The Universe in Six or Seven Numbers

What we assume

Our basic picture for how the universe evolves over large distances or through large stretches of time can be summarized in six numbers—or seven, if we do not make any assumptions about its overall spatial geometry. At first it might seem impressive that so much can be described so succinctly; but this simplicity actually reflects our ignorance of many of the details of our universe and we should be quite glad to add to this list.

Our best explanation for the mutual influence between gravity and the other ingredients of the universe is provided by general relativity. In this theory we describe the behavior of space-time through a metric $ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu$, whose dynamics at each point are determined by the energy and momentum of whatever else happens to be there. At large distances and at early times, the universe appears to be extremely smooth, with hardly any spatial variation at any particular time, the only change being a uniform expansion of space. This geometry can be described through the following metric,

$$ds^2 = dt^2 - a^2(t)\,d\omega_k^2. \tag{1}$$

The spatial part of this metric depends on whether the overall geometry of the universe is hyperbolic ($k=-1$), flat ($k=0$), or spherical ($k=1$). For each of these cases we have in turn,

$$\begin{aligned} d\omega_{-1}^2 &= d\chi^2 + \sinh^2\chi\,d\theta^2 + \sinh^2\chi\sin^2\theta\,d\phi^2 \\ d\omega_0^2 &= dx^2 + dy^2 + dz^2 = d\vec{x}\cdot d\vec{x} \\ d\omega_1^2 &= d\chi^2 + \sin^2\chi\,d\theta^2 + \sin^2\chi\sin^2\theta\,d\phi^2. \end{aligned} \tag{2}$$

To all appearances, the universe seems to be spatially flat, so often the $k=0$ form for the metric is assumed from the beginning,

$$ds^2 = dt^2 - a^2(t)\,d\vec{x}\cdot d\vec{x}. \tag{3}$$

The universe also appears to have been always expanding, so the *scale factor* $a(t)$ was smaller in the past. If we denote the time today by t_0, then we can normalize our coordinates so that $a(t_0)=1$, and as a consequence, $a(t)<1$ for all earlier stages of the universe.

The evolution of this scale factor is influenced by what the universe contains. We shall treat these ingredients as approximately uniform fluids, whose total energy density is $\rho(t)$. One of the components of the Einstein equation, called the *Friedmann equation*, determines how the scale factor evolves in terms of the total density, and the curvature, if we do not impose $k = 0$ from the start,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(a) - \frac{k}{a^2}. \tag{4}$$

We have written the density as depending on $a(t)$ since any particular component is diluted as the universe expands. For example, if the density of matter in the universe today is ρ_m, then it would have been more closely packed at an earlier time, since the volume it occupied would have been smaller by a factor of $a^3(t)$. The density of radiation will be similarly enhanced by this decrease in volume; moreover, since light is stretched by the expansion of the universe, its wavelength must also have been shorter in the past. The energy of a photon is inversely proportional to its wavelength, $E_\gamma = h/\lambda(t)$, so it is enhanced by a factor of $1/a(t)$, $E_\gamma(t) = E_\gamma(t_0)/a(t)$, compared with its value today. Accounting for the change in the volume too, the energy density of radiation thus scales as $1/a^4(t)$.

Finally, we ought to include a third fluid which does not scale as either matter or radiation. It has been added to the cosmological picture only recently, to account for the apparent acceleration in the expansion of the universe which seems to have begun about five billion years ago. Since we know little about this substance, we shall not completely fix how it scales by including a parameter w_{eff} which is to be determined by observations,

$$\rho_\Lambda(a) = \frac{\rho_\Lambda}{a^{3(1+w_{\text{eff}}(a))}}, \tag{5}$$

where ρ_Λ is the density of this stuff today. To be still yet more general, we have not even assumed that the scaling is fixed over time, but instead we have allowed it to vary, $w_{\text{eff}}(a(t))$. The only requirement on w_{eff} is that it should produce an accelerating scale factor today, which occurs whenever $w_{\text{eff}}(a) < -\frac{1}{3}$. For the special case where $w_{\text{eff}} = -1$, the energy density of this fluid remains constant over time.

