(S' \setminus \{s'\})$, then $s \in (S' \setminus \{s'\}) \setminus \mathcal{C}_c^k(S' \setminus \{s'\})$. If $s' \notin \mathcal{C}_c^k(S')$, then by IRS, $\mathcal{C}_c^k(S' \setminus \{s'\}) = \mathcal{C}_c^k(S')$. But then $(S' \setminus \mathcal{C}_c^k(S')) \setminus \{s'\} = (S' \setminus \{s'\}) \setminus \mathcal{C}_c^k(S' \setminus \{s'\})$. Hence, substitutability of H_c for fixed quotas and $s \in \mathcal{C}_c^{SB(k+1)}(S')$ yield $s \in \mathcal{C}_c^{SB(k+1)}(S' \setminus \{s'\})$. Otherwise $s' \in \mathcal{C}_c^k(S')$ and by substitutability of $\mathcal{C}_c^k$ there exists $s'' \in S' \setminus \mathcal{C}_c^k(S')$ such that $\mathcal{C}_c^k(S' \setminus \{s'\}) = (\mathcal{C}_c^k(S') \setminus \{s'\}) \cup \{s''\}$. But then $(S' \setminus \mathcal{C}_c^k(S')) \setminus \{s''\} = (S' \setminus \{s'\}) \setminus \mathcal{C}_c^k(S' \setminus \{s'\})$. Hence, substitutability of H_c for fixed quotas and $s \in \mathcal{C}_c^{SB(k+1)}(S')$ yield $s \in \mathcal{C}_c^{SB(k+1)}(S' \setminus \{s'\})$. Therefore in all cases, $s \in \overline{\mathcal{C}}_c^k(S' \setminus \{s'\})$.

***q_c*-acceptance:** We need to show $|\mathcal{C}_c^{SB}(S')| = \min\{q_c, |S'|\}$. By construction of $\mathcal{C}_c^{SB}$, $s \in S' \setminus \mathcal{C}_c^{SB}(S')$ implies $|\mathcal{C}_c^{SB}(S')| = q_c$ because otherwise $s \in \mathcal{C}_c^{SB(3)}(S')$. Therefore, a student is rejected only when the school fills its capacity, and $\mathcal{C}_c^{SB}$ satisfies q_c-acceptance. □

Proof of Proposition 1. Suppose, by contradiction, that $\mathcal{C}$ is q_c-acceptant and closer to controlled choice constraints than $\mathcal{C}_c^{SB}$. Hence, for some $\emptyset \neq \tilde{S} \subseteq S$ and some type $t^* \in T$, we have $\max\{\underline{d}_c^{t^*}(\mathcal{C}^{SB}(\tilde{S})), \overline{d}_c^{t^*}(\mathcal{C}^{SB}(\tilde{S}))\} > \max\{\underline{d}_c^{t^*}(\mathcal{C}(\tilde{S})), \overline{d}_c^{t^*}(\mathcal{C}(\tilde{S}))\}$. Then $\max\{\underline{d}_c^{t^*}(\mathcal{C}^{SB}(\tilde{S})), \overline{d}_c^{t^*}(\mathcal{C}^{SB}(\tilde{S}))\} > 0$ and either $\underline{d}_c^{t^*}(\mathcal{C}^{SB}(\tilde{S})) > 0$ or $\overline{d}_c^{t^*}(\mathcal{C}^{SB}(\tilde{S})) > 0$.

(1) If $\underline{d}^{t^*}(\mathcal{C}_c^{SB}(\tilde{S})) > 0$, then we must have $\mathcal{C}_c^{SB}(\tilde{S}) \cap S^{t^*} = \tilde{S} \cap S^{t^*}$. This is because $\sum_{t \in T} \underline{q}_c^t \leqslant q_c$, and $\mathcal{C}_c^{SB}$ first fills the floors. But then, $\underline{d}^{t^*}(\mathcal{C}(\tilde{S})) \geqslant \max\{\underline{q}_c^{t^*} - |\tilde{S} \cap S^{t^*}|, 0\} = \underline{d}^{t^*}(\mathcal{C}_c^{SB}(\tilde{S}))$, and therefore, $\max\{\underline{d}_c^{t^*}(\mathcal{C}^{SB}(\tilde{S})), \overline{d}_c^{t^*}(\mathcal{C}^{SB}(\tilde{S}))\} \leqslant \max\{\underline{d}_c^{t^*}(\mathcal{C}(\tilde{S})), \overline{d}_c^{t^*}(\mathcal{C}(\tilde{S}))\}$, a contradiction.

