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On Sublinear Inequalities for Mixed Integer Conic Programs

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Abstract This paper studies $\mathcal{K}$-sublinear inequalities, a class of inequalities with strong relations to $\mathcal{K}$-minimal inequalities for disjunctive conic sets. We establish a stronger result on the sufficiency of $\mathcal{K}$-sublinear inequalities. That is, we show that when $\mathcal{K}$ is the nonnegative orthant or the second-order cone, $\mathcal{K}$-sublinear inequalities together with the original conic constraint are *always* sufficient for the closed convex hull description of the associated disjunctive conic set. When $\mathcal{K}$ is the nonnegative orthant, $\mathcal{K}$-sublinear inequalities are tightly connected to functions that generate cuts—so called *cut-generating functions*. In particular, we introduce the concept of relaxed cut-generating functions and show that each $\mathbb{R}^n_+$-sublinear inequality is generated by one of these. We then relate the relaxed cut-generating functions to the usual ones studied in the literature. Recently, under a structural assumption, Cornuéjols, Wolsey and Yıldız established the sufficiency of cut-generating functions in terms of generating all nontrivial valid inequalities of disjunctive sets where the underlying cone is nonnegative orthant. We provide an alternate and straightforward proof of this result under the same assumption as a consequence of the sufficiency of $\mathbb{R}^n_+$-sublinear inequalities and their connection with relaxed cut-generating functions.

Keywords Valid inequalities · Sublinear inequalities · Cut-generating functions · Mixed integer conic programming

1 Introduction

In this paper, we consider *disjunctive conic sets* of the form

$$\mathcal{S}(A,\mathcal{K},\mathcal{B}) := \{x \in E : \ Ax \in \mathcal{B}, \ x \in \mathcal{K}\},$$

where $\mathcal{K} \subset E$ is a *regular* (full-dimensional, closed, convex, and pointed) cone in a finite dimensional Euclidean space $(E, \langle \cdot,\cdot \rangle)$ with inner product $\langle \cdot,\cdot \rangle$, $A : E \to \mathbb{R}^m$ is a linear map, and $\emptyset \neq \mathcal{B} \subseteq \mathbb{R}^m$ is a set of right hand side vectors. Important examples of regular cones include the nonnegative orthant $\mathbb{R}^n_+$, the second-order cone $\mathcal{L}^n := \left\{x \in \mathbb{R}^n : \ x_n \geq \sqrt{x_1^2 + \ldots + x_{n-1}^2}\right\}$, and the positive semidefinite

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cone $\mathcal{S}_+^n := \{x \in \mathbb{R}^{n\times n} : a^T x\ a \geq 0\ \forall a \in \mathbb{R}^n\}$. Without loss of generality, we assume that $\mathcal{B} \neq \emptyset$. We make no structural assumptions on $\mathcal{B}$, in particular, $\mathcal{B}$ may be either finite or infinite. We are interested in studying the structure of valid linear inequalities defining the closed convex hull of $\mathcal{S}(A,\mathcal{K},\mathcal{B})$, denoted by $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. The cases $\mathcal{S}(A,\mathcal{K},\mathcal{B}) = \emptyset$ or $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B})) = \mathcal{K}$ are trivial. Hence, throughout the paper, we often restrict our attention to the interesting cases, where $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$ is nonempty and not equal to $\mathcal{K}$.

The disjunctive conic sets $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ can represent multi-term disjunctions on $\mathcal{K}$ and thus naturally arise in relaxations for Mixed Integer Conic Programs (MICPs) with a regular cone $\mathcal{K}$. Therefore, these sets are instrumental in derivation and analysis of general cutting planes for MICPs (see [21]). For example, such sets cover the simpler setups commonly studied such as the two-term disjunctions or split disjunctions on regular cones or their cross-sections. The particular case of two-term or split disjunctions on $\mathcal{K} = \mathcal{L}^n$ has recently attracted a lot of attention [1,2,7,6,9,12,15, 23,24,25,26,30].

When $\mathcal{K} = \mathbb{R}_+^n$ and $\mathcal{B}$ is a finite set, Johnson [20] initiated the study of $\mathcal{S}(A,\mathbb{R}_+^n,\mathcal{B})$ and introduced and characterized $\mathbb{R}_+^n$-minimal valid linear inequalities for it. Prior to this, minimal inequalities have been studied in a number of setups related to Mixed Integer Linear Programs (MILPs). For example, in the context of group problem, a characterization of minimal inequalities via the so-called subadditivity, symmetry, and periodicity conditions was given in [16,19]. Jeroslow [18] extended this characterization from group theoretic framework to arbitrary MILPs under the assumption that the feasible region of the MILP is bounded. In particular, an explicit characterization of minimal inequalities based on the value functions of MILPs was given in [18]. Blair [10] broadened the results from [18] to the case where the entries of the constraint matrix in MILP are rational. Bachem et al. [3] further enhanced these characterizations by introducing b-complementarity condition for a master group problem induced by an MILP. We refer the readers to [17] for a survey on the literature for subadditive approach to MILP. The developments in these papers are mainly based on the value functions of MILPs and therefore have strong connections to the subadditive strong duality theory for MILPs. This body of earlier work, with the exception of [20], specifically focus on feasible sets of MILPs in a finite dimensional setup [10,17,18,20] or the associated infinite relaxations [3,16,17,19]. As a result, the framework of these papers are related to disjunctive sets $\mathcal{S}(A,\mathbb{R}_+^n,\mathcal{B})$ with a specific $\mathcal{B}$. More recently, Basu et al. [5], Borozan and Cornuéjols [11] and many others have also studied minimal valid inequalities, but in the context of infinite relaxations associated with MILPs. As opposed to the earlier literature, [5,11] studied more general sets where the set $\mathcal{B}$ is composed of general lattice points [11] or lattice points contained in a rational polyhedron [5] and established strong connections between the minimal inequalities and the gauge functions of maximal lattice-free sets.

Kılınç-Karzan [21] followed up on the general and flexible disjunctive framework of [20] by generalizing the study of $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ to general regular cones $\mathcal{K}$ and removing the finiteness assumption on $\mathcal{B}$. In particular, [21] introduced $\mathcal{K}$-minimal inequalities for disjunctive conic sets and studied their structure in detail in terms of their existence, sufficiency, necessary conditions and sufficient conditions for $\mathcal{K}$-minimality. It was shown in [21] that minimality notion of an inequality is a natural result of identifying dominance relations among valid linear inequalities; and therefore, it should be based on the smallest regular cone $\mathcal{K}$ that can be used to describe $\mathcal{S}(A,\mathcal{K},\mathcal{B})$. This point is important even in the case of a disjunctive set associated with an MILP (see [21, Section 2.2 and Remark 7]); yet it has been overlooked in all of the papers in this literature. Based on the necessary conditions for $\mathcal{K}$-minimality, Kılınç-Karzan [21] also introduced $\mathcal{K}$-sublinear inequalities which contain $\mathcal{K}$-minimal inequalities as a subclass and have easier to characterize algebraic properties.

