SUPPLEMENT TO

"Asymmetric Auctions with Resale"

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Abstract

These notes contain supplementary material associated with the paper "Asymmetric Auctions with Resale."

1 General Distributions

In these notes we relax the assumption that the distribution functions of the two bidders can be stochastically ranked. This assumption was made for expositional and notational ease only. It guarantees that at the resale stage, resale takes place only in one direction: from the weak bidder to the strong bidder.

All of our results generalize, however, to situations where the value distributions of the bidders cross so that the "strong" and "weak" bidder dichotomy is not valid. The only difference this makes is that the direction of trade is endogenously and locally determined. If $\phi_j(b) < \phi_i(b)$, then if j wins the auction, he tries to sell to i. If i wins the auction, however, there are no gains from trade and i does not try to sell to j.

Suppose, as depicted in Figure 1, that $\phi_1(b) < \phi_2(b)$. Then if bidder 1 were to win with a bid of b, he would infer that $V_2 \leq \phi_2(b)$ and since there are potential gains from trade, bidder 1 would make an offer $p(b)$ to bidder 2. On the other hand, if bidder 2 were to win with a bid of b, he would infer that 1's value $V_1 \leq \phi_1(b) < \phi_2(b) = v_2$ and so would not make 1 an offer. But for bid b', $\phi_2(b') < \phi_1(b')$, again as in the figure, the opposite holds. Upon winning with a bid of b', bidder 2 would make an offer $p(b')$ to bidder 1 but not the other way around. The value of b thus uniquely determines the identity of the seller j who sets the price $p(b)$.

In *all* of the proofs, it is possible to make the arguments pointwise once the identities of the seller and the buyer have been determined as above. So we no longer use the "strong" and "weak" bidder terminology, referring to the two bidders merely as bidder 1 and bidder 2, with value distributions F_1 over $[0, a_1]$ and F_2 over $[0, a_2]$, respectively. Without loss of generality, we suppose that $a_1 \geq a_2$.

As in the main body of the paper, we maintain the assumption that both distributions are regular, however.

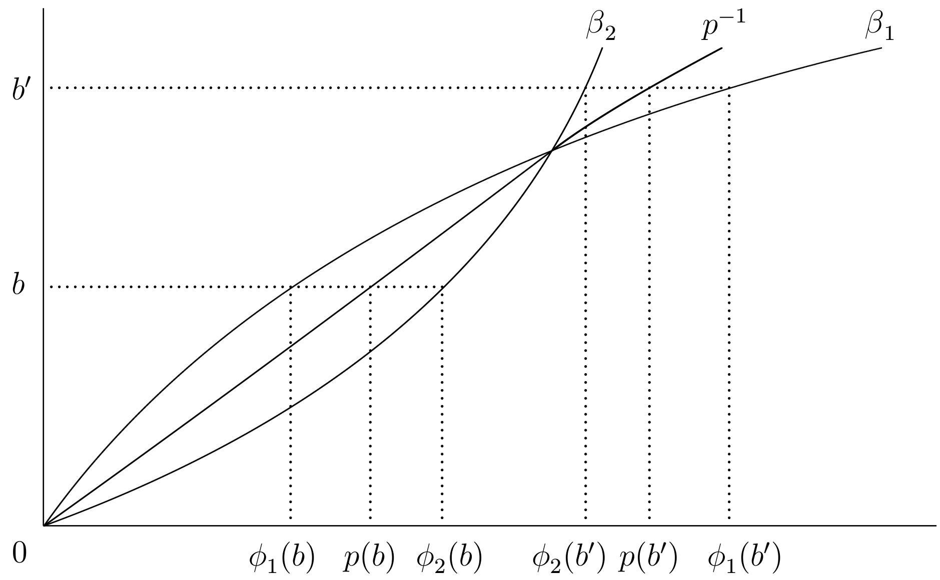

Figure 1: Bidding and Pricing Functions

2 First-Price Auction with Resale

In this section, we study equilibria of first-price auctions with resale.

