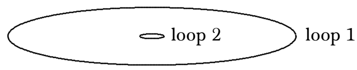
Problem Set 10. Inductors and waves | Due: 21 April 2023
Electromagnetism and Light | at 3:00 pm

Reading: *Sternheim and Kane*, chapter 21, sections 5–6; chapter 23, sections 1–9; chapter 24, sections 1–2;
Lecture Notes, chapter XVI, chapter XVII, chapter XVIII, and chapter XIX, sections 1–2.

Please show all of the necessary steps in solving the following problems. Full credit will only be given for complete solutions.

1. Two concentric circular wires are placed in a plane. One is very, very large, with radius b, and the other is very tiny, with a radius $a \ll b$. A current $I(t)$ flows in a counterclockwise direction in the large circle. Calculate the inductance M associated with a current loop 1 (the big one) that generates an induced emf on a loop 2 (the little one) defined by

$$\mathscr{E}_2 = -M\frac{dI_1}{dt}.$$



Note that in this case the magnetic field of the big loop is approximately constant over the area of the small loop.

2. Suppose that you have two loops of wire. The mutual inductance M_{21} associated with a current loop 1 that is generating an induced emf in a loop 2 is defined by

$$\mathscr{E}_2 = -M_{21}\frac{dI_1}{dt}.$$

The mutual inductance has an interesting property that we shall illustrate through an example. Suppose that our 'loops' are two solenoids, both wrapped around a common cylinder. Both solenoids have the same cross-sectional area A and both have a length ℓ, but one has N_1 coils whereas the other has N_2 coils.

a. Find the mutual inductance M_{21}.
b. Now calculate the opposite mutual inductance M_{12},

$$\mathscr{E}_1 = -M_{12}\frac{dI_2}{dt}.$$

How are M_{12} and M_{21} related? This turns out to be true of *all* mutual inductances.

3. Suppose that a giant solenoid were constructed so that it contained 1000 loops of wire formed into the shape of a cylinder 100 m high and 100 m in radius.

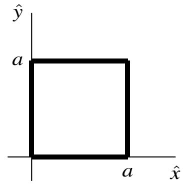
a. Treating it as an ideal solenoid, estimate its inductance, L.
b. How much energy would be stored in this solenoid if a current of 150,000 A were flowing through it? Here we are imagining that it is made of superconducting wires so that the resistance is negligible.
c. How long could the energy stored in the solenoid provide 18.3 GW of power, which is the peak power provided by the Three Gorges Dam.

4. Light is really a **travelling electromagnetic wave**. We shall learn much more about waves in the next part of the course, but for now let us study electric and magnetic fields that are oscillating in the following pattern,

$$\vec{E}(t,x) = E_0 \cos(kx - \omega t)\,\hat{y} \qquad \text{and} \qquad \vec{B}(t,x) = B_0 \cos(kx - \omega t)\,\hat{z}.$$

Show that these fields satisfy Faraday's law, written in the form

$$\oint_C d\vec{\ell}\cdot\vec{E} = -\frac{d}{dt}\int_\Sigma d\vec{A}\cdot\vec{B},$$



for the case of a square loop in the xy plane. How must the amplitudes of the electric and magnetic fields, E_0 and B_0, be related?

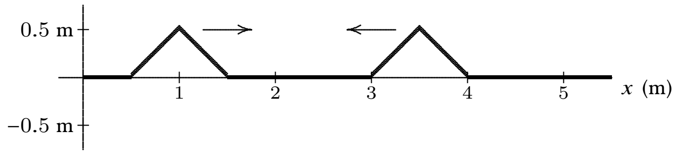
5. A general sinusoid for a given wavelength $\lambda = 2\pi/k$ can always be written in terms of two parameters. For example, in lecture we wrote $A\sin(kx-\phi)$ where A is the amplitude and ϕ is the phase shift. Another way to write a sinusoid is as

$$a\sin(kx) - b\cos(kx).$$

a. What are a and b in terms of A and ϕ?
b. What are A and ϕ in terms of a and b?

6. Two triangular pulses in a rope stretched along the x axis are heading towards each other at speeds of 1.00 $\mathrm{m\,s^{-1}}$, as shown at $t = 0$ s. Sketch the shapes of the rope at

a. $t = 1.00$ s,
b. $t = 1.25$ s, and
c. $t = 1.50$ s.



7. **Combining waves**

a. Express the superposition of two sinusoidal waves travelling in opposite directions with a **relative** phase shift of 2ϕ,

$$y(t,x) = A\sin(kx - \omega t - \phi) + A\sin(kx + \omega t + \phi),$$

as a single standing wave; that is, as a single product of separate sinusoids for the t and the x dependence. Where do the nodes occur? Where do the antinodes occur?

b. In lecture we showed the general form for the interference between two waves. To gain a little more intuition about how waves interfere, consider these two waves,

$$y_1(t,x) = A\sin\big(kx - \omega t - \tfrac{\pi}{8}\big) \qquad \text{and} \qquad y_2(t,x) = A\sin\big(kx - \omega t + \tfrac{\pi}{8}\big).$$

Make a rough sketch of these two waves for $t = 0$ and from $x = 0$ to $x = 3\lambda$, showing their sum on the same graph.

8. Two waves with slightly different frequencies, $y_1(t) = A\cos\omega_1 t$ and $y_2(t) = B\cos\omega_2 t$ are interfering with each other.

a. Rewrite the linear combination of these two waves in terms of sines and cosines whose arguments *only* contain

$$\bar{\omega} = \tfrac{1}{2}\big(\omega_1 + \omega_2\big) \qquad \text{or} \qquad \Delta\omega = \omega_1 - \omega_2.$$

b. What is the expression when $A = B$?
c. If the beat frequency is 4 Hz, and $f_2 = 300$ Hz what are the possible values for $f_1 = \omega_1/2\pi$?