The purpose of this paper is to answer open questions surrounding $\mathcal{K}$-sublinear inequalities, their sufficiency status in important specific cases, and their connections to the cut-generating functions. Previously, in [21], it was established that whenever there is at least one $\mathcal{K}$-minimal inequality, $\mathcal{K}$-minimal inequalities together with the original $x \in \mathcal{K}$ constraint are sufficient for describing $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. However, the existence of $\mathcal{K}$-minimal inequalities depends on certain structural properties of $\mathcal{S}(A,\mathcal{K},\mathcal{B})$; and it is possible to have nontrivial and interesting sets $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ for which none of the describing inequalities are $\mathcal{K}$-minimal (see [21, Example 6]). Despite this, and the fact [21, Theorem 1] that $\mathcal{K}$-sublinear inequalities subsumes $\mathcal{K}$-minimal inequalities, the question of whether $\mathcal{K}$-sublinear inequalities would be sufficient in general (independent of any structural assumption on $\mathcal{S}(A,\mathcal{K},\mathcal{B})$) was left open in [21]. In this paper, we show that whenever $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B})) \neq \mathcal{K}$, $\mathcal{K}$-sublinear inequalities always exist. Moreover, we establish a stronger connection between $\mathcal{K}$-sublinear inequalities and extreme inequalities. That is, for general regular cones $\mathcal{K}$, we show that every extreme inequality is either $\mathcal{K}$-sublinear or can be captured by the conic constraint $x \in \mathcal{K}$ along with all valid equations. On a related note, we also show that not every extreme inequality is $\mathcal{K}$-sublinear or captured by the constraint $x \in \mathcal{K}$. This demonstrates that there can be valid equations which are neither $\mathcal{K}$-sublinear or implied by the constraint $x \in \mathcal{K}$. Nevertheless, we provide a positive answer to the open question on the sufficiency of $\mathcal{K}$-sublinear inequalities in two important cases of interest, namely when $\mathcal{K} = \mathbb{R}^n_+$ or $\mathcal{K} = \mathcal{L}^n$. In the case of $\mathcal{K} = \mathbb{R}^n_+$, there is a strong connection between $\mathbb{R}^n_+$-sublinear inequalities and the functions that generate cut coefficients—so called cut-generating functions (CGFs). Motivated by the connection between $\mathbb{R}^n_+$-sublinear inequalities and the support functions of certain structured sets, we introduce the concept of relaxed CGFs and relate them to the usual CGFs studied in [13]. Then, through this connection and the sufficiency of $\mathbb{R}^n_+$-sublinear inequalities in defining $\overline{\text{conv}}(\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B})$, we also provide an alternative and simplified proof of a recent result by Cornuéjols et al. [14] on the sufficiency of CGFs. The second part of the paper considers the case when $\mathcal{K} = \mathcal{L}^n$. In this case, we show that $\mathcal{L}^n$-sublinearity definition can be considerably simplified. As a consequence, we also establish that any valid inequality for $\mathcal{S}(A,\mathcal{L}^n,\mathcal{B})$ with $n \geq 3$ is always $\mathcal{L}^n$-sublinear.

The remainder of the paper is organized as follows. Section 2 introduces our notation and describes previous results. Section 3 studies some properties of the $\mathcal{K}$-sublinear inequalities for general $\mathcal{K}$. Section 4 presents our results on the sufficiency of $\mathbb{R}^n_+$-sublinear inequalities along with the constraint $x \geq 0$. In this section, we also discuss the connections with CGFs and the implications of the sufficiency of $\mathbb{R}^n_+$-sublinear inequalities on the sufficiency of CGFs. Finally, Section 5 examines $\mathcal{K}$-sublinear inequalities for second-order cones $\mathcal{K} = \mathcal{L}^n$.

2 Notation and Preliminaries

We start by introducing our notation. For a set S, we denote its topological interior with $\text{int}(S)$, its closure with $\overline{S}$ and its boundary with $\partial S = \overline{S} \setminus \text{int}(S)$. We use $\text{conv}(S)$ to denote the convex hull of S, $\overline{\text{conv}}(S)$ for its closed convex hull, and $\text{cone}(S)$ to denote the convex cone generated by the set S. We define the kernel of a linear map $A : E \to \mathbb{R}^m$ as $\text{Ker}(A) := \{u \in E : \ Au = 0\}$ and its image as $\text{Im}(A) := \{Au : \ u \in E\}$. We use A^* to denote the conjugate linear map[1)] given by the identity

$$y^T Ax = \langle A^* y, x\rangle \quad \forall (x \in E, y \in \mathbb{R}^m).$$

For a given cone $\mathcal{K} \subset E$, we let $\text{Ext}(\mathcal{K})$ denote the set of its extreme rays and $\mathcal{K}^*$ denote its dual cone given by $\mathcal{K}^* := \{y \in E : \ \langle x, y\rangle \geq 0 \ \ \forall x \in \mathcal{K}\}$. Note that the cones $\mathbb{R}^n_+$, $\mathcal{L}^n$, and $\mathcal{S}^n_+$ are all self-dual, i.e., $\mathcal{K}^* = \mathcal{K}$; and, in the first two cases, the corresponding Euclidean space E is just $\mathbb{R}^n$ with dot product as the corresponding inner product.

Throughout the paper, we use Matlab notation to denote vectors and matrices and all vectors are to be understood in column form. We will use e_i for the i^{th} unit vector of $\mathbb{R}^n$. We use the notation $a \succeq_{\mathcal{K}} b$ to denote the relation $a - b \in \mathcal{K}$, that is, the *conic inequality* between vectors a and b.

[1)] Note that when $E = \mathbb{R}^n$, and a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ is just an $m \times n$ real-valued matrix, and its conjugate is given by its transpose, i.e., $A^* = A^T$.

2.1 Classes of Valid Linear Inequalities

Given $\mathcal{S}(A,\mathcal{K},\mathcal{B})$, we are interested in the valid linear inequalities for $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. Consider the set of all vectors $\mu \in E$ such that $\vartheta(\mu)$ defined as

$$\vartheta(\mu) := \inf_x \{\langle \mu, x\rangle : \; x \in \mathcal{S}(A,\mathcal{K},\mathcal{B})\} \tag{1}$$

is finite. Then any nonzero vector $\mu \in E$ and a number $\mu_0 \le \vartheta(\mu)$ gives a *valid linear inequality* of the form $\langle \mu, x\rangle \ge \mu_0$ for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$. We denote the corresponding valid inequality by $(\mu;\mu_0)$. We say that a valid inequality $(\mu;\mu_0)$ is *tight*[2)] if $\mu_0 = \vartheta(\mu)$. If both $(\mu;\mu_0)$ and $(-\mu;-\mu_0)$ are valid inequalities, then $\langle \mu, x\rangle = \mu_0$ holds for all $x \in \mathcal{S}(A,\mathcal{K},\mathcal{B})$; and in this case, we refer to $(\mu;\mu_0)$ as a *valid equation* for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$. For any vector $\delta \in \mathcal{K}^* \setminus \{0\}$, the inequality $(\delta;0)$ is always valid for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$. We refer to these inequalities as *cone-implied inequalities*. The constraint $x \in \mathcal{K}$ captures all of the cone-implied inequalities. We define $C(A,\mathcal{K},\mathcal{B}) = \{(\mu;\mu_0) \in E \times \mathbb{R} : \; \mu_0 \le \vartheta(\mu)\}$, the convex cone of all valid linear inequalities. We let $L(A,\mathcal{K},\mathcal{B})$ be the largest linear subspace contained in the cone $C(A,\mathcal{K},\mathcal{B})$, also referred to as the *lineality space* of $C(A,\mathcal{K},\mathcal{B})$. In order to understand the linear inequalities necessary for the description of $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$, we study the generators of $C(A,\mathcal{K},\mathcal{B})$.

Note that any convex cone K can be written as the sum of its lineality space L and a pointed convex cone C. A unique representation of K in the form of $K = L + C$ can be obtained by requiring that C is contained in the orthogonal complement of L. A *generating set* (G_L, G_C) for a cone K is defined to be a minimal set of elements $G_L \subseteq L$, $G_C \subseteq C$ such that

$$K = \left\{ \sum_{w \in G_L} \alpha_w w + \sum_{v \in G_C} \lambda_v v : \; \lambda_v \ge 0 \right\}.$$

Given a generating set (G_L, G_C) of $C(A,\mathcal{K},\mathcal{B})$, we refer to the vectors in G_L as *generating valid equalities* and the vectors in G_C as *generating valid inequalities* of $C(A,\mathcal{K},\mathcal{B})$. Note that the trivial inequality $(0;-1)$ always belongs to $C(A,\mathcal{K},\mathcal{B})$; thus we will always take it as one of the generators of $C(A,\mathcal{K},\mathcal{B})$. Other than this trivial vector, without loss of generality, we can assume that all of the vectors in G_L are orthogonal to each other; and each vector in G_C is orthogonal to all vectors in G_L (see [21, Remark 2]). A nontrivial inequality $(\mu;\mu_0) \in C(A,\mathcal{K},\mathcal{B})$ is called an *extreme inequality* if there exists a generating set for $C(A,\mathcal{K},\mathcal{B})$ including $(\mu;\mu_0)$ as a generating inequality either in G_L or in G_C.

