Discriminatory Auctions with Resale*

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Abstract

We consider multi-unit discriminatory auctions where ex-ante symmetric bidders have single-unit demands and resale is allowed after the bidding stage. When bidders use the optimal auction to sell the items in the resale stage, the equilibrium without resale is not an equilibrium. We find a symmetric and monotone equilibrium when there are two units for sale, and, interestingly, we show that there may not be a symmetric and monotone equilibrium if there are more than two units.

JEL-Classification: D44, C72

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1 Introduction

The discriminatory, or "pay-your-bid," auction is a popular mechanism to sell many important goods, including treasury bills or bonds, electricity, foreign exchange, airport landing slots, and, more recently, carbon emissions. In a discriminatory auction, each bidder submits a demand curve for multiple items of a good and the auctioneer acts as a perfectly discriminating monopolist by charging each bidder his or her winning bid.

In many of these applications, bidders can freely engage in a post-auction market (resale market) in which they can resell some or all of the items they get in the auction stage. The anticipation of a resale market can influence bidders' bidding behavior, which,

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in turn, can affect the outcome of the auction stage. Nevertheless, the vast majority of the single-item or multi-item auction theory literature has neglected the effect of post-auction resale on the allocation of given auction formats. To fill this gap, we ask what are the consequences of a post-auction resale?

In this paper, we study multi-unit discriminatory auctions where items sold in the auction are identical to each other. Moreover, bidders have single-unit demands–i.e., they value only one of units, and bidders' valuations are independently and identically distributed.[1] We model the resale stage as a game in which the winners of the auction stage can sell their (excess) objects optimally. We call these winners *resellers*. In our main model, two or three units are sold in the auction stage, and since we consider symmetric equilibria, on the equilibrium path there might be only one reseller in the resale market. In the resale stage, the resellers use an optimal auction.

As is well known, in a model with symmetric single-unit demand, discriminatory auctions have a symmetric and monotone equilibrium that results in an efficient allocation (see Section 13.5.2 of Krishna (2002)). Therefore, one might think that adding a resale stage to this setup should not alter the equilibrium outcome: all winners have higher valuations than all losers, and so there would be no incentive for resale. We show that this intuition is incorrect: when resale is allowed and resellers have all the bargaining power and can design any mechanism to sell the items, then the symmetric, monotone and efficient "no resale equilibrium" is no longer an equilibrium (Proposition 1). This is so because auction prices may be too low to attract "speculative behavior"–i.e., buying and selling in the resale market.

The auctioneer of our model sells units via discriminatory price auctions that have no minimum prices (reserve prices) on units. In the resale stage we assume that resellers can use optimal mechanisms. Hence, they can use reserve prices. Our main results are the following. When there are two units for sale, we find an equilibrium in which resellers make zero profit (Theorem 1). When more than two units are for sale, surprisingly, there might not be a symmetric and monotone equilibrium (Theorem 2). This is so because when two units could be sold in the resale stage, the expected revenue of selling one unit is different from the expected revenue of selling two units, which results in contradicting requirements for the bid for the first unit. To the best of our knowledge, Theorem 2 is the first result that shows the non-existence of a symmetric and monotone equilibrium in a standard multi-object auction setup (that has independent private values, risk-neutrality, and single-unit demands).

The balance of this section discusses the related literature. Section 2 formally intro-

[1] In the literature this type of framework is called symmetric single-unit demand environment.

duces the model and ends with a motivating example (Example 1). In Section 3, we establish our results. Section 4 concludes. Appendix A and Appendix B contain omitted proofs and some extensions as well as extra results.

Related Literature Research on the theory of multi-unit or multi-item auctions is not as large and advanced as the research on single-item auctions. Noussair (1995) and Engelbrecht-Wiggans and Kahn (1998a) examine the derivation of equilibrium bidding behavior in private value uniform-price auctions. Engelbrecht-Wiggans and Kahn (1998b) analyze equilibrium strategies in discriminatory auctions. In another important paper, Reny (1999) shows the existence of a pure strategy equilibrium for discriminatory auctions that have independent private values. Ausubel, Cramton, Pycia, Rostek, and Weretka (2014) study the issue of demand reduction in multi-unit auctions.

The literature on single-unit auctions with resale is quite large. Most of this literature studies environments in which resale takes place due to inefficient allocation (such as in asymmetric first price auctions) at the end of the bidding stage. Gupta and Lebrun (1999), Haile (2000), Haile (2001), Zheng (2002), Haile (2003), Garratt and Troger (2006), Pagnozzi (2007), and Hafalir and Krishna (2008) are earlier notable examples of this literature.[2] In contrast these papers, ours look at an environment in which the equilibrium without resale is efficient, yet post-auction resale can take place. This phenomenon also occurs in the online supplement to Garratt and Troger (2006). They show that when there is one speculator (who values the item at 0) and n symmetric bidders in a first-price auction, under some conditions, the speculator can play an active role (buy in the auction stage and resell in the resale stage) [3,4]. In our setup, it turns out that no resale equilibrium *always* gives rise to speculative behavior.

Finally, we discuss the theoretical literature on multi-unit auctions with resale. In an earlier work, Bukhchandani and Huang (1989) analyze a multi-item (discriminatory or uniform price) auction with *common values*. In the resale market, bidders receive information about the bids submitted in the auction. They examine the information linkage

[2]This literature has grown more in recent years. See, for instance, Hafalir and Krishna (2009), Lebrun (2010), Pagnozzi (2010), Cheng and Tan (2010), Cheng (2011), Xu, Levin, and Ye (2013), Virag (2013), and Zheng (2014).

[3]These conditions depend on the number of regular bidders and value distribution. For instance, when value distribution is uniform speculators do not play an active role. In our paper, in contrast, speculative behavior always exists. To the best of our knowledge, Garratt and Troger (2006) are the first to point out that speculation can affect equilibrium behavior in single-unit auctions. We further advance research on this topic by analyzing how speculation affects equilibrium behavior in multi-unit auctions.

[4]Garratt and Troger (2006) also show that speculators play an active role in second-price or English auctions. In English auctions, because there are many equilibria (some of which are inefficient), resale can affect equilibrium behavior more easily. See also Garratt, Troger, and Zheng (2009).

between the auction and resale stage and compare expected revenues in two auction formats. Recently, Filiz-Ozbay, Lopez-Vargas, and Ozbay (2015) have studied multi-unit auctions with resale in which bidders have either single or a multi-unit demand. More specifically, they consider environments in which there are k local markets, k local bidders, and 1 global bidder. They analyze the equilibrium of Vickrey auctions and simultaneous second-price auctions. In another recent work, Pagnozzi and Saral (2013) analyze different bargaining mechanisms at the resale stage following a uniform price auction in which bidders are ex-ante asymmetric. More recently, Dworczak (2015) has studied a multi-unit auction in which bidders have multi-unit demands and the auction is followed by bargaining between bidders. In this environment, standard auctions fail to allocate the good efficiently if some bids are announced after the auction. In our model, in contrast, all bidders are ex-ante symmetric with private values, demand only one item, and participate in the same discriminatory price auction.

2 Model

An auctioneer sells k units to n risk neutral bidders and each bidder has a single-unit demand; i.e., each bidder has use for at most one unit. Bidder i's value for the first unit is v_i, which is independently and identically distributed (i.i.d.) from a continuously differentiable and regular (in Myerson's sense) function F over $[0,1]$ with density f; that is, $x - \frac{1-F(x)}{f(x)}$ is increasing in x. Let $\psi(x) := x - \frac{1-F(x)}{f(x)}$ denote Myerson's *virtual valuation* such that $\psi'(x) > 0$.[5] We denote $\psi_x(\cdot)$ as a virtual valuation of $F(\cdot \mid x)$ where $F(y \mid x) := \frac{F(y)}{F(x)}$ denotes the conditional distribution for $y \in [0,x]$ and $f(\cdot \mid x)$ denotes its conditional density. We assume that $\psi_x(y)$ is increasing in y for all x.

The auctioneer uses a discriminatory auction in which the highest k bids are awarded the objects, and winners have to pay their bids to the auctioneer. Ties are broken randomly. No information is revealed after the auction stage.[6] The bidders can engage in the post-auction market (resale stage) with no discounting between auction and resale stages. We assume that *resellers* sell their (excess) units optimally.

[5] Myerson's regularity assumption is satisfied by many distributions and is commonly made in auction theory and mechanism design literature.

[6] Revealing information about winning bids does not alter results. This is because winner (reseller) already knows her value and bid while choosing optimal reserve price. Not revealing information about the losing bids, however, is crucial for our results. The effects of revealing or partially revealing information on bids is beyond the scope of this paper. See Hafalir and Krishna (2008), who argue that there is no symmetric and monotone equilibrium in single-unit first-price auctions with resale when the losing bid is announced. Also see Calzolari and Pavan (2006)'s discussion of a monopolist who optimally designs the beliefs in the resale market by partially revealing information in order to maximize revenue.

We study the perfect Bayesian Nash equilibrium (PBNE) of this game in which players are sequentially rational and update their beliefs according to Bayes' rule and equilibrium behavior. We restrict our attention to the symmetric and monotone PBNE. Hence, we consider an equilibrium such that each bidder with value v bids $(\beta_1(v), \beta_2(v), ..., \beta_k(v))$. Without a loss of generality, we assume $\beta_1(v) \geq \beta_2(v) \geq ... \geq \beta_k(v)$ where β_l denotes the l^{th} highest bid of a bidder with value v. We also assume that β_l is a non-decreasing and continuously differentiable function for each $l = 1, ..., k$, which allows bidders to make zero bids for excess units.

