Logical Topology and Axiomatic Cohesion

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Lawvere proposes to continue the following dialogue:

“What is a space?”
“It is an object of a category of spaces.”
“Then what is a category of spaces?”

Lawvere’s *wu wei* axiomatization of “space”: modalities that remove all “spatial cohesion” in three different ways.

- ♯: whose modal types are the codiscrete spaces.
- ♭: whose modal types are the discrete spaces.
- $\int$: whose modal types are the discrete spaces (but whose action is different).
Models of Cohesion

Some *gros topos* of interest are cohesive toposes:

- **Continuous Sets** as in Shulman’s *Real Cohesion*.
- **Dubuc’s Topos** and **Formal Smooth Sets** as in Synthetic Differential Geometry and Schreiber’s *Differential Cohesion*.
- **Menni’s Topos** (similar to the big *Zariski Topos*) as in algebraic geometry.*

In all of these models, there are suitably nice spaces

- continuous manifolds,
- smooth manifolds,
- (suitable) schemes,

which have topologies (via open sets) on their underlying sets.
Penon’s Logical Topology

In his thesis, Penon defined a **Logical Topology** held by any type.

**Definition (Penon)**

A subtype $U : A \to \text{Prop}$ is **logically open** if

- For all $x, y : A$ with $x$ in $U$, either $x \neq y$ or $y$ is in $U$.

Penon and Dubuc proved that in the three examples

- **Continuous Sets**: Logical opens on continuous manifolds are $\epsilon$-ball opens.
- **Dubuc’s Topos**: Logical opens on smooth manifolds are $\epsilon$-ball opens.
- **Zariski Topos**: Logical opens on (suitable) separable schemes are Zariski opens.
Motivating Question:

*How does the logical topology on a type compare with its cohesion?*

We will see two glimpses today:

- The path connected components $\int_0 A$ (defined through cohesion) are the same as the logically connected components of $A$.
- A set is **Leibnizian** (defined through cohesion) if and only if it is de Morgan (a logical notion).
Cohesive Type Theory Refresher

In his *Real Cohesion*, Shulman gave a type theory for axiomatic cohesion. Cohesive type theory uses two kinds of variables:

- Cohesive variables, which vary “continuously”.
- Crisp variables, which vary “discontinuously”.

Following Shulman, we assume the following:

**Axiom (LEM)**

If $P :: \text{Prop}$ is a crisp proposition, then either $P$ or $\neg P$ holds.

*Every discontinuous proposition is either true or false.*
Cohesive Type Theory Refresher

We will also assume that $\int$ is given by nullifying some “basic contractible space(s)”.

Axiom (Punctual Local Contractibility)

There is a type $\mathbb{A} :: \textbf{Type}$ such that:

- A crisp type $X$ is discrete if and only if it is homotopical – the inclusion of constants $X \to (\mathbb{A} \to X)$ is an equivalence, and
- There is a point $0 :: \mathbb{A}$ in each of these types.

We can consider a map $\gamma : \mathbb{A} \to X$ to be a path in $X$.

- This means that $\int A$ is the homotopy type (or fundamental $\infty$-groupoid) of $A$, considered as a discrete type.
- And, therefore,

$$\int_0 A \equiv \parallel \int A \parallel_0$$

is the set of path connected components of $A$. 
Path components $\equiv$ Connected components?

- So,
  \[ \mathcal{P}_0 A \equiv ||A||_0 \]
  is the set of path connected components of $A$.
- Is it also the set of *logical* connected components of $A$?
The Powerset of a Type

Definition

Given a type $A$, its powerset $\mathcal{P} A : \equiv A \to \text{Prop}$ is the set of propositions depending on an $a : A$. The order on subtypes is given by:

$$P \subseteq Q : \equiv \forall a. \, Pa \Rightarrow Qa$$

We define the usual operations on subtypes point-wise:

$$P \cap Q : \equiv \lambda a. \, Pa \land Qa$$
$$P \cup Q : \equiv \lambda a. \, Pa \lor Qa$$
$$\neg P : \equiv \lambda a. \, \neg Pa$$
Logical Connected Components