Putting all of these ingredients together, the Friedmann equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left[\frac{\rho_m}{a^3} + \frac{\rho_r}{a^4} + \frac{\rho_\Lambda}{a^{3(1+w_{\text{eff}}(a))}}\right] - \frac{k}{a^2}. \tag{6}$$

The expansion rate is often called the *Hubble scale* and is defined by

$$H(a) \equiv \frac{\dot{a}}{a}. \tag{7}$$

If we denote the value of the Hubble scale today by H_0, then we can redefine each of the densities as a fractional contribution to the total density of the universe today,

$$\Omega_X \equiv \frac{8\pi G}{3H_0^2}\rho_X. \tag{8}$$

Even though the effect of the curvature is not produced by an actual fluid, we can also describe its fractional effect on the evolution by defining

$$\Omega_k \equiv -\frac{k}{H_0^2}. \tag{9}$$

Before writing the final form of the Friedmann equation in terms of these new parameters, we should divide the matter into two contributions—one for the ordinary atoms, or baryonic matter Ω_b, and the other for the dark matter, Ω_c. The subscript refers to the fact that the dark matter is assumed to be *cold*, that is, nonrelativistic. Although both have the same effect on the expansion of the universe, they can be readily distinguished from each other and measured separately. The expansion of the universe is then given by the following equation,

$$H^2(a) = H_0^2\left[\frac{\Omega_b}{a^3}+\frac{\Omega_c}{a^3}+\frac{\Omega_r}{a^4}+\frac{\Omega_\Lambda}{a^{3(1+w_{\rm eff}(a))}}+\frac{\Omega_k}{a^2}\right]. \tag{10}$$

Not all of these parameters are independent; if we evaluate this equation today, $t = t_0$, then we have the following sum rule,

$$\Omega_b+\Omega_c+\Omega_r+\Omega_\Lambda+\Omega_k = 1. \tag{11}$$

One of the favored current models assumes that the new fluid responsible for the accelerated expansion is simply a constant vacuum energy, which as we saw corresponds to setting $w_{\rm eff} = -1$. Sometimes it is further assumed that the universe is in fact spatially flat, so that $\Omega_k = 0$. In this case, we need only to measure four numbers to understand the expansion of the universe—at least its spatially independent part—over all the past that we can observe,

$$\{H_0, \Omega_b, \Omega_c, \Omega_\Lambda\}; \tag{12}$$

if we relax the assumption of spatial flatness, then we can add Ω_k to this list.

So far we have been speaking of the universe as though it only varied with time, which is certainly not true today and which, even if the amount of inhomogeneity grew over time, could not have been entirely true even at very early times. Therefore we should include a tiny spatially dependent piece in the metric. Restricting to the flat universe, $k = 0$, and introducing a *conformal time* η by allowing time to expand along with space, $d\eta = a^{-1}(t)\,dt$, then we can write the metric as

$$g_{\mu\nu} = a^2(\eta)\,\eta_{\mu\nu} + \delta g_{\mu\nu}(\eta(t),\vec{x}), \tag{13}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. Nominally, it would seem that we have greatly enlarged the number of parameters that we must first specify to understand the evolution of the universe. Since $\delta g_{\mu\nu}(t,\vec{x})$ is a symmetric tensor, it contains ten independent functions. The background is still invariant under spatial translations and rotations, so we can classify these ten functions by how they transform under these symmetries. Four transform as scalar functions, four more as the components of two divergenceless 3-vectors, and two as the components of a traceless, divergenceless spin-two tensor. However, not all of these functions have a physical meaning; some are only the artifacts of how we have chosen our coordinates.

The most important components are the scalar perturbations. They correspond to regions where the overall gravitational pull is slightly stronger or slightly weaker than at others. The matter in the universe will be attracted to the former; over time, as more and more matter accumulates in a region, stars or galaxies or clusters of galaxies can form there. Among the four scalar components of $\delta g_{\mu\nu}$, only two are independent under small changes of coordinates and so correspond to real physical effects. Of these two, one can usually be removed by one of the equations of motion—one of components of the Einstein equations—leaving a single scalar function. We shall can it $\mathscr{R}(t,\vec{x})$ to suggest a small spatially dependent variation in the scalar curvature.