The behavior in the resale stage is straightforward. In equilibrium, any bidder who wins j units in the bidding stage sells only $j-1$ units in the resale stage. This is because his value for the first unit is greater than the expected return of selling that unit, and this, in turn occurs because bidders follow the symmetric nondecreasing equilibrium strategies. When a bidder is the only reseller, she would use the optimal auction (a uniform-price auction with the optimal reserve price), and buyers in the resale stage would bid their valuations.[7] We do not need to specify what happens if there is more than one reseller, because in the equilibria we will find there is always one reseller.[8]

Before we move on to our motivating example, we first introduce some notations. Let the random variable Y_k^n represent the k^{th} highest random value among n random variables i.i.d. from F, and let F_k^n denote the distribution function for Y_k^n. Also assume that $\mathbf{y_n} := (y_1, y_2, \ldots, y_n)$ is the vector of ordered values of bidder and $g(\mathbf{y_n}) := n! \prod_{l=1}^{n} f(y_l)$ represents the joint density distribution of ordered n values.

Finally, we denote

$$t(x) = \begin{cases} y & \text{if } x < \psi^{-1}(0), \text{ where } \psi_y^{-1}(0) = x \\ 1 & \text{if } x \geq \psi^{-1}(0), \end{cases}$$

which implies that any bidder with value greater than $t(x)$ would charge the optimal reserve price greater than x in the resale market (so a bidder with value x will not be able to buy from that bidder).

[7]The price of a unit in the uniform price auction with a reserve price is the maximum of the highest loser bid and the reserve price.

[8]The main contributions of this paper are, first, to solve for an equilibrium when there are two units and, second, to show that there may not be a symmetric and monotone equilibrium when there are three units. In both cases, in a symmetric and monotone equilibrium there can be at most one reseller in the resale market.

2.1 Motivating Example for a Resale Market

The next example illustrates how a potential resale market changes the equilibrium behavior of bidders.

Example 1. Suppose that an auctioneer sells two units to three risk neutral bidders who are single-unit demand and whose values are *uniformly* distributed over $[0,1]$. According to Section 13.5.2 of Krishna (2002), a symmetric and monotone equilibrium of the discriminatory auction without resale market is $\left(\frac{3x-2x^2}{6-3x}, 0\right)$ for all $x \in [0,1]$.

Consider that bidders can engage in the resale market. Suppose that $\left(\frac{3x-2x^2}{6-3x}, 0\right)$ is an equilibrium. The utility of the bidder with value 1 under this strategy is $\frac{2}{3}$. This utility is lower than the utility when she bids $\left(\frac{1}{3}, \frac{1}{3}\right)$ because the utility from the first unit is $\frac{2}{3}$ and the expected utility from resale market is positive: the cost, $\frac{1}{3}$, is less than the expected revenue, $\frac{5}{12}$, which we determine by using a second-price auction with the optimal reserve price of $\frac{1}{2}$.[9] Since bidding $\left(\frac{1}{3}, \frac{1}{3}\right)$ is a profitable deviation, $\left(\frac{3x-2x^2}{6-3x}, 0\right)$ cannot be an equilibrium. Then, what is an equilibrium of this game?

We claim that $\left(\frac{5}{12}x, \frac{5}{12}x\right)$ for all $x \in [0,1]$ is an equilibrium of this game. First, note that if a bidder wins the second unit at a bid of $\frac{5}{12}z$, this implies that the other bidders' valuations are between 0 and z. By running an optimal auction for selling the second unit, the expected revenue is exactly $\frac{5}{12}z$.[10] Therefore, the expected utility from winning the second unit is 0 no matter what they bid for it, and, hence, bidders cannot benefit from deviating in the bid for the second unit. Second, we need to check that deviating for the first unit is not profitable. Consider a bidder whose value is x and whose bid is $\frac{5}{12}z$ (for the first unit, her bid for the second unit will always bring 0 expected utility). Her expected utility is given by

$$\begin{aligned}\Pi(x,z) &= (x - \frac{5}{12}z)\Pr(z > Y_1^2) + \left(\mathbb{E}[x - \max\{Y_2^2, r\} \mid z < Y_1^2]\right) \Pr(z < Y_1^2) \\ &= (x - \frac{5}{12}z)z^2 + 2\int_z^{\min\{1,2x\}} \left(\int_0^{\frac{y_1}{2}} \left(x - \frac{y_1}{2}\right) dy_2 + \int_{\frac{y_1}{2}}^{\min\{y_1,x\}} (x - y_2)\, dy_2\right) dy_1\end{aligned}$$

where the second summand represents the expected utility from buying in the resale stage.[11] To see this, note that for this bidder to win the unit in the resale stage, the highest

[9] See Footnote 10 for the calculation of the revenue.

[10] The optimal auction is to run a second-price auction with a reserve price $\frac{z}{2}$. Then, with probability $\frac{1}{2}$, the object will be sold at the reserve price, and when the object is sold higher than the reserve price (with probability $\frac{1}{4}$) the expected selling price is $\frac{2}{3}z$ (the expectation of second highest of two random variables uniformly distributed between $\frac{z}{2}$ and z). Hence the optimal revenue is $\frac{1}{2} \times \frac{z}{2} + \frac{1}{4} \times \frac{2z}{3} = \frac{5}{12}z$.

[11] In this example, and also in the proof of Theorem 1, we implicitly assume that buyers will consume

value among two competitors should be greater than z (so that she would win the auction) but not more than $2x$ (so that the reserve price he would charge is not more than x). Moreover, the price she would pay in the resale market is the maximum of the reserve price $\frac{y_1}{2}$ or the second-highest value in the resale market y_2. The profit of deviation is:

$$\Pi(x,z)-\Pi(x,x)=(x-\frac{5}{12}z)z^2-(x-\frac{5}{12}x)x^2+2\int_{[z,x]}\int_0^{\frac{y_1}{2}}\left(x-\frac{y_1}{2}\right)dy_2dy_1$$
$$+2\int_{[z,x]}\int_{\frac{y_1}{2}}^{\min\{y_1,x\}}(x-y_2)\,dy_2dy_1=\begin{cases}0 & \text{if } z\leq x\\ -\frac{1}{3}(z-x)^3 & \text{if } z\geq x\end{cases}\leq 0.$$

Hence, $\left(\frac{5}{12}x,\frac{5}{12}x\right)$ is an equilibrium.

Example 1 suggests that the symmetric, monotone, and efficient equilibrium of discriminatory auctions (without resale) is no longer an equilibrium when bidders engage in post-auction markets, and there is another equilibrium when resale is possible. In the next section, we start by generalizing the results pointed out by Example 1.

3 Results

Section 13.5.2 of Krishna (2002) states that $\beta^n(x)=(\beta_1(x),0,\ldots,0)$ is an equilibrium of a discriminatory auction without resale when bidders have single-unit demand and their valuations are i.i.d. where $\beta_1(x)=\mathbb{E}\left[Y_k^{(n-1)}\mid Y_k^{(n-1)}<x\right]$.

We call $\beta^n(x)$ the *no resale equilibrium strategy*. $\beta_1(x)$ is increasing and, consequently, k bidders who have the highest valuations will be awarded the units. Therefore, if this was the equilibrium of the auction stage, there would be no transactions in the resale stage. Yet, as Example 1 illustrates, no resale equilibrium bid functions do not constitute an equilibrium when $n=3$, $k=2$ and F is uniform. We first generalize this for arbitrary n, k, and F.

Proposition 1. *The bidding strategy of the game with no resale is not part of an equilibrium of discriminatory auctions with resale when resellers can use the optimal auction in the resale market.*

The proof arguments, which are similar to those provided in Example 1, are relegated to the Appendix A.

Next, we find an equilibrium of a game in which there are $k=2$ units for sale and $n\geq 3$ bidders.

(and not try to resale) the first item they receive. If we assume otherwise, our equilibrium still can be shown to satisfy the local first-order conditions. The arguments are available upon request.

3.1 Two Units for Sale

We first denote the expected revenue of the optimal auction when there are $n-1$ buyers whose values are smaller than x where the contribution to the revenue is exactly equal to the virtual value:[12]

$$\gamma(x) := \int_{\psi_x^{-1}(0)}^{x} \psi_x(z) dF_1^{(n-1)}(z \mid x) = \frac{1}{F(x)^{n-1}} \int_{\psi_x^{-1}(0)}^{x} \left(z - \frac{F(x) - F(z)}{f(z)} \right) dF(z)^{n-1} \quad (1)$$

We also provide the derivative of $\gamma(x)$, which will be used in the proof of the next theorem:

$$\gamma'(x) = (n-1) \left[\frac{f(x)}{F(x)} (x - \gamma(x)) - \frac{f(x)}{F(x)^{n-1}} \int_{\psi_x^{-1}(0)}^{x} F(y)^{n-2} dy \right]. \quad (2)$$

Theorem 1. *The bidding strategy $(\gamma(x), \gamma(x))$ for all $x \in [0,1]$ is an equilibrium.*

Proof. We follow similar steps to those in Example 1. First, when a bidder wins the unit with a bid of b, she knows that all $n-1$ bidders have values smaller than $\gamma^{-1}(b)$. By running an optimal auction to sell the second unit her expected revenue is $\gamma\left(\gamma^{-1}(b)\right) = b$. Hence this bidder cannot benefit from deviating in the bid for the second unit. Second, we need to determine whether a deviation for the first unit is not profitable. Note that the interim expected utility of a bidder with value x whose first bid is $\gamma(z)$:

$$\Pi(x,z) := F(z)^{n-1}(x - \gamma(z)) + RR(x,z)$$

where

$$RR(x,z) = \int_{z}^{t(x)} \int_{0}^{\min\{x,y_1\}} \int_{0}^{y_2} \dots \int_{0}^{y_{n-2}} \left(x - \max\{\psi_{y_1}^{-1}(0), y_2\} \right) g(\mathbf{y_{n-1}}) d\mathbf{y_{n-1}}$$

is her expected utility from resale when she is a buyer. The highest value among competitors, y_1, has to be between z and $t(x)$ so that she will be a buyer in the resale stage and the reserve price in the resale market is not greater than her value. The second highest value among competitors, y_2, should be smaller than her value so that she would win the unit in the resale market. Finally, she would pay $\max\{\psi_{y_1}^{-1}(0), y_2\}$ when she receives the unit in the resale stage.