Definition

1. A subtype $U : \mathcal{P}A$ is *merely inhabited* if there is merely an $a : A$ such that $Ua$.

2. A subtype $U : \mathcal{P}A$ is *detachable* if for all $a : A$, $Ua$ or $\neg Ua$.

3. A subtype $U : \mathcal{P}A$ is *logically connected* if for all $P : \mathcal{P}A$, if $U \subseteq P \cup \neg P$, then $U \subseteq P$ or $U \subseteq \neg P$.

Definition

A subtype $U : \mathcal{P}A$ is a *logical connected component* if it is merely inhabited, detachable, and logically connected.

Lemma

If $U$ and $V$ are logical connected components of $A$, and $U \cap V$ is non-empty, then $U = V$. 
\( \int_0 \) gives the Logical Connected Components

We let \( \int_0 A \equiv \| \int A \|_0 \), and \( \sigma_0 : A \to \int_0 A \) be its unit.

**Lemma**

For any type \( A \) and any \( u : \int_0 A \), the proposition \( \sigma_0^* u \equiv \lambda a. \sigma_0 a = u \) is a logical connected component of \( A \).

**Proof.**

- \( \sigma_0^* u \) is merely inhabited because \( \sigma_0 \) is merely surjective (PLC).
- Since \( \int_0 A \) is a discrete set, it has decideable equality (LEM). Therefore, \( \sigma_0^* u \) is detachable.
- If \( \sigma_0^* u \subseteq P \cup \neg P \), then we can define \( \bar{P} : (a : A) \times \sigma_0^* u(a) \to \{0, 1\} \) by cases. But \( (a : A) \times \sigma_0^* u(a) \equiv \text{fib}_0(u) \) and so is \( \int_0 \)-connected; therefore, \( \bar{P} \) is constant, and \( \sigma_0^* u \subseteq P \) or \( \sigma_0^* u \subseteq \neg P \).
\[ \int_0 \] gives the Logical Connected Components

**Theorem**

For a type \( A \), the map \( \sigma_0^* \) gives an equivalence between \( \int_0 A \) and the set of logical connected components of \( A \).
Infinitesimals and Double Negation

In his paper *Infinitesimaux et Intuitionisme*, Penon makes the following claims:

**Proposition (Kock)**

In the big Zariski or étale topos, with $\mathbb{A}$ the affine line,

$$\neg\neg\{0\} = \text{Spec}(\mathbb{Z}[[t]]) = \{a : \mathbb{A} \mid \exists n. a^n = 0\}$$

is the set of nilpotent infinitesimals.

**Proposition (Penon)**

In Dubuc’s topos, with $\mathbb{A}$ the sheaf co-represented by $C^\infty(\mathbb{R})$,

$$\neg\neg\{0\} = \uparrow(C^\infty_0(\mathbb{R}))$$

is co-represented by the germs of smooth functions at 0.
Ainsi donc l'écriture

\[ \neg \neg \{ 0 \} = \{ \text{Infinitésimaux} \} \]

est justifiée.
Neighbors and Germs

**Definition**

Let $A : \text{Type}$, and let $a, b : A$. We say $a$ and $b$ are **neighbors** if they are not distinct:

$$a \approx b \equiv \neg \neg (a = b).$$

**Proposition**

The neighboring relation is reflexive, symmetric, and transitive, and is preserved by any function $f : A \to B$.