Given two vectors, $u, v \in K$ where K is a cone with lineality space L, u is said to be an *L-multiple* of v if $u = \alpha v + \ell$ for some $\alpha > 0$ and $\ell \in L$. From this definition, it is clear that if u is an L-multiple of v, then v is also an L-multiple of u. Thus, whenever the lineality space $L(A,\mathcal{K},\mathcal{B})$ of the cone $C(A,\mathcal{K},\mathcal{B})$ is nontrivial, i.e., $L(A,\mathcal{K},\mathcal{B}) \ne \{0\}$, the generating valid inequalities are only defined uniquely up to the L-multiples.

Understanding the structure of extreme valid linear inequalities is critical in terms of understanding both the structure of $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$ and the dominance relations among valid inequalities. On the other hand, characterizing all extreme valid inequalities can be quite difficult for an arbitrary set $\mathcal{S}(A,\mathcal{K},\mathcal{B})$. A middle ground is obtained by studying the structure of slightly larger classes of inequalities. In particular, classes of *minimal* and *sublinear* inequalities, where these notions are defined with respect to the cone $\mathcal{K}$, had received quite some interest in the recent years. For example, such a study on the structure of tight, $\mathcal{K}$-minimal inequalities underlies the developments of nonlinear disjunctive cuts in [23,24] for the case of two-term linear disjunctions applied to a second-order cone.

[2)] We note that our definition of *tightness* of an inequality does not require the corresponding hyperplane to have a nonempty intersection with the feasible region, as is sometimes the definition used in the literature.

The cone $\mathcal{K}$ in the description of $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ plays a critical role in identifying dominance relations among valid linear inequalities. Consider two valid linear inequalities for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ given by $(\mu;\mu_0)$ and $(\rho;\rho_0)$ such that $\rho \preceq_{\mathcal{K}^*} \mu$ and $\rho \neq \mu$. Then we say that $(\rho;\rho_0)$ *dominates* $(\mu;\mu_0)$ whenever $\rho_0 \geq \mu_0$. In fact, when $(\rho;\rho_0)$ dominates $(\mu;\mu_0)$, we have $\mu = \rho + \delta$ for some $\delta \in \mathcal{K}^* \setminus \{0\}$ and $\rho_0 \geq \mu_0$. Moreover, in this case, the inequality $(\rho;\rho_0)$ together with the conic constraint $x \in \mathcal{K}$ imply the inequality $(\mu;\mu_0)$ because

$$\langle \mu, x\rangle = \langle \rho + (\mu-\rho), x\rangle = \underbrace{\langle \rho, x\rangle}_{\geq \rho_0} + \underbrace{\langle \mu-\rho, x\rangle}_{\geq 0} \geq \rho_0 \geq \mu_0,$$

where the first inequality follows from $x \in \mathcal{K}$ and $\mu - \rho = \delta \in \mathcal{K}^*$.

In the case of $\mathcal{K} = \mathbb{R}^n_+$ and associated finite and infinite relaxations for MILPs, minimality of a valid inequality is traditionally defined with respect to $\mathcal{K} = \mathbb{R}^n_+$. That is, a valid inequality $(\mu;\mu_0)$ is minimal if for all $\rho \leq \mu$ and $\rho \neq \mu$, the inequality given by $(\rho;\mu_0)$ is not valid (see [20]). This means reducing any μ_i for $i \in \{1,\ldots,n\}$ in a minimal inequality $(\mu;\mu_0)$ will lead to a strict reduction in its right hand side value.[3] The following natural extension of the minimality definition to disjunctive conic sets, and thus to the MICP case, was introduced in [21]:

Definition 1 *($\mathcal{K}$-minimal inequality)* A valid linear inequality $(\mu;\mu_0)$ for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ with $\mu \neq 0$ and $\mu_0 \in \mathbb{R}$ is *$\mathcal{K}$-minimal* if for all valid inequalities $(\rho;\rho_0)$ for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ satisfying $\rho \neq \mu$, and $\rho \preceq_{\mathcal{K}^*} \mu$, we have $\rho_0 < \mu_0$.

In particular, the definition of $\mathcal{K}$-minimality implies that if $(\mu;\mu_0)$ with $\mu \neq 0$ is valid but not $\mathcal{K}$-minimal then there exists another valid inequality $(\rho;\rho_0)$ that dominates it. That is, $(\mu;\mu_0)$ is a consequence of the inequality $(\rho;\rho_0)$ and the conic constraint $x \in \mathcal{K}$. Note that such dominance relations are of great interest in obtaining smaller yet sufficient sets of valid linear inequalities. In these dominance relations, the cone $\mathcal{K}$ plays a central role; as a result, the selection of cone $\mathcal{K}$ in the representation of $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ is critical in defining more refined dominance relations among valid linear inequalities. For further discussion on the topic, we refer the reader to [21, Remarks 5 and 7]. On a related note, the concepts of tightness and $\mathcal{K}$-minimality are intrinsically different. Specifically, $\mathcal{K}$-minimality of an inequality in general does not imply the tightness of the inequality (see [21, Propositions 3 and 4, Example 7, and Remark 8] for further discussion and illustrations).

For a given $\mathcal{S}(A,\mathcal{K},\mathcal{B})$, we denote the set of all $\mathcal{K}$-minimal inequalities by $C_m(A,\mathcal{K},\mathcal{B})$. It was shown in [21, Proposition 1] (see [20] for the case of $\mathcal{K} = \mathbb{R}^n_+$) that $C_m(A,\mathcal{K},\mathcal{B}) \neq \emptyset$ only if the following assumption holds:

Assumption 1: For each $\delta \in \mathcal{K}^* \setminus \{0\}$, there exists some $x_\delta \in \mathcal{S}(A,\mathcal{K},\mathcal{B})$ such that $\langle \delta, x_\delta\rangle > 0$.

Moreover, by [21, Proposition 2 and Corollary 2], under **Assumption 1**, $\mathcal{K}$-minimal inequalities together with the original conic constraint $x \in \mathcal{K}$, are sufficient to describe $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. This motivated the further study of the properties of $\mathcal{K}$-minimal inequalities. By isolating a number of algebraic necessary conditions for $\mathcal{K}$-minimality, [21] suggested the following class of *$\mathcal{K}$-sublinear* inequalities:

Definition 2 *($\mathcal{K}$-sublinear inequality)* Given $\mathcal{S}(A,\mathcal{K},\mathcal{B})$, a linear inequality $(\mu;\mu_0)$ with $\mu \neq 0$ and $\mu_0 \in \mathbb{R}$ is *$\mathcal{K}$-sublinear* if it satisfies the following conditions $(\mathbf{A.1}(\alpha))$ for all $\alpha \in \text{Ext}(\mathcal{K}^*)$ and $(\mathbf{A.2})$ where

$$(\mathbf{A.1}(\alpha)) \quad 0 \leq \langle \mu, u\rangle \text{ for all } u \text{ s.t. } Au = 0 \text{ and } \langle \alpha, v\rangle u + v \in \mathcal{K} \quad \forall v \in \text{Ext}(\mathcal{K}),$$
$$(\mathbf{A.2}) \quad \mu_0 \leq \langle \mu, x\rangle \text{ for all } x \in \mathcal{S}(A,\mathcal{K},\mathcal{B}).$$

[3] We note that the valid inequalities that are referred as minimal in [3,4,10,18] correspond to *tight* and *$\mathbb{R}^n_+$-minimal* inequalities with respect to the definitions in this paper.