[12] The optimal auction allocates the object to the bidder who has the highest (positive) virtual value.

Our objective is to show that $D(x,z) := \Pi(x,z) - \Pi(x,x) \leq 0$. First, we rewrite $RR(x,z)$ as:

$$\begin{aligned}
RR(x,z) =&(n-1)\int_z^{t(x)}\left(\int_0^w \left(x-\max\{\psi_{y_1}^{-1}(0), y_2\}\right) dF(y_2)^{n-2}\right) f(y_1)\, dy_1\\
=&(n-1)\int_z^{t(x)}\left[\left(x-\psi_{y_1}^{-1}(0)\right) F\left(\psi_{y_1}^{-1}(0)\right)^{n-2} + \left(\int_{\psi_{y_1}^{-1}(0)}^{w}(x-y_2)\, dF(y_2)^{n-2}\right)\right] f(y_1)\, dy_1\\
=&(n-1)\int_z^{t(x)}\left(F(w)^{n-2}(x-w) - \int_{\psi_{y_1}^{-1}(0)}^{w} F(y_2)^{n-2}\, dy_2\right) f(y_1)\, dy_1
\end{aligned}$$

where $w := \min\{x, y_1\}$. This implies that the profit of deviation is:

$$\begin{aligned}
D(x,z) = & F(z)^{n-1}(x-\gamma(z)) - F(x)^{n-1}(x-\gamma(x))\\
&+\int_z^{t(x)}\left(F(w)^{n-1}(x-w) + \int_{\psi_{y_1}^{-1}(0)}^{w} F(y_2)^{n-1}\, dy_2\right) f(y_1)\, dy_1.
\end{aligned}$$

Note that $D(x,x) = 0$. Therefore we only need to show:

$$\frac{\partial}{\partial z} D(x,z) \begin{cases} \leq 0 & \text{if } z > x \\ \geq 0 & \text{if } z < x \end{cases} \Rightarrow D(x,z) \leq 0.$$

First, consider $z > x$ (by which $w = x$):

$$\begin{aligned}
\frac{\partial}{\partial z} D(x,z) &= (n-1)F(z)^{n-2} f(z)(x-\gamma(z)) - F(z)^{n-1}\gamma'(z) - (n-1) f(z)\int_{\psi_z^{-1}(0)}^{x} F(y_2)^{n-2}\, dy_2\\
&= f(z)(n-1)\left(F(z)^{n-2}(x-\gamma(z)) - \int_{\psi_z^{-1}(0)}^{x} F(y_2)^{n-2}\, dy_2\right) - F(z)^{n-1}\gamma'(z)\\
&< f(z)(n-1)\left(F(z)^{n-2}(z-\gamma(z)) - \int_{\psi_z^{-1}(0)}^{z} F(y_2)^{n-2}\, dy_2\right) - F(z)^{n-1}\gamma'(z) = 0
\end{aligned}$$

where the inequality in the third row is obtained by $\frac{\partial\left(F(z)^{n-2}(x-\gamma(z)) - \int_{\psi_{y_1}^{-1}(0)}^{x} F(y_2)^{n-2} dy_2\right)}{\partial x} \geq 0$ for all $z \geq x$ and the last equality is obtained by Equation (2).

Second, we consider the latter case $z < x$ (by which $w = y_1$):

$$\frac{\partial}{\partial z} D(x,z) = (n-1) F(z)^{n-2} f(z)(x - \gamma(z)) - F(z)^{n-1} \gamma'(z) - (n-1)\left(F(z)^{n-2}(x-z) + \int_{\psi_z^{-1}(0)}^{z} F(y_2)^{n-2} dy_2 \right) f(z) = 0$$

by Equation (2). Thus, deviation is not profitable and $(\gamma(x), \gamma(x))$ is an equilibrium. □

3.2 Three Units for Sale

In this subsection, we show, interestingly, that when there are three units for sale, there might not be a symmetric and monotone equilibrium. We show this by considering a specific example that features with three units and four bidders and by showing that there is no symmetric and monotone equilibrium for this case.

Theorem 2. *When there are three units for sale to four bidders who have single-unit demands that are distributed according to a* uniform *distribution on unit interval, there is no symmetric and monotone equilibrium.*[13]

Proof. In a symmetric and monotone equilibrium, each bidder with value x submits three bids, $(\beta(x), \delta(x), \theta(x))$, where β, δ, and θ are non-decreasing, continuously differentiable, and satisfy $\beta(x) \geq \delta(x) \geq \theta(x) \geq 0$. It is not difficult to see that $\beta(0) = \delta(0) = \theta(0) = 0$ and $\beta(\cdot)$ is strictly increasing.[14]

To prove our result we proceed in three steps. In step 1, we argue that $\theta(1) = 0$ with the help of four lemmas. In step 2, we suppose that $\delta(1) > 0$ and argue that this case implies that in a neighborhood of 0, (i) $\beta(x) \leq \frac{5}{12}x$ and (ii) $\beta(x) = \delta(x)$. Then, by supposing $\beta(x) = \delta(x)$ in a neighborhood of 0 and solving a differential equation, we get a contradiction with $\beta(x) \leq \frac{5}{12}x$ and, hence, conclude $\delta(1) = 0$. Lastly, in step 3, we show that $\theta(1) = \delta(1) = 0$ cannot happen in an equilibrium because the bidder with value 1 has a profitable deviation.

[13] We generalize this result for a power distribution, $F(x) = x^a$, in Appendix B.1.

[14] To see the first claim: suppose that $\beta(0) > 0$, then there exists ε such that $\beta(\varepsilon) > \varepsilon$, and there is a strictly positive probability that the agent with ε value will get the unit at price $\beta(\varepsilon) > \varepsilon$ while not being able to sell the unit to higher value agents in a symmetric equilibrium. This agent would make a negative payoff, which gives a contradiction. To see the second claim, suppose that $\beta(x) = c$ for all $x \in [a, b]$ for some $a > b$. Then, for any bidder with value $x \in [a, b]$, by bidding c, there is a strictly positive probability of a tie. Moreover, (i) in case of a tie she will get the object with $\frac{1}{2}$ or $\frac{1}{3}$ probability, and (ii) her expected utility for this case has to be positive. Next, one can argue that bidding $c + \varepsilon$ for a small enough ε rather than c would strictly increase her expected utility. This is so because with this deviation, she increases her chance of winning (for the cases where there was a tie) from $\frac{1}{2}$ or $\frac{1}{3}$ to 1. Hence we reach a contradiction.

Step 1. Suppose that we have an equilibrium in which $\theta(1) > 0$. Let us denote $\beta^{-1}(\theta(1))$ by c. The following four lemmas provide us with the conditions to prove that we cannot have an equilibrium of this kind. The proofs of the lemmas are relegated to Appendix A.

Lemma 1. *For all $x \in [0, c]$, we have $\beta(x) \leq \frac{23}{64}x$.*

Lemma 2. *(i) For all $t \in [0, \theta(1)]$, if $\left(\beta^{-1}(t)\right)' > \frac{32}{23}$, then $\delta^{-1}(t) = \theta^{-1}(t)$, and (ii) there exists $d \in (0,1]$ such that for all $x \in [0, d]$, then we have $\beta(x) \in \left[0, \frac{23}{64}x\right]$ and $\delta(x) = \theta(x)$.*

Lemma 3. *There exists $e \in (0,1]$ such that for all $x \in [0, e]$, we have $\beta(x) \in \left[0, \frac{23}{64}x\right]$ and $\beta(x) = \delta(x) = \theta(x)$.*

Lemma 4. *Suppose that we have an equilibrium such that $\beta(x) = \delta(x) = \theta(x)$ for all $x \in [0, e]$ for some $e \in (0,1]$. Then, we need $\beta(x) = \frac{3}{8}x$ for all $x \in [0, e)$.*

In what follows we discuss how the lemmas are obtained. Lemma 1 follows by noting that if β is high (specifically when $\beta(x) > \frac{23}{64}x$), then there is a bidder who is making a loss by getting three units and who, hence, would be better off decreasing his δ and θ to 0. Lemma 2 follows from finding a condition that makes a bidder better off increasing his θ a little, and this condition, together with Lemma 1, implies $\delta(x) = \theta(x)$ in a neighborhood around 0. Lemma 3 follows by writing the expected utility of a bidder when $\delta(x) = \theta(x)$ and noting that in a neighborhood around zero, increasing δ would bring a strict benefit, and it would be feasible if $\delta(x) < \beta(x)$. Lastly, Lemma 4 follows by writing expected utility of a bidder when $\beta(x) = \delta(x) = \theta(x)$ and solving the differential equation that provides that they will be all equal to $\frac{3}{8}x$.

Finally, we get a contradiction because Lemma 1 contradicts Lemma 4 ($\frac{23}{64} < \frac{3}{8}$). Therefore $\theta(1) = 0$.