Boundary conditions First, notice that in any equilibrium with continuous and increasing bidding strategies, we must have $\beta_1(0) = \beta_2(0) = 0$.

To see this, first notice that if $\beta_j(0) > \beta_i(0) \geq 0$, then the price p that bidder j with value 0 would set, must be greater than $\beta_j(0)$. Otherwise, bidder j with value 0 would have a negative payoff. But then bidder i with value p has an incentive to deviate and bid greater than $\beta_j(0)$—by bidding $\beta_j(0)$ or less, he obtains an overall payoff of 0, whereas by bidding slightly higher, he obtains a strictly positive payoff. So we must have, $\beta_s(0) = \beta_w(0) \geq 0$.

Next, suppose that $\beta_i(0) = \beta_j(0) > 0$. Without loss of generality, suppose that there is a sequence $v^n \downarrow 0$ such that $\beta_i(v^n) \geq \beta_j(v^n)$. For n large enough, v^n is less than $\beta_i(0)$ and if bidder i with value v^n wins and does not resell then his net gain is less than zero. If bidder i with value v^n wins and resells to bidder j, the price he will receive is also less than $\beta_i(0)$. This is because if he wins with bid of $\beta_i(v^n)$ then j's value $V_j \leq \beta_j^{-1}\beta_i(v^n)$ and by continuity, this is close to zero, when v^n is close to zero. Thus his gain from winning is less than $\beta_i(0)$ whether or not the object is resold. Moreover, if bidder i with value v^n loses, then $v^n \leq V_j$ and there will be no resale. Thus his payoff from losing is 0. As a result, his overall payoff is negative and it is better for bidder i with value v^n to bid 0. Hence, $\beta_i(0) > 0$ is not possible.

Symmetrization The symmetrization result, Proposition 1, in the main paper, of course, continues to hold. The proof of Proposition 1 made no use of the strong and weak bidder dichotomy. The symmetrizing function F may be obtained from F_1 and F_2 as follows:

Lemma 1 (S) *Given distributions F_1 and F_2, define F as follows: if $F_i(p) \leq F_j(p)$, then*

$$F(p) = F_j\left(p - \frac{F(p) - F_i(p)}{f_i(p)}\right) \tag{1}$$

Then F is a uniquely determined distribution function such that $F_i(p) \leq F(p) \leq F_j(p)$. Moreover, if $F_i(p) < F_j(p)$, then $F_i(p) < F(p) < F_j(p)$.

Proof. Fix a p and note that

$$\Psi(p, q) \equiv F_j\left(p - \frac{q - F_i(p)}{f_i(p)}\right)$$

is a strictly decreasing function of q, where $F_i(p) \leq q \leq 1$.

If $F_i(p) = F_j(p)$, then $F(p) = F_i(p)$ also. If $F_i(p) < F_j(p)$, then $\Psi(p, F_i(p)) > F_i(p)$ and $\Psi(p, F_j(p)) < F_j(p)$. Thus for every p, there exists a unique fixed-point $q \in (F_i(p), F_j(p))$ such that $\Psi(p, q) = q$ and by (1) $F(p) \equiv q$. Thus, $F_i(p) \leq F(p) \leq F_j(p)$ and the inequalities are strict if $F_i(p) < F_j(p)$.

Clearly, $F(0) = 0$. We now argue that F is strictly increasing. It suffices to show that $F(p') < F(p'')$ for $p' < p''$ such that for all $p \in [p', p'']$, $F_i(p) \leq F_j(p)$. The regularity of F_i implies that for all q,

$$p'' - \frac{q - F_i(p'')}{f_i(p'')} > p' - \frac{q - F_i(p')}{f_i(p')}$$

Thus, if $p' < p''$, then for all q, $\Psi(p'', q) > \Psi(p', q)$. This implies that if $\Psi(p', q') = q'$ and $\Psi(p'', q'') = q''$, then $q'' > q'$. So $F(p'') > F(p')$. Finally, note that since $a_1 \geq a_2$, F has support $[0, \overline{p}]$, where $\overline{p}$ satisfies

$$a_2 = \overline{p} - \frac{1 - F_1(\overline{p})}{f_1(\overline{p})}$$

We have thus shown that F is a well-defined distribution function over $[0, \overline{p}]$. ∎

2.1 Existence and Uniqueness of Equilibrium

In this section, we establish that the first-price auction with resale has a pure strategy equilibrium in which each bidder follows a strictly increasing bidding strategy.