When an inequality satisfies (**A.1**(α)) for all $\alpha \in \text{Ext}(\mathcal{K}^*)$, we say that it satisfies condition (**A.1**). In this definition, while condition (**A.2**) is simply true for every valid inequality, condition (**A.1**) is not very intuitive. A particular and simple case of (**A.1**) was identified in [21]; and it is of interest: Any valid inequality $(\mu;\mu_0)$ satisfying (**A.1**) also satisfies the condition

$$(\textbf{A.0}) \;\; 0 \le \langle \mu, u\rangle \text{ for all } u \in \mathcal{K} \text{ such that } Au = 0.$$

Note that condition (**A.0**) is in fact equivalent to

$$\mu \in (\mathcal{K} \cap \text{Ker}(A))^* = \mathcal{K}^* + (\text{Ker}(A))^* = \mathcal{K}^* + \text{Im}(A^*),$$

where the last equation follows from the facts that $(\text{Ker}(A))^* = \text{Ker}(A)^{\perp} = \text{Im}(A^*)$ and $\mathcal{K}^* + \text{Im}(A^*)$ is closed whenever $\mathcal{K}$ is closed [29, Corollary 16.4.2]. In fact, it is known [21, Proposition 6] that conditions (**A.0**) and (**A.2**) are satisfied by any valid inequality $(\mu;\mu_0)$ with $\mu \neq 0$ and $\mu_0 \in \mathbb{R}$. Therefore, condition (**A.1**) is the main condition defining the interesting structure of $\mathcal{K}$-sublinear inequalities. When $\mathcal{K} = \mathbb{R}^n_+$, conditions (**A.0**)-(**A.2**) underlie the definition of *subadditive inequalities* from [20] (see e.g., [21, Remark 9]).

We next have the following immediate corollary regarding specific $\mathcal{K}$-sublinear inequalities.

Corollary 1 *Any valid inequality $(\mu;\mu_0)$ with $\mu \in \text{Im}(A^*)$ is $\mathcal{K}$-sublinear.*

Proof Condition (**A.2**) is satisfied since $(\mu;\mu_0)$ is valid. Because $\mu \in \text{Im}(A^*)$, using the fact that all of the u's in condition (**A.1**) are from $\text{Ker}(A)$, we arrive at $\langle \mu, u\rangle = 0$; thus μ satisfies condition (**A.1**) as well. □

Let $C_s(A,\mathcal{K},\mathcal{B})$ denote the cone of $\mathcal{K}$-sublinear inequalities. Conditions (**A.1**)-(**A.2**) imply that $C_s(A,\mathcal{K},\mathcal{B})$ is indeed a convex cone. By [21, Theorem 1], we have $C_m(A,\mathcal{K},\mathcal{B}) \subseteq C_s(A,\mathcal{K},\mathcal{B}) \subseteq C(A,\mathcal{K},\mathcal{B})$. Under **Assumption 1**, based on the sufficiency of $\mathcal{K}$-minimal inequalities, this automatically implies that the valid inequalities from $C_s(A,\mathcal{K},\mathcal{B})$ together with $x \in \mathcal{K}$ are sufficient to describe $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. Nonetheless, the question of whether **Assumption 1** is needed in this sufficiency result for $\mathcal{K}$-sublinear inequalities was not studied in [21]. We pursue this question in the next section.

3 Relation between $\mathcal{K}$-sublinear Inequalities and Extreme Inequalities

In this section, for general regular cones $\mathcal{K}$, we study the connection between $\mathcal{K}$-sublinear inequalities and extreme inequalities describing $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. Without relying on **Assumption 1**, we first establish a stronger structural result between $\mathcal{K}$-sublinear inequalities and extreme inequalities. The following lemma from [20] is useful in our analysis:

Lemma 1 *Let $K = L + C$ where C is a pointed cone and L is the largest linear subspace contained in K. Suppose v is in a generating set for cone K and there exists $v^1, v^2 \in K$ such that $v = v^1 + v^2$, then v^1, v^2 are L-multiples of v.*

Proposition 1 *Every extreme inequality for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ is either $\mathcal{K}$-sublinear or an L-multiple of a cone-implied inequality.*

Proof Suppose $(\mu;\mu_0)$ is an extreme inequality. Let (G_C, G_L) be a generating set for $C(A,\mathcal{K},\mathcal{B})$ containing $(\mu;\mu_0)$. We will show that if $(\mu;\mu_0)$ is not $\mathcal{K}$-sublinear, then it is an L-multiple of a cone-implied inequality. Assume that $(\mu;\mu_0)$ is valid but not $\mathcal{K}$-sublinear. Then it violates condition (**A.1**) and hence for some $\alpha \in \text{Ext}(\mathcal{K}^*)$, there exists u satisfying $\langle \mu, u\rangle < 0$ such that $Au = 0$ and $\langle \alpha, v\rangle u + v \in \mathcal{K}\ \forall v \in \text{Ext}(\mathcal{K})$ holds.

Define the linear map $Z : E \to E$ by

$$Zx = \langle x, u\rangle\alpha + x \quad \text{for any } x \in E.$$

We first show that $(Z\mu; \mu_0)$ is a valid inequality. Since $\langle Z\mu, x\rangle = \langle \mu, Z^*x\rangle\ \forall x \in E$, it is enough to show that $Z^*x \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})$ for any $x \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})$; then the validity of $(\mu; \mu_0)$ will give $\langle Z\mu, x\rangle = \langle \mu, Z^*x\rangle \geq \mu_0$.

Because $A : E \to \mathbb{R}^m$, its conjugate $A^* : \mathbb{R}^m \to E$. Without loss of generality let $A^*e_i =: A^i \in E$ for $i = 1, \dots, m$, where e_i is the i^{th} unit vector in $\mathbb{R}^m$. Then from $u \in \mathrm{Ker}(A)$, we have $\langle A^i, u\rangle = 0$, which implies $ZA^*e^i = \langle A^i, u\rangle\alpha + A^i = A^i$ for all $i = 1, \dots, m$. Therefore, we arrive at $ZA^* = A^*$. Moreover, because $A : E \to \mathbb{R}^m$ and $Z : E \to E$ are linear maps, ZA^* is a linear map and its conjugate is given by $AZ^* = A$.

Consider any $x \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})$; then we have $Ax \in \mathcal{B}$ and $x \in \mathcal{K}$. Hence, $AZ^*x = Ax \in \mathcal{B}$. Also, because u satisfies $\langle \alpha, v\rangle u + v \in \mathcal{K}\ \forall v \in \mathrm{Ext}(\mathcal{K})$, we get $\langle \alpha, x\rangle u + x \in \mathcal{K}$ holds for all $x \in \mathcal{K}$. Thus, for all $w \in \mathcal{K}^*$, we have $\langle w, Z^*x\rangle = \langle Zw, x\rangle = \langle((\langle w, u\rangle\alpha + w), x\rangle = \langle w, u\rangle\langle \alpha, x\rangle + \langle w, x\rangle = \langle w, \underbrace{\langle \alpha, x\rangle u + x}_{\in \mathcal{K}}\rangle \geq 0$.

Therefore, $Z^*x \in \mathcal{K}$, proving that for any $x \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})$, we have $Z^*x \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})$.