Step 2. Next, let $\theta(x) = 0$ for all $x \in [0,1]$ and $\delta(1) > 0$. Because the arguments in this step are very similar to Step 1, we only give a sketch of the arguments. First, as in the case in Lemma 1, we can argue that $\beta(x) \leq \frac{5}{12}x$.[15] Then we can argue that $\beta(x) = \delta(x)$ for a neighborhood around zero as in the case in Lemma 3. Now, consider an equilibrium that satisfies $\beta(x) \in \left[0, \frac{5}{12}x\right]$ and $\beta(x) = \delta(x)$ for all $x \in [0, e]$. Consider a bidder with value $x \in (0, e)$ who bids as if his value is z very close to x. His expected utility is given by

$$u(x,z) = z^3\left(x - 2\beta(z) + \frac{5}{12}z\right) + 3z^2(1-z)(x - \beta(z)) + RR(x,z)$$

[15] The optimal mechanism is a uniform price auction with a reserve price $\frac{v}{2}$ when there is one unit to be sold to two bidders whose valuations are uniformly distributed in $[0, v]$. Therefore the revenue is $2 \times \frac{1}{2} \times \frac{1}{2} \times \frac{v}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{2v}{3} = \frac{5}{12}v$.

where $RR\left(x,z\right)$ is his expected utility from the resale stage when he is a buyer: [16]

$$RR(x,z) = 6\int_{z}^{\min\{1,2x\}}\int_{l}^{\min\{1,2x\}}\left(\int_{\frac{k}{2}}^{x}(x-m)dm + \int_{0}^{\frac{k}{2}}\left(x-\frac{k}{2}\right)dm\right)dkdl.$$

A necessary condition $(\beta(x),\gamma(x),0)$ to be an equilibrium is $\left.\frac{\partial u(x,z)}{\partial z}\right|_{z=x} = 0$. The partial of the expected utility from the resale stage is:

$$\begin{aligned}\left.\frac{\partial RR(x,z)}{\partial z}\right|_{z=x} &= -6\left(\int_{z}^{\min\{1,2x\}}\int_{\frac{k}{2}}^{x}(x-m)\,dmdk + \int_{z}^{\min\{1,2x\}}\int_{0}^{\frac{k}{2}}\left(x-\frac{k}{2}\right)dmdk\right)\\ &= \frac{1}{4}\min\{1,2x\}^{3} - 3\min\{1,2x\}x^{2} + \frac{11}{4}x^{3}.\end{aligned}$$

Hence, for $x \leq \frac{1}{2}$ the necessary condition is:

$$\left.\frac{\partial u\left(x,z\right)}{\partial z}\right|_{z=x} = \left.\frac{\partial}{\partial z}\left(z^{3}\left(x-2\beta\left(z\right)+\frac{5}{12}z\right)+3z^{2}\left(1-z\right)\left(x-\beta\left(z\right)\right)\right)\right|_{z=x} = \frac{5}{4}x^{3}$$

with a boundary condition $\beta(0) = 0$. The unique solution is

$$\beta(x) = x\frac{96-67x}{48(3-x)},$$

which is greater than $\frac{5}{12}x$ for all $x \in \left[0,\frac{1}{2}\right]$. This contradicts $\beta\left(x\right) \in \left[0,\frac{5}{12}x\right]$ and $\beta\left(x\right) = \delta\left(x\right)$ for all $x \in [0,e]$ for some $e \in [0,1]$. Therefore, $\delta(1) = 0$.

Step 3. Finally, suppose there can be an equilibrium of $(\beta(x),0,0)$. In this case, a bidder with value x who mimics a bidder with value z receives a payoff of

$$u(x,z) = \left(1-(1-z)^{3}\right)\left(x-\beta\left(z\right)\right) = z\left(z^{2}-3z+3\right)\left(x-\beta\left(z\right)\right).$$

The solution of the necessary condition $\left.\frac{\partial u(x,z)}{\partial z}\right|_{z=x} = 0$ is:

$$\beta\left(x\right) = \frac{\frac{3}{4}x^{4}-2x^{3}+\frac{3}{2}x^{2}}{x^{3}-3x^{2}+3x}.$$

[16] The variables in integrals k, l, m denote the realizations for the highest, the second highest, and the third highest values among the competitors. The first term in the summation represents the case in which the bidder with value x pays the third highest value, and the last term represents the case in which the bidder with value x pays the reserve price.

Now the bidder with value 1 has a payoff of $1 - \frac{1}{4}$. Yet, if she bids $\left(\frac{1}{4}, \frac{1}{4}, 0\right)$, her payoff will be $1 - \frac{1}{4} + \frac{5}{12} - \frac{1}{4} > \frac{1}{4}$. Therefore $(\beta(x), 0, 0)$ cannot be an equilibrium.

Hence, there is no symmetric and monotone equilibrium for this example. □

There is no symmetric and monotone equilibrium in this example for the following reason. When two units could be sold in the resale stage, selling one or two units results in different expected revenues ($\frac{23}{32}x$ and $\frac{5}{12}x$ for the example in the above Theorem), which results in contradicting requirements for the bid for the first unit. That is why we are led to the following conjecture.

Conjecture 1. *For $n > k \geq 3$, when there are k units for sale to n bidders who have single-unit demands that are distributed according to some continuous distribution F, there is no symmetric and monotone equilibrium.*

Discussion. We explain why the "monotone equilibrium existence results" of Athey (2001), McAdams (2003), and Reny (2011) do not contradict our "no symmetric monotone equilibrium" finding. First, while these three papers consider simultaneous move games, ours is a two stage game. Yet because our equilibrium concept is PBE, we can incorporate resale stage payoffs into the auction stage and consider our game as a simultaneous move Bayesian game. Hence, their results might be applicable in our setup. However, the results in Athey (2001) and McAdams (2003) concern the existence of a monotone equilibrium and not a symmetric monotone equilibrium. The only papers that give existence results for a symmetric equilibrium in a symmetric game are Reny (1999) and Reny (2011). Reny (2011) has shown that (i) if the game satisfies 6 assumptions, G.1-G.6, then a monotone equilibrium exists, and (ii) if, in addition, the game is symmetric, then a symmetric monotone equilibrium exists. In our game, the assumptions G.1-G.5 are satisfied, but G.6—the continuity assumption—is not. Hence, the main result in Reny (2011) does not directly apply to our setup. In his applications section, Reny (2011) also shows that some Bayesian games that are not continuous (most relevantly discriminatory multi-unit auctions with CARA bidders) also have a monotone equilibrium. However, Reny (2011) does not establish the existence of a symmetric monotone equilibrium in these applications.[17] Hence, these methods are not applicable to our setup.

Lastly, we discuss some variations of our model that we consider in Appendix B.2. When the resale market has to be efficient, and, hence, when resellers cannot use reserve

[17]The proof in Reny (2011) is done by appealing to Remark 3.1 in Reny (1999) and showing that this game is "better-reply secure." From Reny (2011)'s extensions, one may conjecture that if a game (i) is symmetric, (ii) is better-reply secure, and (iii) satisfies G.1-G.5, then there exists a symmetric monotone equilibrium. However, this conjecture is wrong because our game can be shown to be better-reply secure.

prices in the resale stage (like the original seller who does not use a reserve price), "no resale equilibrium" remains an equilibrium with resale (Proposition O.1). Yet there exists another "resale equilibrium" in which one bidder buys all the units and sells all but one of them in the resale market (Proposition O.2). Moreover, the resale equilibrium is revenue equivalent to the no resale equilibrium (Proposition O.3). Furthermore, as a corollary to this result, we note that when there are two units for sale and resellers can use reserve prices in the resale market, banning the resale market strictly decreases the expected revenue in a discriminatory price auction (Corollary O.1). Finally, when reserve prices can be used both in the auction stage and the resale stage, the no resale equilibrium remains an equilibrium (Proposition O.4).

4 Conclusion

In this paper, we consider an environment in which ex-ante symmetric bidders who have private single-unit demands can engage in post-auction resale after participating in a discriminatory (pay-your-bid) auction. This environment without resale opportunities result in an efficient allocation of units. Hence, one might expect that adding resale opportunities will not change the equilibrium behavior. We prove this intuition wrong by observing that the auction results in low prices that attract speculative behavior (buying and then selling in the resale stage). We find an equilibrium when there are two units for sale, and we show that there may be no symmetric and monotone equilibrium when there are three or more units for sale.

Overall, we establish that the possibility of resale–even when the equilibrium without resale is efficient–can have significant effects on the auction outcome: equilibrium without resale is not an equilibrium, and for some cases there may not be any symmetric and monotone equilibrium. How to find a (non-symmetric monotone, or mixed strategy) equilibrium when there is no symmetric and monotone equilibrium remains an open question.

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A Appendix

Proof of Proposition 1. The proof follows from an argument that is similar to the one made in Example 1. Suppose every bidder other than bidder 1 uses no resale equilibrium strategy. Consider bidder 1 with value 1 and the alternative strategy $\left(\beta_1^N(1), \beta_1^N(1), 0, ..., 0\right)$

(similar strategies can be found for other values in $(0,1)$). With this strategy, this bidder will receive two units. She can sell the second unit by using a second-price auction with a reserve price $\frac{1}{2}$, which gives her an expected revenue that is strictly higher than $\mathbb{E}\left[Y_k^{(n-1)}\right]$. This is because $\mathbb{E}\left[Y_k^{(n-1)}\right]$ is the expected revenue of the second price auction *with no reserve price*, and the expected revenue of a second price auction with optimal reserve price is strictly higher than that. Since this bidder has paid $\mathbb{E}\left[Y_k^{(n-1)}\right]$ for the second unit and gets strictly more than $\mathbb{E}\left[Y_k^{(n-1)}\right]$ in the resale stage, this deviation strictly increases her utility. $\beta^N(x)$ is not an equilibrium of discriminatory auctions with resale. □

Proof of Lemma 1. First, by method of contradiction, suppose that $\beta(x) > \frac{23}{64}x$ for some $x \in [0,c]$. Consider a bidder with value $y = \theta^{-1}(\beta(x))$. When this bidder receives three units from the auctioneer, his total payment for second and third object is $\delta(y) + \theta(y) \geq 2\theta(y) = 2\beta(x) > \frac{23}{32}x$, whereas his expected revenue from resale for this case is only $\frac{23}{32}x$.[18] Therefore he makes a loss when he receives 3 units. So, he is better off by deviating to $(\beta(y), 0, 0)$. □

Proof of Lemma 2. For (i), by method of contradiction, suppose that there exist $t \in [0, \theta(1)]$ such that $\left(\beta^{-1}(t)\right)' > \frac{32}{23}$ and $\delta^{-1}(t) < \theta^{-1}(t)$. Then we can argue that type $\theta^{-1}(t) \equiv y$ strictly benefits by deviating to $(\beta(y), \delta(y), t+\varepsilon)$ for a small enough ε. This is because, by deviating to $t+\varepsilon$ from t for his third bid (and this is feasible because $\delta^{-1}(t) < \theta^{-1}(t)$), this bidder (i) increases the probability of getting three units and (ii) increases his net utility when he sells two units to unassigned bidders (his payment increases by ε and his expected revenue increases by strictly more than $\frac{23}{32} \times \frac{32}{23} \times \varepsilon = \varepsilon$.)

