Problem Set 4. Capacitors and symmetries
Electromagnetism and Light

Due: 10 February 2023
at 3:00 pm

Reading: *Sternheim and Kane*, chapter 16, sections 12–13;
Electromagnetism and Optics, chapter IV–V.

Please show all of the necessary steps in solving the following problems. Full credit will only be given for complete solutions.

1. The cell membranes in our bodies are naturally occurring examples of capacitors. The capacitance per unit area of a certain type of axon, which is a part of a nerve cell, is $0.0100\ \mathrm{F\,m^{-2}}$.

a. For a cylindrical axon 0.100 m long and 2.50 μm in radius, find the surface area and the capacitance of this axon.

b. If the potential difference across the membrane is 0.0700 V, how much electric energy is stored in the axon?

2. A parallel plate capacitor has a capacitance of 2.00 μF when the plates are separated by a 1.00 mm vacuum. The capacitor is then connected to a battery that generates a 50.0 V potential difference between the plates.

a. How much charge is on the plates?

b. What is the magnitude of the electric field between the plates?

c. If the capacitor is filled with a material whose dielectric constant is $K = 5.00$, what is the capacitance?

d. With a 50.0 V potential difference again, what is the magnitude of the electric field inside the slab of dielectric between the plates?

3. Consider a pair of concentric spherical shells of radii $a < b$, where the inner shell has a charge of $-Q$ and the outer shell has a charge Q. The charge is spread uniformly over each shell.

a. Calculate the potential difference between the inner and the outer shells.

b. What is the capacitance C of these spheres?

c. Now suppose that the radii of the spheres are nearly equal, $b = a + \ell$, where $\ell \ll a$. Express the capacitance in terms of ℓ and the surface area of a sphere, A, to *leading order* in the $\ell \ll a$ limit. How does your formula compare with the capacitance of a square, parallel-plate capacitor?

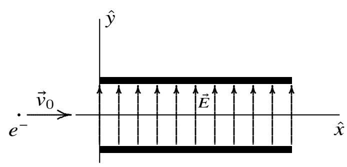
4. An electron moves with an initial horizontal velocity, $\vec{v}_0 = v_0\,\hat{x}$, with $v_0 = 2.00\times10^7\ \mathrm{m\,s^{-1}}$. It passes through a region 20.0 cm long in which there is a uniform electric field $\vec{E} = E\,\hat{y}$, where $E = 5.00\times10^3\ \mathrm{N\,C^{-1}}$.

a. How long with the electron be in the electric field?

b. How far will the electron be deflected up or down when it leaves the field?

c. What is the final velocity of the electron after it has left the electric field?

d. What is the angle between the final and the initial velocity of the electron?



5. Let us examine the effects of some symmetry transformations on the electric field, $\vec{E}(\vec{r})$. For each case, start from a general electric field at $\vec{r}_1$ of the form $\vec{E}(\vec{r}_1) = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$. Imagine a **symmetry** transformation that sends $\vec{r}_1$ to $\vec{r}_2$. If these two points are connected by a symmetry, the electric field at $\vec{r}_2$, $\vec{E}(\vec{r}_2) = \begin{pmatrix} E'_x \\ E'_y \\ E'_z \end{pmatrix}$, must be directly related to the field $\vec{r}_1$—that is, we can determine $\{E'_x, E'_y, E'_z\}$ in terms of $\{E_x, E_y, E_z\}$.

a. If $\vec{r}_2$ is related to $\vec{r}_1$ by a **translation**, what is $\vec{E}(\vec{r}_2)$?

b. If $\vec{r}_2 = r\,\hat{y}$ is related to $\vec{r}_1 = r\,\hat{x}$ by a **rotation** by 90° about the $\hat{z}$-axis, what is $\vec{E}(\vec{r}_2)$?

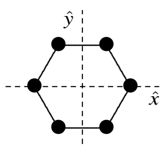
6. Consider a point charge Q at the origin. Through a single rotation about the $\hat{x}$-axis by 180°, we deduced that the electric field at a point on the $\hat{x}$-axis, $\vec{r}_0 = x\,\hat{x}$, must have the form $\vec{E}(\vec{r}_0) = E(x)\,\hat{x}$.

a. Which of the **reflection** symmetries of a point charge at the origin leave a point $\vec{r}_0 = x\,\hat{x}$ on the $\hat{x}$-axis fixed.

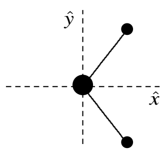
b. Using *only* reflection symmetries, show again that $\vec{E}(\vec{r}_0) = E(x)\,\hat{x}$. Is it possible to do so with just *one* reflection?

7–8. **Symmetric shapes** **[10 points]**

We explained how we could infer properties of the electric field—usually its direction—from the symmetries of the charge distribution producing it. Consider these two shapes:



shape 1



shape 2

You can assume each is symmetric about the $z = 0$ plane. Notice that the first is a simple model for a benzene molecule (a regular hexagon) and the second is a model for a water molecule. Each shape is placed in three-dimensional space.

a. What are the symmetries of shape 1, *i.e.* the transformations of space that would not alter the picture of the charge distribution? [Note: you do not need to describe the transformations in a mathematical form, but you should explain them precisely enough that it is clear what you mean.]

b. What are the symmetries of shape 2?

c. In what direction does the electric field point along the $\hat{x}$-axis for each of the shapes? Explain why. Which of the symmetries did you use?

d. In what direction does the electric field point along the $\hat{z}$-axis for each of the shapes? Explain why. Which of the symmetries did you use?

e. Explain why you know that the electric field in the xy plane has no $\hat{z}$-component for all three of the shapes.

f. Give an example of a line in the xy plane (and not the $\hat{x}$ or $\hat{y}$-axes!) where you know the *exact* direction of the electric field for shape 1.

This problem shows you why symmetries are so important; you can deduce many of the properties about the electric field for each shape without having performed any integrals.