Since $\mathscr{R}(t,\vec{x})$ is rather small empirically in the early universe, a useful method from describing it is to break it up into moments, [1]

$$\langle \mathscr{R}(t,\vec{x}_1)\mathscr{R}(t,\vec{x}_2)\cdots\mathscr{R}(t,\vec{x}_n)\rangle. \tag{14}$$

If we could measure all of these moments, then we could completely reconstruct $\mathscr{R}(t,\vec{x})$. But since $\mathscr{R}(t,\vec{x})$ is very small, most of these moments, those with many factors of $\mathscr{R}(t,\vec{x})$, will be impossible to measure with any precision, so we can approximate the scalar perturbation by measuring just the lowest-order moments. The mean fluctuation is assumed to vanish, $\langle \mathscr{R}(t,\vec{x})\rangle = 0$, since any nonvanishing result could always be cancelled by adding a spatially constant function, which presumably we should have already included in the scale factor, $a(t)$. Therefore, the first nonvanishing moment is the two-point function, which we write in terms of its Fourier transform,

$$\langle \mathscr{R}(t,\vec{x})\mathscr{R}(t,\vec{y})\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\, \frac{2\pi^2}{k^3}\Delta^2_{\mathscr{R}}(k). \tag{15}$$

The function $\Delta^2_{\mathscr{R}}(k)$ is called the *power spectrum*. It is typically assumed that this function scales as some power of the spatial momentum, k, so that once we

[1] The notation $\langle\cdots\rangle$ refers to spatially averaging the function, or in the case of a quantum operator, taking the expectation value in a state. In the former case we have in the $n = 2$ case, $\langle \mathscr{R}(t,\vec{x})\mathscr{R}(t,\vec{y})\rangle = \int d^3\vec{r}\,\mathscr{R}(t,\vec{x}+\vec{r})\mathscr{R}(t,\vec{y}+\vec{r})$, for example.

have chosen some particular value k_0 at which to normalize it, it can be written as

$$\Delta^2_{\mathscr{R}}(k) = \Delta^2_{\mathscr{R}}(k_0)\left(\frac{k}{k_0}\right)^{n_s(k_0)-1+\frac{1}{2}\frac{dn_s}{d\ln k}}. \tag{16}$$

Thus, in this model we have reduced the general function $\mathscr{R}(t,\vec{x})$ to just three numbers. The number $n_s = n_s(k_0)$ is called the *tilt* of the power spectrum and its derivative $\frac{dn_s}{d\ln k}$ is called the *running index*. In practice the running index is quite small and so far only the tilt has been observed unambiguously. Therefore, in this model we need measure only two more numbers to understand how the scalar perturbations behave,

$$\left\{\Delta^2_{\mathscr{R}}(k_0), n_s(k_0)\right\}. \tag{17}$$

What we see

Gathering the numbers needed to describe the smooth part of the universe together with those needed to describe the small spatial variations, we arrive at a list of six or seven numbers that specify the basic picture,

$$\left\{H_0, \Omega_b, \Omega_c, \Omega_\Lambda, [\Omega_k], \Delta^2_{\mathscr{R}}(k_0), n_s(k_0)\right\}. \tag{18}$$

In the various cosmological experiments that we shall mention here, not all of these parameters are measured directly. For example, the fractional amounts of baryonic matter and cold dark matter are measured in a form that depends in part on the Hubble scale. And even the Hubble scale is not itself measured directly in these experiments. A different sixth independent parameter, the *optical depth*, is used instead.[2]

The observations that we shall use to constrain these numbers come from three distinct sources:

1. The WMAP satellite, which has been carefully measuring the cosmic microwave background radiation for five years,

2. Observations of the baryon acoustic oscillations [BAO] in the distribution of distant galaxies made by the Sloan Digital Sky Survey and 2dF Galaxy Redshift Survey, and

3. Precise observations of the dimming of distant supernovae [SN] performed by three groups, the Hubble Space Telescope and the SNLS and ESSENCE surveys.

[2]The optical depth, τ, tells the fraction ($e^{-\tau}$) of light that has been absorbed or scattered between us and the time when the universe was reionization by its first stars. Its measured value is $\tau = 0.084 \pm 0.016$.