The equilibrium is unique in the class of pure strategy equilibria with nondecreasing bidding strategies.

The proof that there is a strictly increasing equilibrium is constructive. Given regular distribution functions F_1 and F_2, construct F as in Lemma 1. Consider a symmetric first-price auction in which each bidder draws values independently from F. In symmetric auctions, it is known that a symmetric equilibrium β exists and is strictly increasing. We will use the equilibrium β to construct equilibrium bidding strategies β_1 and β_2 for the asymmetric first-price auction with resale.

2.1.1 Existence of Equilibrium

The existence of a monotone equilibrium is established in the main paper and again, does not use the "strong" and "weak" dichotomy.

2.1.2 Uniqueness of Equilibrium

In this section we show that the equilibrium constructed in Theorem 1 (in the main body of the paper) is, in fact, the only equilibrium in which bidders follow *nondecreasing* bidding strategies. We first show that it is unique in the class of equilibria with strictly increasing strategies. The proof is completed by showing that if both equilibrium bidding strategies are nondecreasing, then they must be strictly increasing and continuous.

The following proposition is a key step. It shows that the distribution of resale prices in any equilibrium is given by F. Since F is determined without reference to the equilibrium, this shows that the distribution of resale prices is uniquely determined.

Proposition 1 (S) *Suppose ϕ_1 and ϕ_2 are strictly increasing equilibrium inverse bidding strategies. Then F, defined in* (1)*, is the distribution of equilibrium resale prices and for all b, $F(p(b)) = F_j(\phi_j(b))$, $j = 1, 2$.*

Proof. Let the random variable P denote the resale price resulting from ϕ_1 and ϕ_2. Fix a b such that $\phi_j(b) < \phi_i(b)$ and suppose bidder j wins the auction with a bid of b. Then we have.

$$\begin{aligned} \Pr[P \leq p(b)] &= \Pr\left[V_j \leq \phi_j(b)\right] \\ &= F_j\left(\phi_j(b)\right) \end{aligned}$$

Because in equilibrium, for all b, $F_1(\phi_1(b)) = F_2(\phi_2(b))$, it is also the case that $\Pr[P \leq p(b)] = F_i(\phi_i(b))$. And since $p(b)$ is the monopoly resale price when the winning bid is b, it must satisfy the first-order condition

$$\phi_j(b) = p(b) - \frac{F_i(\phi_i(b)) - F_i(p(b))}{f_i(p(b))}$$

So

$$\begin{aligned}\Pr[P \le p(b)] &= F_j\left(\phi_j(b)\right) \\ &= F_j\left(p(b) - \frac{\Pr[P \le p(b)] - F_i(p(b))}{f_i(p(b))}\right)\end{aligned}$$

where we have used the fact that $\Pr[P \le p(b)] = F_i(\phi_i(b))$ also. Now (1) implies that

$$\Pr[P \le p(b)] = F(p(b))$$

and this completes the proof. ■

Since $F(p(b))) = F_j\left(\phi_j(b)\right)$, Proposition 2 implies that

$$\frac{d}{db}(F(p(b))) = \frac{1}{p(b) - b}$$

This means that $p(b)$ satisfies the differential equation characterizing the equilibrium inverse bidding strategy in the symmetric first-price auction in which both bidders draw values independently from the same distribution F. But given F, the equilibrium inverse bidding strategy in a symmetric first-price auction is uniquely determined.

We have thus argued that given F_1 and F_2, the equilibrium distribution of resale prices, F, and the function $p(b)$ are uniquely determined. Now it follows from $F_i(\phi_i(b)) = F(p(b))$ that ϕ_1 and ϕ_2 are uniquely determined. Thus there is only one equilibrium in which bidders follow *strictly increasing* bidding strategies.