Additionally, $(\mu - Z\mu) = -\langle \mu, u\rangle\alpha \in \mathcal{K}^*$ since $\langle \mu, u\rangle < 0$. So $(\mu - Z\mu; 0)$ is a cone-implied inequality. Finally, because $\mu = Z\mu + (\mu - Z\mu)$, the original inequality $(\mu; \mu_0)$ is a conic combination of the valid inequality $(Z\mu; \mu_0)$ and the cone-implied inequality $(\mu - Z\mu; 0)$. Since we have assumed that $(\mu; \mu_0)$ is extreme, by Lemma 1, these inequalities must be L-multiples of each other, implying that $(\mu; \mu_0)$ is an L-multiple of a cone-implied inequality. □

Consider $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ and an inequality $(\mu; \mu_0)$ for it. We associate with μ the following set

$$D_\mu := \{\lambda \in \mathbb{R}^m :\ A^*\lambda \preceq_{K^*} \mu\}, \tag{2}$$

and its support function given by $\sigma_{D_\mu}(z) := \sup_{\lambda \in \mathbb{R}^m}\{\langle z, \lambda\rangle : \lambda \in D_\mu\}$. These entities play critical roles in the characterization of $\mathcal{K}$-sublinear inequalities.

We now show that as long as nontrivial valid linear inequalities for $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ exist, $\mathcal{K}$-sublinear inequalities must exist as well.

Proposition 2 *Suppose* $\overline{\mathrm{conv}}(\mathcal{S}(A, \mathcal{K}, \mathcal{B})) \neq \mathcal{K}$. *Then,* $\mathcal{K}$*-sublinear inequalities always exist.*

Proof Because $\overline{\mathrm{conv}}(\mathcal{S}(A, \mathcal{K}, \mathcal{B})) \neq \mathcal{K}$, there exists at least one extreme inequality $(\mu; \mu_0)$ that is non-cone-implied. Then by Proposition 1, it is either $\mathcal{K}$-sublinear or an L-multiple of a cone-implied inequality. If it is $\mathcal{K}$-sublinear, then we are done. So, suppose it is an L-multiple of a cone-implied inequality, that is $(\mu; \mu_0) = (\rho; \rho_0) + (\delta; 0)$ where $(\rho; \rho_0)$ is a valid equation and $\delta \in \mathcal{K}^* \setminus \{0\}$. Since $(\mu; \mu_0)$ is non-cone-implied, we have $(\rho; \rho_0)$ is also non-cone-implied.

Because $(\rho; \rho_0)$ is a valid equation, both $(\rho; \rho_0)$ and $(-\rho; -\rho_0)$ satisfy condition (**A.0**). Thus, $\rho \in \mathrm{Im}(A^*) + \mathcal{K}^*$ and $-\rho \in \mathrm{Im}(A^*) + \mathcal{K}^*$. Then there exists $\lambda^+, \lambda^- \in \mathbb{R}^m \setminus \{0\}$ and $\delta^+, \delta^- \in \mathcal{K}^*$ such that $\rho = A^*\lambda^+ + \delta^+ = -A^*\lambda^- - \delta^-$. Therefore, $-A^*(\lambda^+ + \lambda^-) = \delta^+ + \delta^-$ implying $\delta^+ + \delta^- \in \mathrm{Im}(A^*) \cap \mathcal{K}^*$. If $\mathrm{Im}(A^*) \cap \mathcal{K}^* = \{0\}$, then $\delta^+ = -\delta^-$. Because $\mathcal{K}^*$ is pointed and both $\delta^+ = -\delta^- \in \mathcal{K}^*$ and $\delta^- \in \mathcal{K}^*$, we have $\delta^+ = \delta^- = 0$. Then $\rho \in \mathrm{Im}(A^*)$; and using Corollary 1, we conclude that $(\rho; \rho_0)$ is $\mathcal{K}$-sublinear. When $\mathrm{Im}(A^*) \cap \mathcal{K}^* \neq \{0\}$, there exists $0 \neq \xi \in \mathrm{Im}(A^*) \cap \mathcal{K}^*$. For that ξ, we have $0 \in D_\xi$ where the set D_ξ is defined as in (2). Thus, the support function $\sigma_{D_\xi}(\cdot)$ is nonnegative, i.e., $\sigma_{D_\xi}(b) \geq 0$ for all b. From [21, Proposition 7], we have $\vartheta(\xi) \geq \inf_{b \in \mathcal{B}} \sigma_{D_\xi}(b)$. Then using $\inf_{b \in \mathcal{B}} \sigma_{D_\xi}(b) \geq 0$ and by Corollary 1, this leads to $(\xi; 0)$ is $\mathcal{K}$-sublinear. □

Proposition 1 suggests that a detailed study of the structure of valid equations for $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ is key to understanding the sufficiency of $\mathcal{K}$-sublinear inequalities. In order to completely argue for the sufficiency of $\mathcal{K}$-sublinear inequalities for general regular cones $\mathcal{K}$, one may conjecture that every extreme valid equation is either $\mathcal{K}$-sublinear or cone-implied. Unfortunately, this is not true as we demonstrate with a counter example in the next section.

4 Sufficiency of $\mathcal{K}$-sublinear Inequalities for $\mathcal{K} = \mathbb{R}^n_+$

We first note that due to the decomposable structure of $\mathbb{R}^n_+$, following [21, Remark 9], the conditions (**A.1**(α)) required for $\mathbb{R}^n_+$-sublinearity simplify and become (**A.0**) and the following conditions (**A.1i**) for $i = 1, \ldots, n$:

$$(\mathbf{A.1i}) \quad 0 \leq \mu^T u \text{ for all } u \text{ such that } Au = 0 \text{ and } u + e_i \in \mathbb{R}^n_+,$$

where e_i denotes the i^{th} unit vector in $\mathbb{R}^n$.

In addition, for a given $\mu \in \mathbb{R}^n$, μ satisfies condition (**A.0**) if and only if $\mu \in \mathbb{R}^n_+ + \text{Im}(A^T)$, and thus, $D_\mu \neq \emptyset$. We also recall the following result [21, Proposition 8]:

Remark 1 Consider $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$. Then for any $\mu \in \mathbb{R}^n_+ + \text{Im}(A^T)$, we have $\vartheta(\mu) = \inf_{b \in \mathcal{B}} \sigma_{D_\mu}(b)$. $\diamondsuit$

Furthermore, Theorem 4 and Proposition 10 from [21] are handy in our developments (see also [20, Theorems 9-10] and [21, Remarks 9, 10, and 11]). We summarize these here:

Theorem 1 *Consider $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$. Then any nontrivial valid inequality $(\mu; \mu_0)$ satisfies $\mu \in \mathbb{R}^n_+ + \text{Im}(A^T)$. Moreover, $(\mu; \mu_0)$ is an $\mathbb{R}^n_+$-sublinear inequality if and only if it is valid ($\mu_0 \leq \vartheta(\mu)$) and $\sigma_{D_\mu}(A_i) = \mu_i$ for all $i = 1, \ldots, n$ where A_i denotes the i-th column of the matrix A.*

We start by considering an illustrative example of an extreme inequality that is not $\mathbb{R}^n_+$-sublinear. Then we derive a $\mathbb{R}^n_+$-sublinear inequality that dominates it, using methods that are later on generalized.

Example 1 Consider the set $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ given by

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \ \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \text{ and } \mathcal{K} = \mathbb{R}^3_+.$$

Note that $\mathcal{S}(A, \mathcal{K}, \mathcal{B}) = \{[1; 0; 0]\}$. A generating set $(G_C; G_L)$ for the corresponding cone of valid inequalities $C(A, \mathcal{K}, \mathcal{B})$ is given by $G_C = \{(0; -1)\}$ and $G_L = \{(e_1; 1), (e_2; 0), (e_3; 0)\}$. Consider the valid inequality $(\mu; \mu_0) = (-e_3; 0) \in G_L$. We will show that this extreme inequality is not $\mathbb{R}^3_+$-sublinear. Given $(\mu; \mu_0) = (-e_3; 0)$, let us examine condition **(A.1i)** for $i = 2$, which requires $u_2 \geq 0$ for all u such that $Au = 0$ and $u + e_2 \in \mathbb{R}^3_+$. Yet, the vector $u = [1; -1; 1] \in \text{Ker}(A) \cap (-e_2 + \mathbb{R}^3_+)$ violates this condition, implying that $(\mu; \mu_0) = (-e_3; 0)$ is a valid inequality that is not $\mathbb{R}^3_+$-sublinear.