(2) If $\overline{d}_c^{t^*}(\mathcal{C}^{SB}(\tilde{S})) > 0$, then both $|\mathcal{C}_c^{SB}(\tilde{S}) \cap S^{t^*}| > \overline{q}_c^{t^*}$ and $|\mathcal{C}_c^{SB}(\tilde{S}) \cap S^{t^*}| > |\mathcal{C}(\tilde{S}) \cap S^{t^*}|$. This implies that for each $t \neq t^*$, either $\mathcal{C}_c^{SB}(\tilde{S}) \cap S^t = \tilde{S} \cap S^t$ or $|\mathcal{C}_c^{SB}(\tilde{S}) \cap S^t| \geqslant \overline{q}_c^t$. By construction of $\mathcal{C}_c^{SB}$: for each type t with $|\mathcal{C}_c^{SB}(\tilde{S}) \cap S^t| > \overline{q}_c^t$, the top $\overline{q}_c^t$ number of students in $\tilde{S}^t$ according to $\succ_c$ are chosen before the $(\overline{q}_c^{t^*} + 1)$st top student of type t^*.

Now, we argue that $|\mathcal{C}_c^{SB}(\tilde{S}) \cap S^t| \geqslant |\mathcal{C}(\tilde{S}) \cap S^t|$ for all $t \neq t^*$. If $\mathcal{C}_c^{SB}(\tilde{S}) \cap S^t = \tilde{S} \cap S^t$, then obviously $|\mathcal{C}_c^{SB}(\tilde{S}) \cap S^t| \geqslant |\mathcal{C}(\tilde{S}) \cap S^t|$. If $|\mathcal{C}_c^{SB}(\tilde{S}) \cap S^t| \geqslant \overline{q}_c^t$, then either (i) $|\mathcal{C}_c^{SB}(\tilde{S}) \cap S^t| > \overline{q}_c^t$ and we have $|\mathcal{C}_c^{SB}(\tilde{S}) \cap S^t| \geqslant |\mathcal{C}(\tilde{S}) \cap S^t|$; or (ii) if $|\mathcal{C}_c^{SB}(\tilde{S}) \cap S^t| = \overline{q}_c^t$, then we have $|\mathcal{C}_c^{SB}(\tilde{S}) \cap S^t| = \overline{q}_c^t \geqslant |\mathcal{C}(\tilde{S}) \cap S^t| \geqslant \underline{q}_c^t$.

Hence, $|\mathcal{C}_c^{SB}(\tilde{S}) \cap S^{t^*}| > |\mathcal{C}(\tilde{S}) \cap S^{t^*}|$ and $|\mathcal{C}_c^{SB}(\tilde{S}) \cap S^t| \geqslant |\mathcal{C}(\tilde{S}) \cap S^t|$ for all $t \neq t^*$. Summing up the inequalities yields

$$\left|\mathcal{C}_c^{SB}(\tilde{S})\right| = \sum_{t \in T} \left|\mathcal{C}_c^{SB}(\tilde{S}) \cap S^t\right| > \sum_{t \in T} \left|\mathcal{C}(\tilde{S}) \cap S^t\right| = \left|\mathcal{C}(\tilde{S})\right| \geqslant \min\left\{q_c, |\tilde{S}|\right\}$$

where the last inequality follows from the fact that $\mathcal{C}$ is q_c-acceptant. But now $|\mathcal{C}_c^{SB}(\tilde{S})| > \min\{q_c, |\tilde{S}|\}$ which contradicts the fact that $\mathcal{C}_c^{SB}$ is q_c-acceptant (Lemma 1). □

Proof of Theorem 7. First, we prove the following lemma.

Lemma 7. *Let $S'' \subseteq S' \subseteq S$ be such that $|S''| = q_c$ for some school c. Then, $\underline{d}_c(\mathcal{C}_c^{SB}(S')) \leqslant \underline{d}_c(\mathcal{C}_c^{SB}(S''))$ and $\overline{d}_c(\mathcal{C}_c^{SB}(S')) \leqslant \overline{d}_c(\mathcal{C}_c^{SB}(S''))$.*

Proof. Since $S'' \subseteq S'$, $\underline{d}_c^t(\mathcal{C}_c^{SB}(S')) \leqslant \underline{d}_c^t(\mathcal{C}_c^{SB}(S''))$ for all $t \in T$. Therefore, $\underline{d}_c(\mathcal{C}_c^{SB}(S')) \leqslant \underline{d}_c(\mathcal{C}_c^{SB}(S''))$. Again since $S'' \subseteq S'$,

$$\left|\mathcal{C}_c^{SB(1)}(S') \cup \mathcal{C}_c^{SB(2)}(S')\right| \geqslant \left|\mathcal{C}_c^{SB(1)}(S'') \cup \mathcal{C}_c^{SB(2)}(S'')\right|.$$