Theorem 1 (S) *The first-price auction with resale has a unique equilibrium in the class of equilibria with nondecreasing bidding strategies.*

It has already been established that the equilibrium constructed in Theorem **??** is unique in the class of equilibria with strictly increasing strategies. Here we complete the proof Theorem 1 by showing that every equilibrium in nondecreasing strategies must, in fact, have strictly increasing bidding strategies.

Lemma 2 (S) *If β_1 and β_2 are nondecreasing equilibrium bidding strategies in the first-price auction with resale, then β_1 and β_2 are continuous.*

Proof. Suppose that there exists an $v_i > 0$ such that $\lim_{v \uparrow v_i} \beta_i(v) = b' < b'' = \lim_{v \downarrow v_i} \beta_i(v)$. First, note that in that case bidder j also does not bid between b' and b''; that is, there does not exist an v_j such that $b' < \beta_j(v_j) < b''$. Otherwise, bidder j with value v_j could increase his payoff by decreasing his bid to $\beta_j(v_j) - \varepsilon > b'$. This change does not affect his payoff if he were to lose but increases it by ε if he were to win (which happens with positive probability since $v_i > 0$). Second, note that bidder j bids b' or lower with positive probability. Otherwise, $\lim_{v \to 0} \beta_j(v) \ge b''$ and bidder

j with value v_j close to zero can improve his payoff by reducing his bid to b'. This is because bidder j would gain at least $b'' - b'$ whenever bidder i's value was between 0 and v_i and suffer only a small loss in the cases when bidder i's value is just slightly above v_i. So there does not exist an v_j such that $b' < \beta_j(v_j) < b''$.

Now consider bidder i with a value slightly greater than v_i, say $v_i + \delta$. By reducing his bid from $\beta_i(v_i + \delta) \geq b''$ to b', bidder i could increase his payoff. Once again, this change does not affect his payoff if he were to lose and increases it by at least $b'' - b'$ if he were to win (which happens with positive probability). Thus for δ small enough, bidder i with value $v_i + \delta$ has a profitable deviation.

We have argued that the bidding strategies β_i are continuous at any $v_i > 0$. It remains to argue that they are also continuous at 0.

Suppose that $\lim_{v \to 0} \beta_i(v) = b_0 > 0$ and without loss of generality, suppose that for some small δ, it is the case that for all $v \in (0, \delta)$, $\beta_i(v) \geq \beta_j(v)$. Then we must have that $\lim_{v \to 0} \beta_j(v) = b_0$ also. Otherwise, bidder i with a value close to zero could reduce his bid an improve his payoff. If β_i is increasing in $(0, \delta)$, then the same argument that shows that $\beta_i(0) = 0$ when both strategies are increasing shows that this is impossible. If both β_i and β_j are constant over $(0, \delta)$, then bidder i with value close to zero can improve his payoff by increasing his bid slightly. Thus $\lim_{v \to 0} \beta_i(v) = 0$. ■

Lemma 3 (S) *If β_1 and β_2 are nondecreasing equilibrium bidding strategies in the first-price auction with resale, then β_1 and β_2 are strictly increasing.*

Proof. Suppose that there is an interval $[v', v'']$ such that for all $v_i \in (v', v'')$, $\beta_i(v_i) = b > 0$; that is, bidder i's strategy is constant. Consider bidder j with value v_j such that $\lim_{v \uparrow v_j} \beta_j(v) = b$. For v close to v_j, bidder j can improve his payoff by bidding higher than b. This is because he then wins whenever $V_i \in (v', v'')$ and the loss is arbitrarily small. If $b = 0$, then bidder j with a small value v_j can improve his payoff by reducing his bid. ■

3 Second-Price Auction with Resale

The existence of a robust equilibrium of the second-price auction with resale was established in the main paper.

Uniqueness In a second-price auction (SPA) without resale, it is a dominant strategy for each bidder to bid his value. This, of course, is also a robust equilibrium. But in a SPA there are also *other* robust equilibria.