Let us now construct an $\mathbb{R}^3_+$-sublinear inequality that dominates $(\mu; \mu_0) = (-e_3; 0)$. Given μ, we consider the set $D_\mu = \{\lambda \in \mathbb{R}^2 : A^*\lambda \leq \mu\} = \{\lambda \in \mathbb{R}^2 : \lambda_1 \leq 0, \lambda_2 \leq 0, -\lambda_1 + \lambda_2 \leq -1\}$. Using the support function of D_μ, i.e., $\sigma_{D_\mu}(b) = \sup_\lambda \{b^T\lambda : A^*\lambda \leq \mu\}$, we construct a new vector based on the value of this function for each column A_i of the matrix A, that is, $\eta = [\sigma_{D_\mu}(A_1); \sigma_{D_\mu}(A_2); \sigma_{D_\mu}(A_3)] = [0; -1; -1]$. The validity of $(\eta; \mu_0)$ with $\mu_0 = 1$ is easily seen by the fact that $\mathcal{S}(A, \mathcal{K}, \mathcal{B}) = \{[1; 0; 0]\}$. Moreover, $D_\eta = \{\lambda \in \mathbb{R}^2 : A^*\lambda \leq \eta\} = \{\lambda \in \mathbb{R}^2 : \lambda_1 \leq 0, \lambda_2 \leq -1, -\lambda_1 + \lambda_2 \leq -1\} = D_\mu$. Thus, $\eta_i = \sigma_{D_\eta}(A_i)$ for all $i = 1, 2, 3$. Then using Theorem 1, we conclude that $(\eta; \mu_0)$ with $\mu_0 = 0$ is an $\mathbb{R}^3_+$-sublinear inequality. Finally, note that $\mu = \eta + \delta$ with $\delta = e_2 \in \mathbb{R}^3_+$; and thus $(\eta; \mu_0)$ dominates $(\mu; \mu_0)$. $\diamondsuit$

We next state our main result of this section which establishes that $\mathbb{R}^n_+$-sublinear inequalities along with the constraint $x \geq 0$ are sufficient to describe $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$.

Proposition 3 *Let $\mathcal{K} = \mathbb{R}^n_+$. Then any nontrivial valid inequality $(\mu; \mu_0)$ for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ is equivalent to or dominated by an $\mathbb{R}^n_+$-sublinear inequality given by $(\eta; \mu_0)$ where $\eta_i = \sigma_{D_\mu}(A_i)$ for all $i = 1, \ldots, n$ and the domination is defined with respect to $\mathcal{K} = \mathbb{R}^n_+$.*

Proof Given a nontrivial valid inequality $(\mu;\mu_0)$ for $\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B})$, let us consider the set $D_\mu = \{\lambda \in \mathbb{R}^m : A^T\lambda \leq \mu\}$, and its support function $\sigma_{D_\mu}(\cdot)$. Based on these, we will generate a new vector η by setting $\eta_i = \sigma_{D_\mu}(A_i)$ for all $i = 1,\ldots,n$ where A_i is the i-th column of the matrix A. First, note that because $(\mu;\mu_0)$ is a nontrivial valid inequality, it satisfies condition (**A.0**). Then by [21, Theorem 3], we have $D_\mu \neq \emptyset$ and $\eta_i = \sigma_{D_\mu}(A_i) \leq \mu_i$. Thus, $\mu \geq \eta$. Moreover, since $x \in \mathbb{R}^n_+$ and $\sigma_{D_\mu}(\cdot)$ is a support function and thus is positively homogeneous and subadditive, for any b, we conclude

$$\inf_x \left\{\eta^T x : Ax = b,\, x \in \mathbb{R}^n_+\right\} = \inf_x \left\{ \sum_{i=1}^n \underbrace{\sigma_{D_\mu}(A_i)x_i}_{=\sigma_{D_\mu}(A_i x_i)} : Ax = b,\, x \in \mathbb{R}^n_+ \right\}$$

$$\geq \inf_x \left\{ \sigma_{D_\mu}(\underbrace{\sum_{i=1}^n A_i x_i}_{=b}) : Ax = b,\, x \in \mathbb{R}^n_+ \right\} \geq \sigma_{D_\mu}(b).$$

Also, from the validity of $(\mu;\mu_0)$ we have $\vartheta(\mu) \geq \mu_0$. Then by Remark 1, we conclude $\vartheta(\mu) = \inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b) \geq \mu_0$. This implies

$$\inf_x\{\eta^T x : x \in \mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B})\} = \inf_{b\in\mathcal{B}} \inf_x \left\{\eta^T x : Ax = b,\, x \in \mathbb{R}^n_+\right\} \geq \inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b) = \vartheta(\mu) \geq \mu_0.$$

Hence, $(\eta;\mu_0)$ is a valid inequality that dominates $(\mu;\mu_0)$.

In order to finish the proof, we need to show that $(\eta;\mu_0) \in C_s(A,\mathcal{K},\mathcal{B})$. To see this, let us consider the set D_η and its support function $\sigma_{D_\eta}(\cdot)$. Once again using [21, Theorem 3], we have $D_\eta \neq \emptyset$ and $\sigma_{D_\eta}(A_i) \leq \eta_i$ for all i. Besides, from the definition of η, $\eta_i = \sigma_{D_\mu}(A_i)$; and hence $D_\mu \subseteq \{\lambda \in \mathbb{R}^m : A_i^T\lambda \leq \eta_i\}$ holds for any i. Thus, we reach to $D_\mu \subseteq D_\eta$. Using the monotonicity of the support functions, we then obtain $\sigma_{D_\mu}(a) \leq \sigma_{D_\eta}(a)$ for all $a \in \mathbb{R}^m$. In particular, $\eta_i = \sigma_{D_\mu}(A_i) \leq \sigma_{D_\eta}(A_i)$ for all i. Therefore, we conclude $\eta_i = \sigma_{D_\eta}(A_i)$ for all i. Then Theorem 1 implies that $(\eta;\mu_0)$ is an $\mathbb{R}^n_+$-sublinear inequality. □

A few remarks are in order. The sufficiency result given in Proposition 3 leads to nice consequences. When $\mathcal{K} = \mathbb{R}^n_+$, based on the precise correspondence (see [21, Remark 9]) between subadditive inequalities of Johnson [20] and $\mathbb{R}^n_+$-sublinear inequalities, and the fact that every $\mathbb{R}^n_+$-sublinear inequality is generated by the support function of a certain set (see Theorem 1 and Remark 1), we conclude that the support functions of the nonempty sets of form $D_\mu = \{\lambda \in \mathbb{R}^m : A^*\lambda \leq \mu\}$ are sufficient to generate all valid inequalities for $\overline{\text{conv}}(\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B}))$. As a result of this, we define support functions of nonempty sets $D \subset \mathbb{R}^m$ as *relaxed cut-generating functions*.

Definition 3 Given $\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B})$ and a set $\emptyset \neq D \subset \mathbb{R}^m$, we say that the support function $\sigma_D : \mathbb{R}^m \to (\mathbb{R} \cup +\infty)$ of D is a *relaxed cut-generating function* for $\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B})$.