This implies

$$q_c - \left|\mathcal{C}_c^{SB(1)}(S') \cup \mathcal{C}_c^{SB(2)}(S')\right| \leqslant q_c - \left|\mathcal{C}_c^{SB(1)}(S'') \cup \mathcal{C}_c^{SB(2)}(S'')\right|.$$

But since $\mathcal{C}_c^{SB}$ is q_c-acceptant and $q_c = |S''| \leqslant |S'|$, we obtain

$$\left|\mathcal{C}_c^{SB(3)}(S')\right| \leqslant \left|\mathcal{C}_c^{SB(3)}(S'')\right|.$$

By definition, $|\mathcal{C}_c^{SB(3)}(S')| = \overline{d}_c(\mathcal{C}_c^{SB}(S'))$ and $|\mathcal{C}_c^{SB(3)}(S'')| = \overline{d}_c(\mathcal{C}_c^{SB}(S''))$. Therefore, $\overline{d}_c(\mathcal{C}_c^{SB}(S')) \leqslant \overline{d}_c(\mathcal{C}_c^{SB}(S''))$, the desired conclusion. □

Second, we prove Theorem 7.

Let $c \in C$. If $\mu(c) = \mu^*(c)$, then there is nothing to prove. Suppose otherwise that $\mu(c) \neq \mu^*(c)$. By the rural hospitals theorem, $|\mu(c)| = |\mu^*(c)|$ (see Theorem 8, Hatfield and Milgrom [35]). Then there exists at least one student in $(\mu^*(c) \cup \mu(c)) \setminus \mathcal{C}_c^{SB}(\mu^*(c) \cup \mu(c))$ as $\mathcal{C}_c^{SB}(\mu^*(c) \cup \mu(c)) = \mu^*(c)$ by Theorem 6. Since $\mathcal{C}^{SB}$ is q_c-acceptant, we obtain $|\mathcal{C}_c^{SB}(\mu^*(c) \cup \mu(c))| = |\mu^*(c)| = q_c$. Hence, $|\mu(c)| = q_c$. Applying Lemma 7 to $S'' \equiv \mu(c)$ and $S' \equiv \mu^*(c) \cup \mu(c)$ yields $\underline{d}_c(\mu^*) \leqslant \underline{d}_c(\mu)$ and $\overline{d}_c(\mu^*) \leqslant \overline{d}_c(\mu)$, as desired. □

Proof of Theorem 8. Let μ be a feasible assignment that is strongly-fair across types and non-wasteful. Since μ is a feasible assignment, for every school c and student type t we have $\underline{q}_c^t \leqslant |\mu^t(c)| \leqslant \overline{q}_c^t$. Together with strong-fairness across types, this implies that μ is fair under soft bounds. If μ is also non-wasteful under soft bounds, the conclusion follows from Theorem 4. Suppose otherwise that μ violates non-wastefulness under soft bounds. This means that there exist a school c and a student s such that $c P_s \mu(s)$ and $|\mu(c)| < q_c$. Whenever there exists such a pair we apply the following algorithm to improve students' matches. Note that this algorithm is equivalent to the school-proposing deferred acceptance algorithm if μ is the assignment in which no agent is matched.[33]

Step 1 For school c defined above, find $S^1 \equiv \{s \in S\colon\ c P_s \mu(s)\}$. Among the students in S^1 first match the highest ranked students according to $\succ_c$ until the ceilings are filled or S^1 is exhausted. Then match the best students according to $\succ_c$ up to the capacity or until S^1 is exhausted. Define μ_1 to be the new assignment.

Step k If there is no school with an empty seat that a student prefers to her match in μ_{k-1}, then stop. Otherwise consider one such school, say c_k. Let $S^k \equiv \{s \in S\colon\ c_k P_s \mu_{k-1}(s)\}$. Among the students in S^k first match the highest ranked students according to $\succ_{c_k}$ until the floors are filled or S^k is exhausted. Then match the highest ranked students according to $\succ_{c_k}$ until the ceilings are filled. Finally, match the best students according to $\succ_{c_k}$ if there are more students and seats available. Define μ_k to be the new assignment.

33 This is similar to the vacancy-chain dynamics studied in Blum et al. [12].

This algorithm ends in finite time since it improves the match of at least one student at every step of the algorithm. Let $\hat{\mu}$ denote the assignment produced by this algorithm. It is clear that $\hat{\mu}$ is non-wasteful under soft bounds. We further claim that $\hat{\mu}$ removes justifiable envy under soft bounds.