Example 1 (S) *The values $V_1, V_2 \in [0, 1]$. Bidder 1 bids according to the strategy $\beta_1(v) = 1$ and bidder 2 according to $\beta_2(v) = 0$.*

These strategies, while weakly dominated, nevertheless constitute a robust equilibrium. Thus in the second-price auction without resale, there is a multiplicity of robust equilibria.

When there is resale, however, there is (essentially) a unique robust equilibrium—both bidders bid their values and since the resulting allocation is efficient, there is no resale.

Proposition 2 (S) *Suppose β_1 and β_2 are bidding strategies in a robust equilibrium of the second-price auction with resale. Then for all $v \in [0, a_2]$, $\beta_1(v) = \beta_2(v) = v$.*

The proof that there is a unique robust equilibrium in the second-price auction with resale follows from the lemmas below.

We suppose that equilibrium bidding strategies are right continuous. Since the expected payoff functions of the bidders are continuous in equilibrium, if there is a discontinuity in the bidding strategies, bidders must be indifferent between bidding the right and left limits. Therefore, focusing on equilibria with right-continuous bidding strategies is without loss of generality.

Consider a robust equilibrium of the SPAR: $(\beta_1(\cdot), p_1(\cdot,\cdot), \beta_2(\cdot), p_2(\cdot,\cdot))$.

Lemma 4 (S) *For all v_i and for all $b \leq \beta_i(v_i)$, the set $\{v_j : \beta_j(v_j) = b$ and $v_j > v_i\}$ is either a singleton or empty.*

Proof. Suppose there are two points, say v_j' and v_j'' are in the set $\{v_j : \beta_j(v_j) = b$ and $v_j > v_i\}$. Upon winning, the optimal price $p_i(b, v_i)$ that i will set must then depend on the distribution F_j, contradicting the definition of a robust equilibrium. ■

A simple but important consequence of Lemma 4(S) is that if j loses in the auction, his payoff must be zero. Either bidder i makes no offer to him or makes an offer equal to v_j.

Lemma 5 (S) $\inf_{v_1} \beta_1(v_1) = \inf_{v_2} \beta_2(v_2)$

Proof. Suppose $\inf_{v_1} \beta_1(v_1) < \inf_{v_2} \beta_2(v_2)$. By right continuity, there is an open interval I of values V_1 such that for all $v_1 \in I$, $\beta_1(v_1) < \inf_{v_2} \beta_2(v_2)$. Consider $v_1', v_1'' \in I$ such that $0 < v_1' < v_1''$. For all $v_2 < v_1'$, bidder 2 with value v_2 wins against both v_1' and v_1'' and from Lemma 4(S), offers to sell to both at prices equal to their values. But this means that bidder 1 with value v_1'' is better off by bidding $\beta_1(v_1') \neq \beta_1(v_1'')$. ■

Lemma 6 (S) *For $i = 1, 2$ there exists an v_i^0 such that bidder i with value v_i^0 makes an overall expected payoff of* 0.

Proof. From Lemma 5(S), $\inf_{v_1} \beta_1(v_1) = \inf_{v_2} \beta_2(v_2) = m$, say. Let $v_i^0 = \inf\{v_i : \beta_i(v_i) = m\}$. Since $\beta_i(0) \geq m$, from Lemma 4(S) there is at most one v_j such that $\beta_j(v_j) = m$. This means that bidder i with value v_i^0 wins the object with probability 0. Since the payoff from losing is also 0, his overall payoff is 0. ■

Lemma 7 (S) $\beta_1(v) = \beta_2(v) = v$, *for all* $v \in [0, a_2]$.

Proof. Suppose that $\beta_j(v_j') < v_j'$ for some v_j'. By right continuity, there exists a δ such that for all $v_j \in I = [v_j', v_j' + \delta]$, $\beta_j(v_j) < v_j'$. Thus there exists an $\varepsilon > 0$ such that $v_j' - \beta_j(v_j) \geq \varepsilon$ for all $v_j \in I$. Suppose now that bidder i with value v_i^0 (whose overall profit is zero), bids v_j'. Whenever bidder j's value $v_j \in I$, i will win and make a profit of at least ε. Consider a distribution of values F_j such that $\Pr[V_j \in I]$ is close enough to 1 so that the overall expected profit of v_i^0 is also positive, which is a contradiction. So $\beta_j(v_j') < v_j'$ is impossible.