Note that for fixed n, A and $\mathcal{B}$, and a vector $\mu \in \mathbb{R}^n_+ + \text{Im}(A^T)$, the relaxed CGF associated with $D_\mu \neq \emptyset$ leads to a valid inequality for $\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B})$ via Remark 1. At first sight, the relaxed CGFs might seem to depend on all of the data A, n and $\mathcal{B}, m$ associated with the specific instance $\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B})$ and therefore work only for the given $\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B})$. We next show that due to the sublinearity of relaxed CGFs, their validity is indeed independent of A and n; and thus, they can be used to generate valid inequalities for other instances $\mathcal{S}(A',\mathbb{R}^{n'}_+,\mathcal{B})$ as long as the set $\mathcal{B}$ is kept the same.

Proposition 4 *Suppose $\mathcal{B} \subset \mathbb{R}^m$ is given. Let $\sigma(\cdot)$ be a relaxed CGF for $\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B})$ associated with a nonempty set $D \subset \mathbb{R}^m$. Then, the inequality $\sum_{i=1}^{n'} \sigma(A'_i)x_i \geq \inf_{b\in\mathcal{B}} \sigma(b)$ is valid for any $x \in \mathcal{S}(A',\mathbb{R}^{n'}_+,\mathcal{B})$ where the dimension n' and the matrix $A' \in \mathbb{R}^{m\times n'}$ are arbitrary, and A'_i denotes the i-th column of the matrix A'.*

Proof Because $\sigma(\cdot)$ is a relaxed CGF, it is a support function; and thus, it is sublinear (subadditive and convex). Then for any $x \in \mathcal{S}(A', \mathbb{R}^{n'}_+, \mathcal{B})$, we have $\sum_{i=1}^{n'} \sigma(A'_i)x_i \geq \sigma(\sum_{i=1}^{n'} A'_i x_i) = \sigma(b)$ for some $b \in \mathcal{B}$, where the inequality is due to the sublinearity of $\sigma(\cdot)$ and the equation is due to $x \in \mathcal{S}(A', \mathbb{R}^{n'}_+, \mathcal{B})$. Hence, $\sum_{i=1}^{n'} \sigma(A'_i)x_i \geq \inf_{b\in\mathcal{B}} \sigma(b)$ holds for all $x \in \mathcal{S}(A', \mathbb{R}^{n'}_+, \mathcal{B})$. □

While relatively simple, Proposition 4 has an important consequence. Together with Remark 1, Proposition 4 indicates that the relaxed CGFs σ_{D_μ} obtained from $\mathbb{R}^n_+$-sublinear inequalities $(\mu; \mu_0)$ for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ can be used to generate cuts of the form $\sum_{i=1}^{n'} \sigma_{D_\mu}(A'_i)x_i \geq \inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b) \geq \mu_0$ for any set $\mathcal{S}(A', \mathbb{R}^{n'}_+, \mathcal{B})$ where A' and n' are arbitrary, i.e., they can be taken as varying.

We now turn our attention to cuts of the form $c^T x \geq 1$ separating origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. This setup naturally arises in the context of separating a fractional solution from the feasible region of an MILP [13] and has attracted some attention in the recent literature. In particular, some of the previous literature, including [13,14], has specifically focused on the separation of the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ under the assumption that $\mathcal{B}$ is closed and $0 \notin \mathcal{B}$. In this case, $0 \notin \overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$, which ensures the existence of valid inequalities separating the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. Our analysis to follow does not depend on the closedness of $\mathcal{B}$; and thus we will not make this assumption. Instead, we simply assume that $0 \notin \overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. Proposition 3 implies that for any cut $c^T x \geq 1$ separating the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$, we either have $(c; 1) \in C_s(A, \mathcal{K}, \mathcal{B})$ or there will be at least one $\mathbb{R}^n_+$-sublinear inequality that dominates $(c; 1)$. Therefore, by focusing on the associated relaxed CGFs, we arrive at the following corollary of Proposition 3.

Corollary 2 *Let A_i be the i-th column of the matrix A for all $i = 1, \ldots, n$. Consider the case with $\mathcal{K} = \mathbb{R}^n_+$, $\mathcal{B} \subset \mathbb{R}^m$ and $0 \notin \overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. Then any valid inequality $c^T x \geq 1$ separating the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ is equivalent to or dominated by one of the form $\sum_{i=1}^{n} \sigma(A_i)x_i \geq 1$, obtained from a relaxed CGF $\sigma : \mathbb{R}^m \to (\mathbb{R} \cup +\infty)$.*

Proof Since $(c; 1)$ separates the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$, any valid inequality that is equivalent to or dominates $(c; 1)$ will also separate the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. Because $\mathbb{R}^n_+$-sublinear inequalities along with $x \geq 0$ are sufficient to describe $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ (see Proposition 3), and the cone-implied inequalities $x_i \geq 0$ do not separate the origin, there exists $(\mu; \mu_0) \in C_s(A, \mathcal{K}, \mathcal{B})$ that is equivalent to or dominates $(c; 1)$. Thus, $\mu_0 \geq 1$ and $(\mu; \mu_0)$ is a nontrivial valid inequality. Then, from [21, Proposition 6], $(\mu; \mu_0)$ satisfies condition (**A.0**), i.e., $\mu \in \mathbb{R}^n_+ + \text{Im}(A^T)$. Finally, from Theorem 1, we have, for every $\mathbb{R}^n_+$-sublinear inequality $(\mu; \mu_0)$, the support function σ_{D_μ} of the associated nonempty set D_μ generates the coefficients of this inequality, i.e., $\sigma_{D_\mu}(A_i) = \mu_i$ for all i. □

Recently, Conforti et al. [13] studied a variant of the set $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ where $\mathcal{B} \in \mathbb{R}^m$ is a fixed nonempty and closed set such that $0 \notin \mathcal{B}$, yet n and $A \in \mathbb{R}^{m\times n}$ are varying. It is easy to see [13, Lemma 2.1] that in such a setup, $0 \notin \overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. For this particular setup, the authors are interested in generating cuts that separate the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ and introduce the concept of a *cut-generating function* as follows:

Definition 4 Given a nonempty and closed set $\mathcal{B} \in \mathbb{R}^m$ satisfying $0 \notin \mathcal{B}$, a *cut-generating function* for $\mathcal{B}$ is a function $f : \mathbb{R}^m \to \mathbb{R}$ such that for any natural number $n \in \mathbb{N}$ and any matrix $A \in \mathbb{R}^{m\times n}$, the linear inequality given by $\sum_{i=1}^{n} f(A_i)x_i \geq 1$ is valid for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ where A_i is the i-th column of the matrix A.

In this setup, various properties of cut-generating functions and their relations to $\mathcal{B}$-free sets were studied in [13]. One question left open in [13] was whether or not CGFs are sufficient to give all of the cuts separating the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. In the follow up work [14], by using the value functions of linear programs and arguments from convex analysis, a positive answer to this question was provided under an additional structural assumption. From Remark 1, Proposition 4, and Corollary 2, we immediately deduce a related result. That is, without any further technical assumptions, the relaxed CGFs are always sufficient to generate all cuts separating the origin from $\overline{\text{conv}}(\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B}))$ and A and n can be taken as varying exactly as in the framework of [13]. In contrast to this, Example 6.1 of [13] shows that there are sets of the form $\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B})$ such that CGFs are not sufficient to generate all cuts separating the origin. This disparity between the sufficiency of relaxed CGFs and the insufficiency of regular CGFs is due to the fact that regular CGFs are required to be finite valued everywhere while relaxed CGFs are not. In particular, the relaxed CGFs used in Proposition 4 and Corollary 2 are simply support functions of certain sets; and therefore they may not be finite valued everywhere. We note that it is not necessary to require a function to be finite valued everywhere in order to use it to generate cuts for a given problem instance $\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B})$. As long as the function is finite valued on the set of columns of A, it will lead to nontrivial valid inequalities. Nevertheless, for a function to work (i.e., generate nontrivial valid inequalities) for all instances of the form $\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B})$ with varying A and n, the finite valuedness of the function is important. Within the framework of [13], the sufficiency of CGFs was established in [14] under the additional structural assumption that $\mathcal{B}\subseteq \text{cone}(A)$. We next show that under the same assumption, we can in fact guarantee finite valuedness of relaxed CGFs as well.