Consider a student s and school c such that $c P_s \hat{\mu}(s)$. Let $\tau(s) = t$. For any student $s' \in \hat{\mu}^t(c)$, either s' was already matched with c in strongly-fair across types assignment μ which implies $s' \succ_c s$, or s' got matched with c in the above algorithm which also implies $s' \succ_c s$. Furthermore, because $\underline{q}_c^t \leqslant |\mu^t(c)| \leqslant \overline{q}_c^t$, whenever school c is considered in the above algorithm, school c must fill its floor for type-t students, and once some type-t students leave c, again reconsidering school c the floor $\underline{q}_c^t$ must be filled because $\tau(s) = t$ and $c P_s \hat{\mu}(s)$. Hence, $|\hat{\mu}^t(c)| \geqslant \underline{q}_c^t$. Now we split the rest of the analysis depending on whether type t students fill their ceiling at school c or not.

Case 1: $|\hat{\mu}^t(c)| \geqslant \overline{q}_c^t$. Consider $s' \in \hat{\mu}(c)$ such that $|\hat{\mu}^{\tau(s')}(c)| > \overline{q}_c^{\tau(s')}$. Since μ is feasible it must be that some type $\tau(s')$ students got matched with c in the above algorithm. Moreover, such students must have lower priority compared to other type $\tau(s')$ students who were matched with c in μ. In addition, type $\tau(s')$ students who got matched with c in the above algorithm have a descending priority with respect to the order they were matched. The last type $\tau(s')$ student who got matched with c must have a higher priority than s since type $\tau(s')$ has already filled their ceiling and student s is not admitted to c in this step even though she wants to switch to c. This implies that every student of type $\tau(s')$ is preferred to s.

Case 2: $\overline{q}_c^t > |\hat{\mu}^t(c)| \geqslant \underline{q}_c^t$. In this case, for any $t' \in T \backslash \{t\}$ we cannot have $|\hat{\mu}^{t'}(c) > \overline{q}_c^{t'}|$: At least one student of type t' must have been matched with c during the above algorithm since μ is feasible. Consider the last student of type t' who got matched with c (and say that type t' is the last type for which this happened). At the stage when this student got matched with c, since s is not matched with c, it must be that type t students have filled their ceiling. Later on some type t students in c must have matched with other schools, so that type t students do not fill their ceilings in school c at the end of the algorithm. After the step when type t students do not fill their ceiling anymore, type t students can be admitted without violating school c's capacity (because any other type $t'' \in T \backslash \{t, t'\}$ which did not fill its ceiling at the last step where type t' students exceeded their ceiling, will never increase the number of slots assigned to t'' in the algorithm, and any type $t''' \in T \backslash \{t\}$ which exceeded its ceiling will never exceed again its ceiling). Since s is not matched with c, and that type t students do not fill their ceiling at the end of the algorithm, we get a contradiction. Therefore, $|\hat{\mu}^{t'}(c)| \leqslant \overline{q}_c^{t'}$.

To complete the argument for Case 2, consider type t' such that $\overline{q}_c^{t'} \geqslant |\hat{\mu}^{t'}(c)| > \underline{q}_c^{t'}$. Let s' be the student in $\hat{\mu}^{t'}(c)$ with the least priority among type t' students. If $\mu(s') = c$ and $|\mu^{t'}(c)| > \underline{q}_c^{t'}$, then $s' \succ_c s$ since μ is fair. If $\mu(s') = c$ and $|\mu^{t'}(c)| = \underline{q}_c^{t'}$, then at least one type t' student must be matched with c during the above algorithm. But this gives a contradiction since that student prefers c to her match in μ and she has a higher priority than s'. Finally, if $\mu(s') \neq c$, then s' has been matched with c during the above algorithm. If at the stage when s' is admitted, type t students do not fill their ceilings then $s' \succ_c s$. Otherwise, if type t students fill their ceiling at that stage, then some of these students must have matched with other schools later in the algorithm. Since s is not matched with c, and that type t students do not fill their ceilings at the end of the algorithm, we get a contradiction. Therefore, in all of the possibilities we conclude $s' \succ_c s$.

Thus, $\hat{\mu}$ removes justifiable envy under soft bounds. Hence, $\hat{\mu}$ is both fair under soft bounds and non-wasteful under soft bounds. Since under the student-proposing DA all students are matched to the best outcome among such assignments and $\hat{\mu}$ improves students' matches compared to μ, the conclusion follows. □

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