Similarly, suppose $\beta_j(v_j') > v_j'$ for some v_j'. Again, by right continuity, there exists a δ such that for all $v_j \in I = [v_j', v_j' + \delta]$, $\beta_j(v_j) > v_j' + \delta$. For all $v_i \in I$, if $\beta_i(v_i) > v_j' + \delta$, then bidder i with value v_i makes a negative profit when he faces bidder j with value $V_j \in I$. Consider a distribution of values F_j such that $\Pr[V_j \in I]$ close enough to 1 so that his overall expected profit is also negative, which is a contradiction. This shows that for all $v_i \in I$, $\beta_i(v_i) \leq v_j' + \delta$. Now consider v_i' and v_i'' such that $v_j' < v_i' < v_i'' < v_j' + \delta$. Bidder j with value v_j such that $v_j' < v_j < v_i'$ wins against both v_i' and v_i'' and from Lemma 4(S), offers to sell to both at prices equal to their values. But this means if the distribution of values F_j is such that $\Pr[v_j' < v_j < v_i']$ is close to 1, then bidder i with value v_i'' is better off by bidding $\beta_i(v_i') \neq \beta_i(v_i'')$. ■

If $a_1 > a_2$, then there are robust equilibria in which for $v > a_2$, bidder 1 bids more than v. Since bidder 1 wins for sure in these cases, all such equilibria are outcome equivalent to the one in which for all v, bidder 1 bids his value.

4 Other Resale Mechanisms

Here we discuss how the results may be extended to other resale mechanisms. As discussed in the main paper, in the extensions of the basic model that follow, we assume that *no bids are announced* at the end of the auction. Thus only the identity of the winner is commonly known.

4.1 Monopsony Resale

In this case, the loser of the auction can make an offer to buy the object from the winner.

Suppose that the two bidders follow *continuous* and *strictly increasing* inverse bidding strategies ϕ_1 and ϕ_2, respectively.

We know from Proposition 3 in the main paper that the following characterize the equilibrium bidding when resale is via the monopsony mechanism.

$$\frac{d}{db}\ln F_1(\phi_1(b)) = \frac{1}{r(b)-b} \tag{2}$$
$$\frac{d}{db}\ln F_2(\phi_2(b)) = \frac{1}{r(b)-b} \tag{3}$$

where if $\phi_j(b) \leq \phi_i(b)$,

$$r(b) = \arg\max_r \left[F_j(r) - F_j\left(\phi_j(b)\right)\right](\phi_i(b) - r) \tag{4}$$

It should be noted that although the formulae defining the two are similar, in general, $r(b) \neq p(b)$. This implies of course that the equilibrium bidding strategies when resale is via the monopsony mechanism—that is, the solutions to (2) and (3)—are different from the equilibrium bidding strategies when resale is via the monopoly mechanism. So that there is no ambiguity, let us denote by $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ the equilibrium inverse bidding strategies in a first-price auction in which resale is via monopsony.

The remainder of the analysis parallels that in the case of monopoly *exactly* once the monopoly pricing function $p(b)$ is replaced by the monopsony pricing function $r(b)$. Specifically,

1. As in Proposition 2, the differential equations (2) and (3), together with (4), are both necessary and sufficient for $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ to be an equilibrium.

2. As in Lemma 1(S), F_1 and F_2 uniquely determine a distribution G of monopsony resale prices where if $F_i(r) \leq F_j(r)$, then

$$G(r) = F_i\left(r - \frac{G(r) - F_j(r)}{f_j(r)}\right) \tag{5}$$

   The distribution G has a geometric interpretation similar to that of F in Figure 2. In general, $G \neq F$.