Next, because $\beta(0) = 0$ and $\beta(x) \leq \frac{23}{64}x$ for all $x \in [0,c]$, we have $\beta'(0) \leq \frac{23}{64}$ or $\left(\beta^{-1}(0)\right)' \geq \frac{64}{23} > \frac{32}{23}$. Because β is continuously differentiable, there exists $d \leq c$ such that $(\beta^{-1}(x))' > \frac{32}{23}$ for all $x \in [0,d]$; part (i) implies that $\delta(x) = \theta(x)$ for all $x \in [0,d]$. □

Proof of Lemma 3. By Lemma 2, we know that for all $x \in [0,d]$, we have $\beta(x) \in \left[0, \frac{23}{64}x\right]$ and $\delta(x) = \theta(x)$. Let us define the net utility of a buyer with value $x \in [0,d]$ when he is a seller in the resale stage:[19]

[18] The optimal mechanism is a uniform price auction (where the price is the highest of the highest loser's bid and the reserve price) with reserve price $\frac{v}{2}$ when there are two units to be sold to three bidders whose valuations are uniformly distributed in $[0,v]$. Therefore the revenue is $\frac{3}{8} \times \frac{v}{2} + \frac{3}{8} \times 2 \times \frac{v}{2} + \frac{1}{8} \times 2 \times \frac{5v}{8} = \frac{23}{32}v$.

[19] The optimal mechanism is a uniform price auction with reserve price $\frac{v}{2}$ when there is one unit to be sold to two bidders whose valuations are uniformly distributed in $[0,v]$. Therefore the revenue is $2 \times \frac{1}{2} \times \frac{1}{2} \times \frac{v}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{2v}{3} = \frac{5}{12}v$.

$$s\left(x\right)=\beta^{-1}\left(\theta\left(x\right)\right)^{3}\left(\frac{23}{32}\beta^{-1}\left(\theta\left(x\right)\right)-2\theta\left(x\right)\right)+$$
$$3\beta^{-1}\left(\theta\left(x\right)\right)^{2}\left(x-\beta^{-1}\left(\theta\left(x\right)\right)\right)\left(\frac{5}{12}\beta^{-1}\left(\theta\left(x\right)\right)-\theta\left(x\right)\right).$$

First, if $s'(x)>0$ then we have $\theta(x)=\beta(x)$. This is because whenever $s'(x)$ is positive, a bidder with value x becomes strictly better off by increasing his second and third bids by ε, and doing this would be feasible if $\beta\left(x\right)>\theta\left(x\right)$.

Next, by method of contradiction, suppose that there exists no $e\in(0,1]$ such that for all $x\in[0,e]$, $\beta\left(x\right)=\delta\left(x\right)=\theta\left(x\right)$. This means there exists $f>0$ such that we have $\beta\left(x\right)>\theta\left(x\right)$ for all $x\in(0,f]$. Note that Lemma 1 implies $\frac{23}{32}\beta^{-1}(\theta(x))>2\theta(x)$ and $\frac{5}{12}\beta^{-1}(\theta(x))>\theta(x)$. As a result, $s(x)>0$. This implies that $s'(y)>0$ for some $y\in(0,f]$ since $s(0)=0$. Therefore we have $\theta(y)=\beta(y)$. □

Proof of Lemma 4. Consider a bidder with value $x\in(0,e)$ who bids as if his value is z (which is very close to x.) His expected utility is given by

$$u\left(x,z\right)=z^{3}\left(x-3\beta\left(z\right)+\frac{23}{32}z\right)+R\left(x,z\right)$$

where $R\left(x,z\right)$, first, is his expected utility from the resale stage when he is a buyer and, second, is given by

$$R\left(x,z\right)=6\int_{z}^{\min\{1,2x\}}\int_{\frac{k}{2}}^{x}\int_{\frac{k}{2}}^{l}\left(x-m\right)dmdldk+6\int_{z}^{\min\{1,2x\}}\int_{x}^{k}\int_{\frac{k}{2}}^{x}\left(x-m\right)dmdldk$$
$$+6\int_{z}^{\min\{1,2x\}}\int_{\frac{k}{2}}^{k}\int_{0}^{\frac{k}{2}}\left(x-\frac{k}{2}\right)dmdldk+6\int_{z}^{\min\{1,2x\}}\int_{0}^{\frac{k}{2}}\int_{0}^{l}\left(x-\frac{k}{2}\right)dmdldk$$

where: k,l,m denote the realizations for the highest, the second highest, and the third highest values among the competitors; the first two terms in the summation represent the cases in which the bidder with value x pays the third highest value; and the last two terms in the summation represent the cases in which the bidder with value x pays the reserve price.

A necessary condition for an equilibrium is $\left.\frac{\partial u(x,z)}{\partial z}\right|_{z=x}=0$. Note that

$$\frac{\partial R\left(x,z\right)}{\partial z}=-6\left(\int_{\frac{z}{2}}^{x}\int_{\frac{z}{2}}^{l}\left(x-m\right)dmdl+\int_{x}^{z}\int_{\frac{z}{2}}^{x}\left(x-m\right)dmdl+\int_{\frac{z}{2}}^{z}\int_{0}^{\frac{z}{2}}\left(x-\frac{z}{2}\right)dmdl\right.$$
$$\left.+\int_{0}^{\frac{z}{2}}\int_{0}^{l}\left(x-\frac{z}{2}\right)dmdl\right)=x^{3}+\frac{5}{8}z^{3}-3x^{2}z.$$

The necessary condition can be rewritten as:

$$\frac{\partial}{\partial z}\left(z^3\left(x-3\beta\left(z\right)+\frac{23}{32}z\right)\right)+\left(x^3+\frac{5}{8}z^3-3x^2z\right)\Big|_{z=x}=0$$

This differential equation will have a unique solution, which is $\beta\left(x\right)=\frac{3}{8}x$. $\square$

B Appendix

B.1 Generalization of Theorem 2

In this section, we generalize the Theorem 2 of the main body of the paper for arbitrary power distribution: $F(x) = x^a$. With regards to notation, let $\mathbf{y} = [y_1, \ldots, y_{N-1}]$ where there are N bidders in the auction stage and y_j is the j^{th} highest value. Also, let $g(\mathbf{y})$ represent its probability distribution. Next, we denote bidding functions as (β, δ, γ), which satisfies the bidding requirements in main text.

First, we note the optimal reserve price equals $\psi_x^{-1}(0) = \left(\frac{1}{\alpha+1}\right)^{\frac{1}{\alpha}} x$ for $v \in [0, x]$.[20] The following three lemmas (Lemmas A, B and C) will be helpful in establishing a generalization of Theorem 1.

Lemma A. *If a bidder with value x receives 3 units in the auction, his optimal revenue from resale is linear in $\beta^{-1}(\theta(x))$.*

Proof. Let $y = \beta^{-1}(\theta(x))$. Note that $r = \psi_y^{-1}(0)$. The revenue equals:

$$\rho_2(y) = \frac{\int_r^y \int_0^r \int_0^{y_2} r\, g(\mathbf{y}) d\mathbf{y} \int_r^y \int_r^{y_1} \int_0^r 2\, r\, g(\mathbf{y}) d\mathbf{y} + \int_r^y \int_r^{y_1} \int_r^{y_2} 2\, y_3\, g(\mathbf{y}) d\mathbf{y}}{F(y)^3} = A(a) y$$

where $A(a) = \frac{3a\left(4a^2(a+1)+(6a+1)\left(\frac{1}{a+1}\right)^{1/a}\right)}{(a+1)^2(a(6a+5)+1)}$. □

Lemma B. *Consider $\delta = \theta$. If a bidder with value x receives 2 units in the auction, the optimal revenue from resale is linear in $\beta^{-1}(\delta(x))$.*

Proof. Let $y = \beta^{-1}(\delta(x))$. Note that $r = \psi_y^{-1}(0)$. The revenue equals:

$$\rho_1(y) = \frac{\int_y^x \int_r^y \int_0^r r\, g(\mathbf{y}) d\mathbf{y} + \int_y^x \int_r^y \int_r^{y_2} y_3\, g(\mathbf{y}) d\mathbf{y}}{3(F(x) - F(y))F(y)^2} = \underbrace{2a \left(\left(\frac{1}{a+1}\right)^{\frac{2a+1}{a}} + \frac{a\left(1 - 2\left(\frac{1}{a+1}\right)^{\frac{1}{a}+1}\right)}{(a+1)(2a+1)} \right)}_{:=B(a)} y.$$