- For $a : A$, $a \approx a$,
- For $a, b : A$, $a \approx b$ implies $b \approx a$,
- For $a, b, c : A$, $a \approx b$ and $b \approx c$ imply $a \approx c$,
- For $a, b : A$ and $f : A \to B$, if $a \approx b$, then $f(a) \approx f(b)$. 
Neighbors and Germs

Definition

The **neighborhood** $\mathbb{D}_a$ of $a : A$ is the type of all its neighbors:

$$\mathbb{D}_a : \equiv (b : A) \times a \simeq b.$$

The **germ** of $f : A \to B$ at $a : A$ is

$$df_a : \mathbb{D}_a \to \mathbb{D}_{f(a)}$$

$$(d, \_ ) \mapsto (f(d), \_)$$

Proposition

**(Chain rule)** For $f : A \to B$, $g : B \to A$, and $a : A$,

$$d(g \circ f)_a = dg_{f(a)} \circ df_a.$$
Cohesion Refresher

**Theorem (Shulman)**

\[ \# \text{ is } \text{lex}: \text{ for any } x, y : A, \text{ there is an equivalence } (x^\# = y^\#) \simeq #(x = y) \]

such that the following diagram commutes.

\[
\begin{array}{ccc}
  & x^\# = y^\# & \\
 \text{ap}_{(-)^\#} & & \simeq \\
 x = y & \downarrow & \\
 & ( - )^\# & \\
 & \#(x = y) & \\
\end{array}
\]

**Lemma (Shulman)**

For any \( P : \text{Prop} \), \( \#P = \neg \neg P \), and a proposition is codiscrete if and only if it is not-not stable.
Codiscretes and Infinitesimals

Putting these facts together, we get:

**Proposition**

For a set $A$ and points $a, b : A$,

$$ a \approx b \equiv \neg \neg (a = b) \iff \#(a = b) \iff a^\# = b^\# $$

**Corollary**

$0$ is the only crisp infinitesimal.

*In fact, since*

$$ \text{fib}(\_)^\#(x^\#) : \equiv (y : A) \times x^\# = y^\# $$

$$ \simeq (y : A) \times x \approx y \equiv: \mathbb{D}_x $$

we have that all formal discs $\mathbb{D}_x$ are $\#$-connected.
Leibnizian Sets and the Leibniz Core

Definition (Lawvere)

A set $A$ is Leibnizian if $\#\sigma : \#A \to \# \int A$ is an equivalence, where $\sigma : A \to \int A$ is the unit.

For crisp sets, this is equivalent to the points-to-pieces transform $\sigma \circ (-)_b : \♭ A \to \int A$ being an equivalence.

Every piece contains exactly one crisp point.

Definition

The Leibniz core $\mathcal{L} A$ of a crisp set $A$ is the pullback

$$\mathcal{L} A \equiv (a : \♭ A) \times (b : A) \times a_b^\# = b^\#
\simeq (a : \♭ A) \times \mathbb{D}_{a_b}$$
A Set is Leibnizian if and only if it is de Morgan

**Definition**
A type $A$ is *de Morgan* if for all $a, b : A$,

$$ a \approx b \text{ or } a \napprox b. $$

**Theorem**
A set $A$ is Leibnizian if and only if it is de Morgan

Compare with:

**Theorem (Shulman)**
A set $A$ is discrete if and only if it is decidable – that is,

for $a, b : A$, $a = b$ or $a \neq b$. 
Sketching a Proof

**Theorem**

A set $A$ is Leibnizian if and only if it is de Morgan

If $A$ is Leibnizian, then $\#\sigma_0$ is an equivalence as well. For $a, b : A$, either $\sigma_0 a = \sigma_0 b$ or not; therefore, $(\sigma_0 a)^\# = (\sigma_0 b)^\#$ or not. Naturality then gives us that $\#\sigma_0(a^\#) = \#\sigma_0(b^\#)$ or not. But $\#\sigma_0$ is an equivalence, so $a^\# = b^\#$ or not.

On the other hand, if $A$ is de Morgan we can give an inverse to $\#$ by sending $u : \# \int A$ to $x^\#$ where $\sigma x = u^\#$. This is well defined since we can map $y : \text{fib}_\sigma(\sigma x)$ to $\{0, 1\}$ according to whether or not $y \approx x$; this shows that every $y$ in the fiber of $\sigma x$ is its neighbor, and therefore that $y^\# = x^\#$. 
References