3. As in Theorem 1, there exists an equilibrium with strictly increasing bidding strategies $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ and it is unique in the class of all nondecreasing equilibria.

4. As in Theorem 2, the revenue from a first-price auction with monopsony resale is at least as large as that from a second-price auction with resale.

4.2 Random Proposer Mechanism

In this mechanism, resale takes place as follows. With probability k, the winner of the auction makes a take-it-or-leave-it offer to the loser and with probability $1-k$ the loser makes a take-it-or-leave-it offer to the winner. Either side may decide not to participate, in which case no transaction takes place.

When $k = 1$, this reduces to the monopoly resale mechanism considered earlier. When $k = 0$, it reduces to the monopsony mechanism of the previous subsection.

Once again Proposition 3 in the main paper guarantees that equilibrium bidding strategies when resale is via a k-double auction are characterized by

$$\begin{aligned} \frac{d}{db} \ln F_1 (\phi_1 (b)) &= \frac{1}{t (b) - b} \\ \frac{d}{db} \ln F_2 (\phi_2 (b)) &= \frac{1}{t (b) - b} \end{aligned}$$

where $t (b) = kp (b) + (1 - k) r (b)$ is the *expected* price and $p (b)$ and $r (b)$ are the monopoly and monopsony prices, respectively.

Once again, in general, $p (b) \neq r (b)$ and so for $k \in (0, 1)$, $t (b)$ is distinct from both $p (b)$ and $r (b)$. Thus the equilibrium bidding strategies are now different from both those in the case of monopoly resale and those in the case of monopsony resale. This in turn implies that the pricing functions $p (b)$ and $r (b)$ in the case of a k-double auction are also different from those resulting in the case of a pure monopoly or a pure monopsony.

Define $\widehat{\varphi}_1, \widehat{\varphi}_2$ to be the equilibrium inverse bidding strategies in a first-price auction in which resale is via the random proposer mechanism. Similarly, define $\widehat{p} (b)$ and $\widehat{r} (b)$ to be the monopoly and monopsony pricing functions in the random proposer mechanism. Thus $\widehat{\varphi}_1, \widehat{\varphi}_2, \widehat{p}, \widehat{r}$ are the simultaneous solutions to the two differential equations above together with with the first-order conditions characterizing the monopoly and monopsony pricing functions. For notational consistency, let $\widehat{t} (b) = k\widehat{p} (b) + (1 - k)\widehat{r} (b)$.

4.2.1 Distribution of $\widehat{t} (b)$

We wish to determine the distribution of the random variable $\widehat{t} (b)$. As before, the differential equations imply once again that the distributions of bids of the two bidders are identical; that is, $F_1(\widehat{\varphi}_1 (b)) = F_2(\widehat{\varphi}_2 (b))$. Let $A (b) \equiv F_j(\widehat{\varphi}_j (b))$ be the common distribution of bids.

Assume, without loss of generality, that $\widehat{\varphi}_j (b) < \widehat{\varphi}_i (b)$ and notice that $\widehat{p} (b), \widehat{r} (b)$ and $\widehat{t} (b)$ satisfy:

$$\widehat{\varphi}_j (b) = \widehat{p} (b) - \frac{F_i(\widehat{\varphi}_i (b)) - F_i (\widehat{p} (b))}{f_i (\widehat{p} (b))} \tag{6}$$

$$\widehat{\varphi}_i (b) = \widehat{r} (b) - \frac{F_j(\widehat{\varphi}_j (b)) - F_j (\widehat{r} (b))}{f_j (\widehat{r} (b))} \tag{7}$$

$$\widehat{t} (b) = k\widehat{p} (b) + (1 - k) \widehat{r} (b) \tag{8}$$

Define distributions $\widehat{F}, \widehat{G}$ and L as follows

$$\widehat{F} (\widehat{p} (b)) = \widehat{G} (\widehat{r} (b)) = L \left(\widehat{t} (b)\right) = A (b) \tag{9}$$

F, G and L are well defined since $\widehat{p}(\cdot)$, $\widehat{r}(\cdot)$ and $\widehat{t}(\cdot)$ are increasing functions.