□

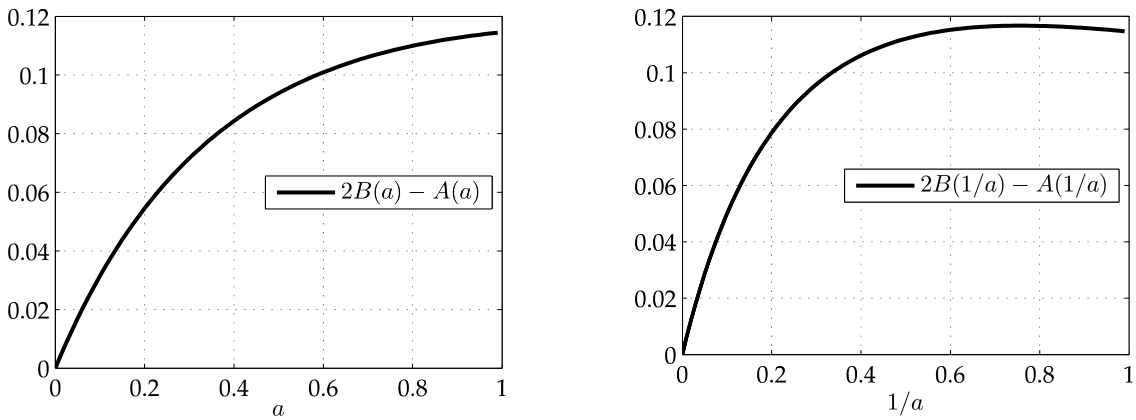
Lemma C. *Assume that $\beta(x) \leq \frac{\rho_2(x)}{2}$. If all bidders follow $\beta(x) \geq \delta(x) = \theta(x)$, then there exists a e such that for all $x \in [0, e]$ we have $\theta(x) \leq \rho_1(\beta^{-1}(\theta(x)))$.*

Proof. Suppose not. Then there exists $x \in [0, e]$ such that $\theta(x) > \rho_1(\beta^{-1}(\theta(x)))$. Consider the bidder $y = \theta^{-1}(\beta(x))$. Then

$$0 > -\theta(x) + \rho_1(\beta^{-1}(\theta(x))) > -\theta(x) + \rho_1(\rho_2^{-1}(2\theta(x)))$$

[20]The virtual value is represented as $\psi_x(v) = v - \frac{1-F(v|x)}{f(v|x)} = v - \frac{F(x)-F(v)}{f(v)}$.

because $\beta^{-1}(x) \geq \rho_2^{-1}(2x)$ for small x. But then since $\rho_2^{-1}(x) = \frac{x}{A(a)}$, we have $\rho_1(\rho_2^{-1}(2x)) = \frac{2B(a)}{A(a)}x > x$ (See Figure 1).



(a) Small Values of a

(b) Large Values of a

Figure 1: $2B(a) - A(a)$ Values

Thus, $-\theta(x) + \rho_1(\rho_2^{-1}(2\theta(x))) > 0$, which is a contradiction. □

Theorem O.2. *Suppose there are three units for sale and four bidders whose values are independently and identically drawn from a power distribution: $F(x) = x^a$, then there is no symmetric and monotone equilibrium.*

Proof. The proof will be obtained in three steps given by three lemmas.

1. First, we show $\theta(x) = 0$ for all $x \in [0, 1]$ in Lemma O.1.
2. Second, we show $\delta(x) = 0$ for all $x \in [0, 1]$ in Lemma O.2.
3. Third, we configure an equilibrium candidate in $(\beta(x), 0, 0)$ format. Then we show the bidder who has value 1 benefits from resale market if she follows $(\beta(1), \beta(1), 0)$ in Lemma O.3.

Lemma O.1. $\theta(x) = 0$ *for all* $x \in [0, 1]$.

Proof. Assume that $\theta(1) > 0$ and denote $c = \beta^{-1}(\theta(1))$. We establish this lemma with the help of 4 sub-lemmas.

Lemma O.1.1. *$\forall x \in [0, c]$, we have* $\beta(x) \leq \frac{\rho_2(\beta^{-1}(\theta(x)))}{2} \leq \frac{\rho_2(x)}{2}$.

Proof. Suppose $\beta(x) > \frac{\rho_2(\beta^{-1}(\theta(x)))}{2}$. Consider the bidder $y = \theta^{-1}(\beta(x))$. If she gets 3 units, the cost of non-valued units is bigger than the benefit of the resale market: $\delta(y) + \theta(y) \geq 2\theta(y) = 2\beta(x) > \rho_2(\beta^{-1}(\theta(x)))$ (see Lemma A). □

Lemma O.1.2. *(i) $\forall t \in [0, \theta(1)]$, if $(\beta^{-1}(t))' > (\rho_2^{-1}(2t))'$ then $\delta^{-1}(t) = \theta^{-1}(t)$, (ii) there exists $d \in (0,1]$ s.t. $\forall x \in [0,d]$ then $\beta(x) \leq \frac{\rho_2(\beta^{-1}(\theta(x)))}{2} \leq \frac{\rho_2(x)}{2}$ and $\delta(x) = \theta(x)$.*

Proof. Suppose there exists a $t \in [0, \theta(1)]$, s.t. $(\beta^{-1}(t))' > (\rho_2^{-1}(2t))'$ and $\delta^{-1}(t) < \theta^{-1}(t)$. Consider a bidder with value $y = \theta^{-1}(t)$. This bidder can be better off increasing her bid for the last unit by ε because the probability of winning 3 units increases and the expected revenue increases more than ε:

$$(\rho_2(\beta^{-1}(\theta(x))))' \times (\beta^{-1}(\theta(x)))' \times \varepsilon > (\rho_2(\rho_2^{-1}(2t)))' \times (\rho_2^{-1}(2t))' \times \varepsilon = \varepsilon.$$

Note that $\beta(0) = 0$ and $\beta(x) \leq \frac{\rho_2(x)}{2}$ for all $x \in [0,c]$. Therefore, $(\beta^{-1}(x))' \geq \left(\rho_2^{-1}(2x)\right)'$ for all $x \in [0,d]$ where $d \leq c$. This also implies that $\delta(x) = \theta(x)$ for all $x \in [0,d]$. □

Lemma O.1.3. *There is $e \in [0,d]$ s.t. $\forall x \in [0,e]$, we have $\beta(x) = \delta(x) = \theta(x)$ and $\beta(x) \leq \frac{\rho_2(x)}{2}$.*

Proof. Suppose $\beta(x) > \delta(x) = \theta(x)$. The expected utility from resale for the bidder is positive: $RU(x) = F(\beta^{-1}(\theta(x)))^3 \left(\rho_2(\beta^{-1}(\theta(x))) - 2\theta(x)\right) + 3F(\beta^{-1}(\theta(x)))^2(F(x) - F(\beta^{-1}(\theta(x)))) \left(\rho_1(\beta^{-1}(\theta(x))) - \theta(x)\right) > 0$ appealing to Lemma C and Lemmas O.1.1-O.1.2. Hence increasing $\delta(x), \theta(x)$ is a profitable deviation. □

Lemma O.1.4. *$\beta = \delta = \theta$ is not an equilibrium strategy for $x \in [0,e)$ where $e < 1$.*

Proof. Suppose it is. The utility of the bidder with value x who bids $b = \beta(z)$ is:

$$u(x,z) = F(z)^3 \left(x - 3\beta(z) + \rho_2(z)\right) + RR(x,z)$$

where $RR(x,z)$ is the utility of getting a unit from resale:

$$\begin{aligned} RR(x,z) = &\int_z^{\Psi(x)} \int_{\psi_{y_1}^{-1}(0)}^{x} \int_{\psi_{y_1}^{-1}(0)}^{y_2} (x - y_3) g(\mathbf{y}) d\mathbf{y} + \int_z^{\Psi(x)} \int_x^{y_1} \int_{\psi_{y_1}^{-1}(0)}^{x} (x - y_3) g(\mathbf{y}) d\mathbf{y} \\ &+ \int_z^{\Psi(x)} \int_{\psi_{y_1}^{-1}(0)}^{y_1} \int_0^{\psi_{y_1}^{-1}(0)} (x - \psi_{y_1}^{-1}(0)) g(\mathbf{y}) d\mathbf{y} + \int_z^{\Psi(x)} \int_0^{\psi_{y_1}^{-1}(0)} \int_0^{y_2} (x - \psi_{y_1}^{-1}(0)) g(\mathbf{y}) d\mathbf{y} \end{aligned}$$

where $\Psi(x) = \max\{1, (a+1)^{\frac{1}{a}} x\}$. The necessary condition is:

$$\frac{\partial u(x,z)}{\partial z}|_{z=x} = 3F(x)^2 f(x) \left(x - 3\beta(x) + \rho_2(x)\right) + F(x)^3 \left(-3\beta'(x) + \rho_2'(x)\right) + \frac{\partial RR(x,z)}{\partial z}|_{z=x} = 0 \tag{NC}$$

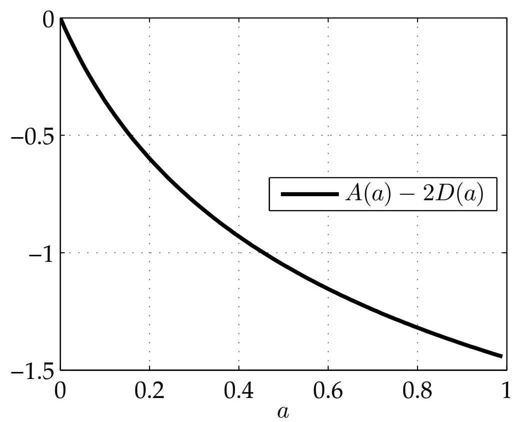
Note that:

$$-\frac{\partial RR(x,z)}{\partial z}|_{z=x} = \int\limits_{\psi_x^{-1}(0)}^{x} \int\limits_{\psi_x^{-1}(0)}^{y_2} (x-y_3)f(y_3,y_2,x)dy_3dy_2 + \int\limits_{\psi_x^{-1}(0)}^{x} \int\limits_{0}^{\psi_x^{-1}(0)} (x-\psi_x^{-1}(0))f(y_3,y_2,x)dy_3dy_2$$
$$+ \int\limits_{0}^{\psi_x^{-1}(0)} \int\limits_{0}^{y_2} (x-\psi_x^{-1}(0))f(y_3,y_2,x)dy_3dy_2 = x^{3a}C(a)$$