Combining the measurements of these experiments, we arrive at the following limits on these six parameters,

Parameter	WMAP+BAO+SN (68% CL)	Class
H_0	$70.1 \pm 1.3\,\text{km/s/Mpc}$	derived
Ω_b	$4.62 \pm 0.15\%$	derived
Ω_c	$23.3 \pm 1.3\%$	derived
Ω_Λ	$72.1 \pm 1.5\%$	primary
$\Delta^2_{\mathscr{R}}(k_0)$	$(2.457^{+0.092}_{-0.093}) \times 10^{-9}$	primary
$n_s(k_0)$	$0.960^{+0.014}_{-0.013}$	primary

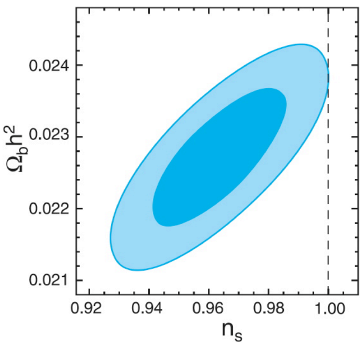
where $\Delta^2_{\mathscr{R}}(k_0)$ and $n_s(k_0)$ are evaluated at $k_0 = 2$ Gpc^{-1}. These values were fit for a universe without any global spatial curvature, $\Omega_k = 0$. If we relax this requirement, we obtain the following bounds on its fractional effect

$$-0.0175 < \Omega_k < 0.0085 \qquad\qquad 95\%\text{CL}. \tag{19}$$

Writing the bounds on these numbers separately as above disguises some of the degeneracies among them. For example, if we look at the tilt of the power spectrum n_s simultaneously with the baryon fraction, normalized in terms of the Hubble scale,

$$h \equiv \frac{H_0}{100\,\text{km/s/Mpc}}, \tag{20}$$

we observe that the difference between the measured value of n_s and 1 is not quite as dramatic as the limits in the table suggest.



Finally we ought to mention one more derived quantity, the age of the universe, which is not independent of the others, since we can determine it by simply evolving the Friedmann equation from the time when $a(t) = 0$ until today, t_0.

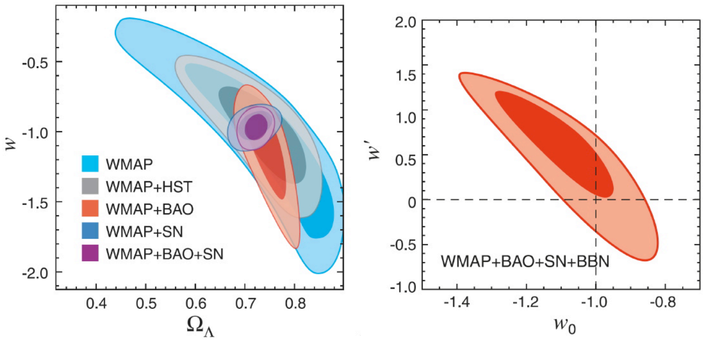
Parameter	WMAP+BAO+SN (68% CL)	Class
t_0	$13.73 \pm 0.12\,\text{Gyr}$	derived

And what comes next

As we mentioned at the beginning, the small number of quantities that have been measured so far reflects our ignorance of many of the finer details of our universe. While introducing the standard cosmological picture, we met a few other parameters that have not been detected yet. For example, there are the running index, which tell how the power spectrum might additionally change with scales, and the function $w_{\rm eff}(a)$ associated with the mysterious ingredient responsible for the current acceleration in the expansion of the universe. As indicated, this function $w_{\rm eff}(a)$ might not be constant over time, and it is sometimes hoped that both its current value and its first derivative might be measured in the relatively near future,[3]

$$w_{\rm eff}(a) = w_0 + \frac{1-a}{a} w' + \cdots, \tag{21}$$

although there very well may be much better ways of modeling how $w_{\rm eff}(a)$ changes over time. Some of the current observational constraints on these parameters are shown in the following figures



The spatial dependence of the metric is still modeled very crudely. Each of the moments tells us a little more about the true form of the scalar perturbations, $\mathscr{R}(t,\vec{x})$. With precise enough measurements we might eventually observe the three-point function as well,

$$\langle \mathscr{R}(t,\vec{x})\mathscr{R}(t,\vec{y})\mathscr{R}(t,\vec{z}) \rangle, \tag{22}$$

which would reveal whether the scalar perturbations contain any *non-Gaussian* structures.