Note that (6) and (7) then can be rewritten as follows:

$$\widehat{F}(p) = F_j\left(p - \frac{\widehat{F}(p) - F_i(p)}{f_i(p)}\right) \tag{10}$$

$$\widehat{G}(r) = F_i\left(r - \frac{\widehat{G}(r) - F_j(r)}{f_j(r)}\right) \tag{11}$$

But now notice that since (10) is the same as (1) and F was uniquely determined there, $\widehat{F} = F$, the distribution of prices when resale is via monopoly. Similarly, (11) is the same as (5) and so $\widehat{G} = G$, the distribution of prices when resale is via monopsony. We thus obtain the conclusion that even though, in general, $\widehat{p} \neq p$ and $\widehat{\varphi}_i \neq \phi_i$, we have

$$F(\widehat{p}(b)) = F_i(\widehat{\varphi}_i(b)) \text{ and } F(p(b)) = F_i(\phi_i(b))$$

Similarly, even though, in general, $\widehat{r} \neq r$ and $\widehat{\varphi}_i \neq \widetilde{\varphi}_i$,

$$G(\widehat{r}(b)) = F_i(\widehat{\varphi}_i(b)) \text{ and } G(r(b)) = F_i(\widetilde{\varphi}_i(b))$$

Finally, since $F = \widehat{F}$ and $G = \widehat{G}$, (9) implies that for all b,

$$L^{-1}(A(b)) = kF^{-1}(A(b)) + (1-k)G^{-1}(A(b))$$

Since $A(b)$ varies from 0 to 1, we obtain

$$L^{-1}(q) = kF^{-1}(q) + (1-k)G^{-1}(q) \tag{12}$$

for all $q \in [0,1]$, where F and G satisfy (10) and (11), respectively.

4.2.2 Revenue from FPA with Resale via the Random Propser Mechanism

Lemma 8 (S)

$$\int_0^{\overline{t}} (1-L(t))^2\,dt = k\int_0^{\overline{p}} (1-F(p))^2\,dp + (1-k)\int_0^{\overline{r}} (1-G(r))^2\,dr$$

Proof. Let $L(t) = q$, then from (12) we obtain

$$t = L^{-1}(q) = kF^{-1}(q) + (1-k)G^{-1}(q)$$

so that

$$dt = \left(\frac{k}{f(F^{-1}(q))} + \frac{1-k}{g(G^{-1}(q))}\right)dq$$

By changing the variable of integration from $t \in \left[0, \bar{t}\right]$ to $q = L(t) \in [0,1]$ we obtain

$$\begin{aligned}
\int_0^{\bar{t}} (1 - L(t))^2 \, dt &= \int_0^1 (1-q)^2 \left(\frac{k}{f(F^{-1}(q))} + \frac{1-k}{g(G^{-1}(q))} \right) dq \\
&= k \int_0^1 \frac{(1-q)^2}{f(F^{-1}(q))} dq + (1-k) \int_0^1 \frac{(1-q)^2}{g(G^{-1}(q))} dq
\end{aligned}$$

Changing the variables again from $q \in [0,1]$ to $p = F^{-1}(q) \in [0, \overline{p}]$ in the first integral and $r = G^{-1}(q) \in [0, \overline{r}]$ in the second, we obtain the required equality. ■

The expected revenue of the original seller when resale is via the k-double auction is given by

$$\int_0^{\bar{t}} (1 - L(t))^2 \, dt$$

where L satisfies (12).

We have already shown in Theorem 2 that

$$\int_0^{\overline{p}} (1 - F(p))^2 \, dp \geq \int_0^{a_2} (1 - F_1(v))(1 - F_2(v)) \, dv$$

and it is similarly the case that

$$\int_0^{\overline{r}} (1 - G(r))^2 \, dr \geq \int_0^{a_2} (1 - F_1(v))(1 - F_2(v)) \, dv$$

Lemma 8 now implies that the revenue from a first-price auction with resale via the random proposer mechanism is also greater than or equal to the revenue from a second-price auction.