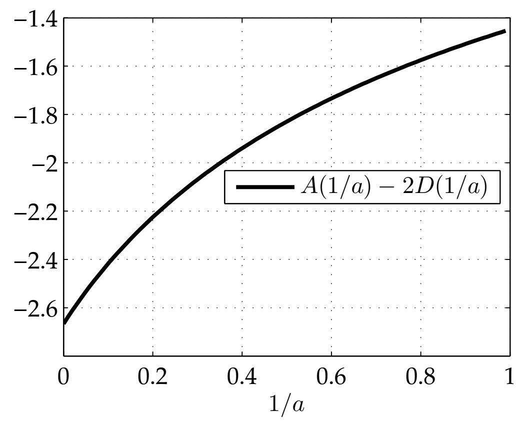
where $C(a) := \frac{3a\left(3a^2-\left(\frac{1}{a+1}\right)^{1/a}-4a\left(\left(\frac{1}{a+1}\right)^{1/a}-1\right)+1\right)}{(a+1)^2(2a+1)}$. Rewrite NC:

$$3x^{2a}ax^{a-1}\left(x-3\beta(x)+A(a)x\right)+x^{3a}\left(-3\beta'(x)+A(a)\right)+C(a)x^{3a}=0$$

The unique solution of this differential equation is $\beta(x) = Dx$ where $D(a) = \frac{3a+(3a+1)A(a)+C(a)}{3(3a+1)}$. Lemma O.1.1 implies $D(a) < \frac{A(a)}{2}$. However, Figure 2 shows that $D(a) > \frac{A(a)}{2}$:



(a) Small Values of a



(b) Large Values of a

Figure 2: $A(a) - 2D(a)$ Values

□

As a result, $\theta(1) = 0$. □

Lemma O.2. *$\delta(x) = 0$ for all $x \in [0,1]$.*

Proof. Let us consider an equilibrium with $\theta(x) = 0$ but $\delta(1) > 0$. We can argue that $\beta(x) \leq B(a)x$ and $\beta(x) = \delta(x)$ for a neighborhood of zero (see Lemmas O.1.1-O.1.2). Consider a bidder with value $x \in (0,e)$ bids as his value is z in the neighborhood of x.

Let $R(a) = \left(\frac{1}{a+1}\right)^{\frac{1}{a}}$. His expected utility is:

$$u(x,z) = F(z)^3(x - 2\beta(z) + B(a)z) + 3F(z)^2(1 - F(z))(x - \beta(z)) + RR(x,z)$$

where

$$RR(x,z) = \int_z^{\max\{1,\frac{x}{R(a)}\}} \int_z^{y_1} \left(\int_{R(a)y_1}^{x} (x - y_3)g(\mathbf{y})d\mathbf{y} + \int_0^{R(a)y_1} (x - R(a)y_1)g(\mathbf{y})d\mathbf{y} \right)$$

is the utility of the bidder who gets a unit from resale. The necessary condition is:

$$\begin{aligned}\frac{\partial u(x,z)}{\partial z}|_{z=x} &= 3F(x)^2 f(x)(x - 2\beta(x) + B(a)x) + F(x)^3(-2\beta'(x) + B(a)) - 3F(x)^2(1 - F(x))\beta'(x) \\ &+ 6F(x)f(x)(1 - F(x))(x - \beta(x)) - 3F(x)^2 f(x)(x - \beta(x)) - \Gamma(x) = 0 \qquad \text{(NC2)}\end{aligned}$$

where

$$\Gamma(x) = -\frac{\partial RR(x,z)}{\partial z}|_{z=x} = x^{3a} \underbrace{\frac{6a\left(a\left(\frac{1}{a+1}\right)^{\frac{a+1}{a}} + (a+1)^2 - 2a - 1\right)}{(a+1)(2a+1)}}_{:=H(a)}$$

After some algebra, we can rewrite NC2:

$$-\beta(x)3a(2 - x^a) - \beta'(x)x(3 - x^a) + 6ax + ((3a+1)B(a) + H(a) - 6a)x^{a+1} = 0.$$

The unique solution of this differential system is:

$$\beta(x) = \frac{2ax\left(3 - x^a \frac{(6a^2+8a+3) - \frac{1}{(a+1)^{1+1/a}}}{(a+1)(3a+1)}\right)}{(2a+1)(3 - x^a)}.$$

In order to have $\beta(x) \leq B(a)x$, the following inequality should hold for all $x \in (0, e]$:

$$\frac{\left(3 - x^a \frac{(6a^2+8a+3) - \frac{1}{(a+1)^{1+1/a}}}{(a+1)(3a+1)}\right)}{(2a+1)(3 - x^a)} < \left(\left(\frac{1}{a+1}\right)^{\frac{2a+1}{a}} + \frac{a\left(1 - 2\left(\frac{1}{a+1}\right)^{\frac{1}{a}+1}\right)}{(a+1)(2a+1)} \right).$$

Let us define $\Omega_1(a) = \frac{(6a^2+8a+3) - \frac{1}{(a+1)^{1+1/a}}}{(a+1)(3a+1)}$ and $\Omega_2(a) = (2a+1)\left(\left(\frac{1}{a+1}\right)^{\frac{2a+1}{a}} + \frac{a\left(1 - 2\left(\frac{1}{a+1}\right)^{\frac{1}{a}+1}\right)}{(a+1)(2a+1)} \right)$.

Then, the inequality can be rewritten as: $\Sigma(a) := \left(\frac{3(1-\Theta_2)}{\Theta_1 - \Theta_2}\right)^{\frac{1}{a}} < x$ for all $x \in (0, e)$. However, Figure 3 suggests that $\Sigma(a) > 0$ for all $a \in (0, \infty)$ and, therefore, we can find an x which violates NC2.

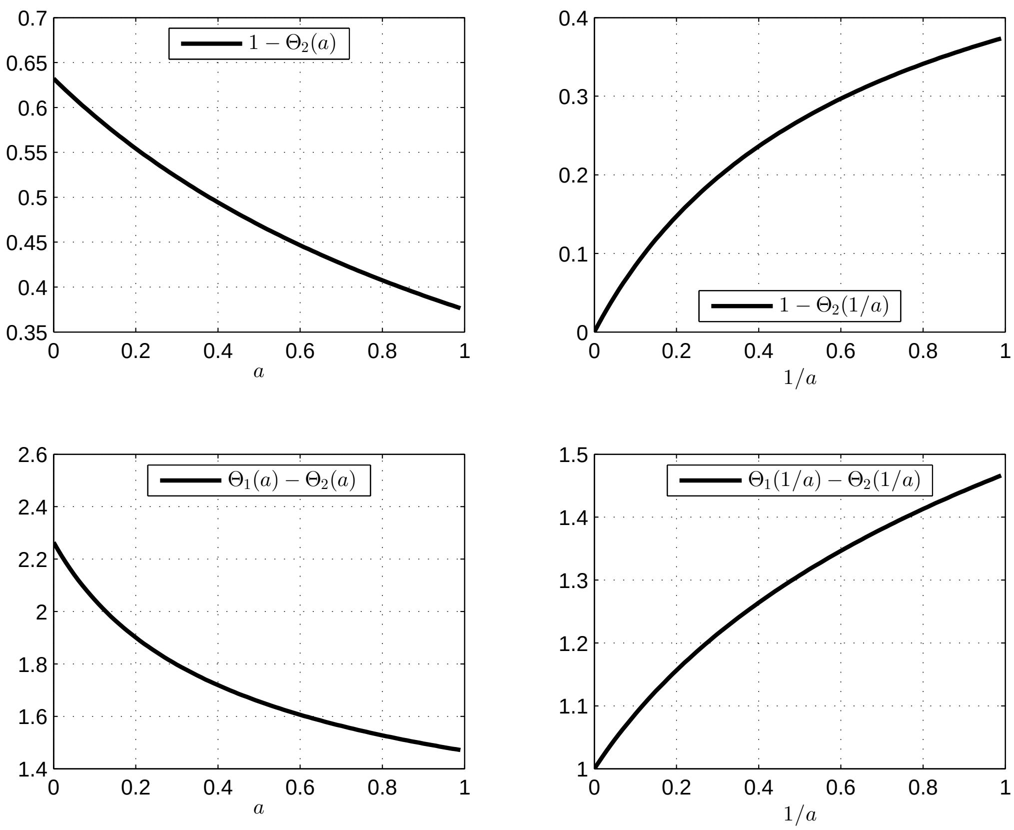


Figure 3: $\Sigma(a) > 0$ for all $a > 0$

Left panels are for $a \in (0, 1]$ and right panels are for $a \in [1, \infty)$.

Therefore $\delta(1) = 0$. □

Lemma O.3. $(\beta(x), 0, 0)$ *is not an equilibrium strategy.*

Proof. Suppose that it is an equilibrium strategy. The utility of the bidder with value x bids $b = \beta(z)$ is:

$$u(x, z) = (1 - (1 - F(z))^3)(x - \beta(z)).$$

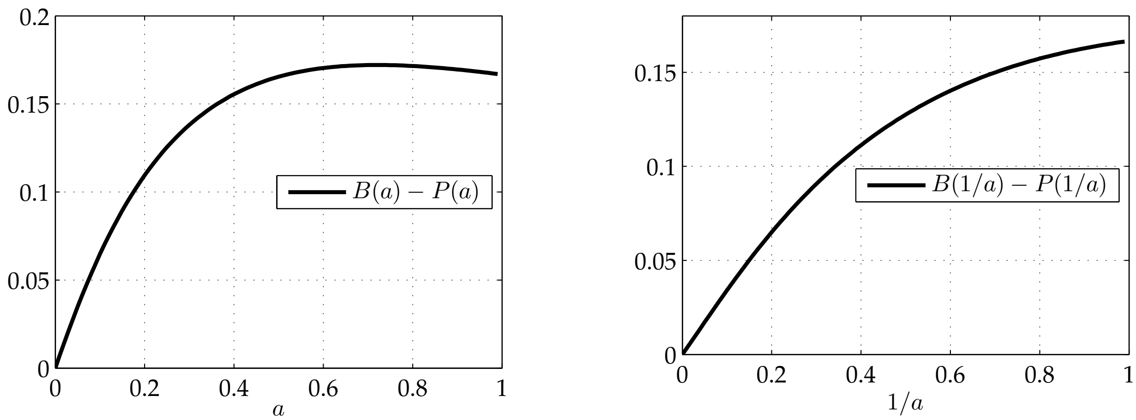
The necessary condition is:

$$-\beta'(x)(1 - (1 - F(x))^3) + (x - \beta(x))3(1 - F(x))^2 f(x) = 0.$$

The solution of this differential equation is :

$$\beta(x) = \frac{3ax\left(\frac{x^{2a}}{3a+1} - \frac{2x^a}{2a+1} + \frac{1}{a+1}\right)}{(x^a - 3)\,x^a + 3}.$$