[3] The expansion corresponds to a Taylor series in the *redshift* z, defined by $1+z = \frac{1}{a}$.

So far we have only concentrated on the scalar part of the perturbations, since it is the most easily measured and it is the most important for understanding the growth of the structures in the universe. However, the spatially dependent part of the metric, $\delta g_{\mu\nu}(t,\vec{x})$ can contain spin-two tensor fluctuations as well,

$$\delta g_{\mu\nu}(t,\vec{x})\, dx^\mu dx^\nu = a^2(t)\, h_{ij}(t,\vec{x})\, dx^i dx^j. \tag{23}$$

where i and j run only over the spatial dimensions $(i,j=1,2,3)$. These perturbations $h_{ij}(t,\vec{x})$ are *gravity waves*. To remove any vector or scalar components from h_{ij} we require that it satisfy

$$\partial_i h^i{}_j = 0 \quad \text{and} \quad h^i{}_i = 0, \tag{24}$$

which leaves only two independent degrees of freedom in h_{ij}. We can then define a power spectrum for the tensor perturbations just as we did earlier for the scalar ones,

$$\langle h_{ij}(t,\vec{x}) h^{ij}(t,\vec{y})\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\, \frac{2\pi^2}{k^3} \Delta_h^2(k), \tag{25}$$

and hope that its power spectrum can be reasonably approximated by a simple power law too

$$\Delta_h^2(k) = \Delta_h^2(k_0)\left(\frac{k}{k_0}\right)^{n_t(k_0)}. \tag{26}$$

In principle we could have included a running index here as well, but since the tensor perturbations have not been observed yet, the tilt n_t should be quite sufficient for now. The bounds on the amplitude $\Delta_h^2(k_0)$ are often not stated directly, but are instead described as a bound on the *scalar to tensor ratio*, r, defined by

$$r \equiv \frac{\Delta_h^2(k_0)}{\Delta_{\mathscr{R}}^2(k_0)}. \tag{27}$$

For the WMAP experiment, for example, $k_0 = 2\ \text{Gpc}^{-1}$ is chosen.

So with enough time, and the aid of new experiments, we might be able to add several more numbers to this minimal list needed to describe our universe. We have mentioned a few these parameters in the course of introducing those that have already been measured, such as

$$\left\{\frac{dn_s}{d\ln k}, w_0, w', r, n_t, \ldots\right\}, \tag{28}$$

together with the numbers needed to describe the non-Gaussianities in the scalar spatial variations of the metric, $\mathscr{R}(t,\vec{x})$—though even this list is hardly exhaustive. The current bounds on some of these numbers, derived from the same three sets of experiments, are

Parameter	Assumption	WMAP+BAO+SN
Running index	no gravity waves	$-0.0728 < dn_s/d\ln k < 0.0087$
$w_{\rm eff}$	constant $w_{\rm eff}$	$-1.11 < w < -0.86$
	evolving $w_{\rm eff}$	$-1.38 < w_0 < -0.86$
Gravity waves	no running index	$r < 0.20$
Neutrino mass		$\sum m_\nu < 0.61$ eV
Neutrino species		$N_{\rm eff} = 4.4 \pm 1.5 (68\%)$

Before concluding, we shall mention one last constraint that these cosmological measurements have placed the neutrino masses. Although we have treated all of the more familiar ingredients of our universe as being either non-relativistic matter or radiation, some particles do not readily fall into either category, at least during the earlier stages of the universe. For neutrinos to have had a relatively negligible effect on the pattens in the cosmic microwave background radiation, they need to have been still relativistic. This requirement constrains how heavy they can be. The mass *differences* between pairs of neutrinos has already been determined by other experiments. Combining them with the results of the cosmological experiments limits the total mass of all the different types of neutrinos to be no more than 0.61 eV. The total number of light neutrino species also affects when the era of matter-radiation occurred and is thereby fixed by cosmological observations to be $N_{\rm eff} = 4.4 \pm 1.5$, agreeing with the value $N_{\rm eff} \approx 3.04$ found in accelerator experiments.

All of the data and figures are taken from the five-year WMAP results provided in E. Komatsu *et al.*, “Five-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation,” astro-ph/0803.0547.