Now suppose the bidder with value 1 bids $(\beta(1), \beta(1), 0)$. He gets the second unit and the payment is $P(a) = 3a\left(-\frac{2}{2a+1} + \frac{1}{3a+1} + \frac{1}{a+1}\right)$ which is lower than the benefit $B(a)$ (see Lemma C):



(a) Small Values of a (b) Large Values of a

Figure 4: $B(a) - P(a)$ for highest value bidder

□

This completes the proof of Theorem O.2. □

B.2 Variations

In this section, we consider some variations of the model where we allow for arbitrary k and n with $n > k$. We consider firsr a case in which the resellers in the resale market cannot use reserve prices (for instance, because of commitment problems). In this variation, if there is one seller in the resale market he would use the "optimal efficient mechanism," which is a uniform price auction that has no reserve price. For this case, we find two equilibria. One is the "no resale equilibrium" $\beta^n(x) = (\beta_1(x), 0, 0, ..., 0)$ where $\beta_1(x) = \mathbb{E}[Y_k^{n-1} \mid Y_k^{n-1} < x]$:

Proposition O.1. *When resellers in the resale stage cannot use reserve prices, no resale equilibrium $\beta^n(x)$ remains an equilibrium of the discriminatory auction with resale.*

Proof. We only need to show that the resale market is not beneficial under this strategy. Suppose that all bidders but the bidder with value 1 are bidding according to the above strategy. If the bidder with value 1 wins one additional unit for a bid b, he expects to sell it for:[21]

$$\mathbb{E}\left[Y_k^{n-1} \mid Y_{k-1}^{n-1} < \beta^{-1}(b)\right],$$

which is less than b. This result can be extended for an arbitrary number of additional units. □

In the second equilibrium, a bidder bid the same amount for all units. In particular, a bidder with value x bids $\beta^{\mathbf{R}}(x) = (\beta^R(x), \ldots, \beta^R(x))$ where $\beta^R(x) = \mathbb{E}[Y_k^{n-1} \mid Y_1^{n-1} < x]$. In this equilibrium there will be one bidder (with the highest value) who will win all k units, and he will sell $k-1$ units in the resale stage using a uniform-price auction.

Proposition O.2. *When resellers in the resale stage cannot use reserve prices, the k-tuple $\beta^{\mathbf{R}}(x)$ is an equilibrium of the discriminatory auction with resale.*

Proof. We first consider deviations for extra units. Note that the expected revenue of selling one unit in resale is exactly $\beta^R(x)$ for the bidder with value x. Therefore, she bids $\beta^R(x)$ for extra units.[22]

Now, consider a deviation for the first bid. Let $z > Y_1 > x$ and the bidder with value x deviates to $(\beta^R(z), \beta^R(x), ..., \beta^R(x))$. The benefit of deviation is:

$$\begin{aligned}
&\Pr(Y_1^{n-1} < z)(x - \beta^R(z)) - \Pr(Y_k^{n-1} < x < Y_1^{n-1})(x - \mathbb{E}[Y_k^{n-1} \mid Y_k^{n-1} < x < Y_1^{n-1}]) \\
&= F_1^{n-1}(z)\left(x - \frac{\int_0^z t f_k^{n-1}(t)dt}{F_1^{n-1}(z)}\right) - \left(x(\Pr(Y_k^{n-1} < x < Y_1^{n-1}) - F_k^{n-1}(x)) + \int_0^x F_k^{n-1}(t)dt\right) \\
&= F_1^{n-1}(z)x - F_1^{(n-1)}(z)z + \int_0^z F_k^{n-1}(t)dt - \int_0^x F_k^{n-1}(t)dt = F_1^{n-1}(z)(x-z) + \int_x^z F_k^{n-1}(t)dt \\
&= F_k^{n-1}(z)(x-z) + \int_x^z F_k^{n-1}(t)dt < 0.
\end{aligned}$$

[21] This is so because when bidder 1 wins two units, he knows, first, the highest losing value is $k-1$ out of $n-1$ opponents, and, second, that in a second price auction he could sell to this person at the k^{th} highest of $n-1$.

[22] It can be easily shown that if she bids less than $\beta^R(x)$, the expected utility gets higher by increasing the bid slightly.

As a result, the k-tuple $(\beta^R(x),\ldots,\beta^R(x))$ is an equilibrium. $\square$

Note that the above two equilibria result in very different allocations after the auction stage. In the first equilibrium, k bidders with the highest values obtain the units, whereas in the second equilibrium the highest-valued bidder obtains all the units. Yet after the resale stage the same allocation is achieved: k bidders with the highest values obtain the units. The auctioneer's revenues in these two equilibria also seem quite different. In the first equilibrium, the revenue is given by $\mathbb{E}\left[\sum_{l=1}^{k}\beta_1\left(Y_l^{n-1}\right)\right]$, whereas in the second one it is given by $k\times\mathbb{E}\left[\beta^R\left(Y_1^{n-1}\right)\right]$.

However, they are equal to each other. We establish this in the following Proposition.

Proposition O.3. *$\beta^n(x)$ and $\beta^R(x)$ are revenue equivalent.*

This result can be obtained by appealing to the revenue equivalence principle. This can be argued by noting that (i) the two equilibria result in the same (efficient allocation), (ii) the expected payment of the bidder with value 0 is 0 under both equilibria, and (iii) the transfers between the bidders in the resale stage aggregate to 0.

As a corollary to Proposition O.3, we establish the following result.

Corollary O.1. *When there are two units for sale and resellers can use reserve prices in the resale market, banning the resale market may strictly decrease the expected revenue in a discriminatory price auction. More specifically, the revenue in the symmetric and monotone equilibria we have found in the model with resale is higher than that of in the model without resale.*

This corollary can be simply obtained by the following observations. First, if there is no resale market and the revenue is equal to the revenue from the symmetric strategies $\left(\beta^R(x),\beta^R(x)\right)$ (Proposition O.3 and the fact that $(\beta_1(x),0)$ is an equilibrium of discriminatory auctions with no resale). Second, with the resale market, the revenue is equal to the revenue from symmetric strategies $(\gamma(x),\gamma(x))$ (Theorem 1 of main text). Third, we have $\gamma(x)>\beta^R(x)$ for all $x\in(0,1]$ because $\gamma(x)$ is the revenue from the optimal auction and $\beta^R(x)$ is the revenue from a second-price auction (when there are $k-1$ buyers who have values smaller than x).

Next, we consider the case in which reserve prices are allowed for both the auction stage and the resale stage. More specifically, we consider a discriminatory price auction with a reserve price $r_1^*=\psi^{-1}(0)$ and wherein a seller in the resale market uses an optimal auction (and hence can use reserve prices). We show the following.

Proposition O.4. *The standard equilibrium $\left(\beta^{RR}(x),0,...,0\right)$ where $\beta^{RR}(x)=\mathbb{E}[\max\{Y_k^{n-1},r_1^*\}\mid Y_k^{n-1}<x]$, is also the equilibrium of the game where the reserve price in the auction stage is r_1^* and any reserve price can be used in the resale stage.*

Proof. We first show that $(\beta^{RR}(x),0,...,0)$ is an equilibrium candidate. Consider a deviation $(\beta^{RR}(z),0,...,0)$. For a bidder with value $x\geq r_1^*$ the expected utility is

$$\Pi(x,z)=(x-\beta^{RR}(z))\Pr(z>Y_k^{n-1})=(x-\beta^{RR}(z))F_k^{n-1}(z).$$

The necessary condition is:

$$\frac{\partial \Pi(x,z)}{\partial z}|_{z=x} = x f_k^{n-1}(x) - \left(\beta^{RR}(x) F_k^{n-1}(x)\right)' = 0.$$

Together with the boundary condition $\beta^{RR}(r) = r$, the unique solution is

$$\beta^{RR}(x) = \mathbb{E}[\max\{Y_k^{n-1}, r_1^*\} \mid Y_k^{n-1} < x].$$

Second we show that the resale market is not beneficial. If the bidder with value x bids $\left(\beta^{RR}(x), b, 0, ..., 0\right)$ where $b > r_1^*$, she will get an additional unit and will sell the unit to the $k-1$ highest value bidder with the price $\max\{r_2^*, Y_k^{n-1}\}$. The expected revenue from resale is $\mathbb{E}[\max\{r_2^*, Y_k^{n-1}\} \mid Y_{k-1}^{n-1} < \left(\beta^{RR}\right)^{-1}(b)]$ which is less than $\mathbb{E}[\max\{r_1^*, Y_k^{n-1}\} \mid Y_k^{n-1} < \left(\beta^{RR}\right)^{-1}(b)] = b$.[23] As a result, the resale market ends up with a negative profit. Following the arguments presented above, we conclude that the general deviation $(b_1, b_2, ..., b_l, 0, ..., 0)$ is not profitable. □

[23] In the bidding stage, the auctioneer choses the optimal reserve price from the interval $[0,1]$ for n bidders. In the resale stage, the auctioneer (or the winner of the bidding stage whose value is x) choses the optimal reserve price from the interval $[0,x]$ for $n-k+1$ bidders. Since $1 \geq x$, it is easy to see that $r_1^* \geq r_2^*$.