Proposition 5 *Suppose* $0\notin\overline{\text{conv}}(\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B}))$ *and* $\mathcal{B}\subseteq\text{cone}(A)$*. Let* $c^Tx\geq 1$ *be a valid inequality separating the origin from* $\overline{\text{conv}}(\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B}))$*. Then*

(*a*) $D_c := \{\lambda\in\mathbb{R}^m : A^*\lambda\leq c\}$ *is nonempty and the support function of* D_c *leads to a relaxed CGF, which then leads to a valid inequality that is equivalent to or dominates* $c^Tx\geq 1$*.*

(*b*) *Let* $\mathcal{V}$ *denote the set of extreme points of the polyhedral set* D_c*, and* ρ *be such that* $\rho\geq\max\left\{\max_{v\in\mathcal{V}}\|v\|_\infty,\ 1+\inf_{\lambda\in D_c}\|\lambda\|_\infty\right\}$*. Then* $D_{c,\rho} := \{\lambda\in\mathbb{R}^m : A^*\lambda\leq c,\ \|\lambda\|_\infty\leq\rho\}$ *is nonempty. Moreover, the support function of* $D_{c,\rho}$ *is finite valued everywhere, piecewise linear, and leads to a valid inequality that is equivalent to or dominates* $c^Tx\geq 1$*.*

Proof (*a*) This part immediately follows from Corollary 2 and Proposition 3.

(*b*) Let $B(0,\rho) := \{\lambda\in\mathbb{R}^m : \|\lambda\|_\infty\leq\rho\}$, where $\rho\geq 1$ is as defined above. For any $v\in\mathcal{V}$, using the definition of ρ, we have $v\in B(0,\rho)$ as well. Since $D_{c,\rho}=D_c\cap B(0,\rho)$, we get $v\in D_{c,\rho}$ for any $v\in\mathcal{V}$, which also proves that $D_{c,\rho}$ is nonempty whenever $\mathcal{V}\neq\emptyset$. Also, if $\mathcal{V}=\emptyset$, then $\rho=1+\inf_{\lambda\in D_c}\|\lambda\|_\infty=1+\min_{\lambda\in D_c}\|\lambda\|_\infty$ because D_c is nonempty and polyhedral. Hence, there exists $\bar\lambda\in D_c$ such that $\|\bar\lambda\|_\infty\leq\rho$; thus $D_{c,\rho}\neq\emptyset$ in this case as well. Because $D_{c,\rho}$ is a nonempty bounded set, its support function is finite valued everywhere. Also, $D_{c,\rho}\subseteq D_c$ implies $\sigma_{D_{c,\rho}}(z)\leq\sigma_{D_c}(z)$ for every $z\in\mathbb{R}^n$. Therefore, $\sigma_{D_{c,\rho}}(A_i)\leq\sigma_{D_c}(A_i)\leq c_i$ where the last inequality follows from part (*a*) and A_i denote the i-th column of the matrix A.

To finish the proof, we need to show that the cut generated using $\sigma_{D_{c,\rho}}(\cdot)$ with a right hand side value of 1 is valid (this will also show that this cut is equivalent to or dominates $c^Tx\geq 1$). Note that, for any $z\in\mathbb{R}^n$ such that $\sigma_{D_c}(z)$ is finite, we have $\sigma_{D_c}(z)=\max_{v\in\mathcal{V}}\{z^Tv\}\leq\sigma_{D_{c,\rho}}(z)\leq\sigma_{D_c}(z)$, which implies $\sigma_{D_c}(z)=\sigma_{D_{c,\rho}}(z)$. Also, the set of recessive directions of D_c are given by $\text{Rec}(D_c)=\{d\in\mathbb{R}^n : A^*d\leq 0\}$. So, $\sigma_{D_c}(z)=+\infty$ if and only if there exists $0\neq d\in\text{Rec}(D_c)$ such that $d^Tz>0$. Next, we show that under the assumption that $\mathcal{B}\subseteq\text{cone}(A)$, for every $b\in\mathcal{B}$, we have $d^Tb\leq 0$ for all $d\in\text{Rec}(D_c)$, implying $\sigma_{D_c}(b)<+\infty$. Because $\mathcal{B}\subseteq\text{cone}(A)$, for any $b\in\mathcal{B}$, $\exists x_b\in\mathbb{R}^n_+$ such that $b=Ax_b$. Also, $d\in\text{Rec}(D_c)$ if and only if $A^*d\in\mathbb{R}^n_-$. Then $0\geq\underbrace{(A^*d)^T}_{\in\mathbb{R}^n_-}\underbrace{x_b}_{\in\mathbb{R}^n_+}=d^TAx_b=d^Tb$, which implies that $\sigma_{D_c}(b)<+\infty$. Hence, $\sigma_{D_c}(b)=\sigma_{D_{c,\rho}}(b)$ for all $b\in\mathcal{B}$. Thus, we have $\inf_{b\in\mathcal{B}}\sigma_{D_{c,\rho}}(b)=\inf_{b\in\mathcal{B}}\sigma_{D_c}(b)\geq 1$, where the last inequality follows from part (*a*). Therefore, using Proposition 4, the relaxed CGF $\sigma_{D_{c,\rho}}(\cdot)$ leads to the valid inequality $\sum_{i=1}^n\sigma_{D_{c,\rho}}(A_i)x_i\geq 1$. Then, because $\sigma_{D_{c,\rho}}(\cdot)$ leads to an inequality which is equivalent to or dominates $(c;1)$, this concludes the proof. □

Proposition 5 establishes that the relaxed CGF $\sigma_{D_{c,\rho}}(\cdot)$ is actually finite valued whenever $\mathcal{B} \subseteq \text{cone}(A)$. Considering this together with Proposition 4, we conclude that $\sigma_{D_{c,\rho}}(\cdot)$ is a CGF in the usual sense. Therefore, Proposition 5 recovers the main result, Theorem 1.1, of [14].

Note that when we ignore the restriction of a CGF being finite valued, the existence of an $\mathbb{R}^n_+$-sublinear inequality equivalent to or dominating $(c;1)$ and hence a relaxed CGF leading to the corresponding $\mathbb{R}^n_+$-sublinear inequality, i.e., Corollary 2, is independent of the condition $\mathcal{B} \subseteq \text{cone}(A)$; and in the proof of Proposition 5, the condition $\mathcal{B} \subseteq \text{cone}(A)$ is used only to ensure that the support function used for the relaxed CGF is finite valued everywhere. We refer the readers to Kılınç-Karzan and Yang [22] for recent developments on more general conditions ensuring the sufficiency of CGFs.

Remark 2 The sufficiency of CGFs for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ has strong connections with the subadditive (superadditive) duality theory for MILPs as well. The feasible set of any MILP problem can be converted into the form of optimizing a linear function over $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ by appropriately selecting A and $\mathcal{B}$. Specifically, we can write the set of feasible solutions of an MILP in disjunctive conic form as follows:

$$\{x \in \mathbb{R}^n_+ : \bar{A}x \geq \bar{b},\, x_i \in \mathbb{Z}_+ \ \forall i = 1, \ldots, \ell\} = \Bigg\{x \in \underbrace{\mathbb{R}^n_+}_{:=\mathcal{K}} : \underbrace{\begin{pmatrix} \bar{A} \\ I_\ell \quad 0 \end{pmatrix}}_{:=A} x \in \underbrace{\begin{pmatrix} \bar{b} + \mathbb{R}^m_+ \\ \mathbb{Z}^\ell_+ \end{pmatrix}}_{:=\mathcal{B}} \Bigg\}.$$

The subadditive (superadditive) strong duality theorem for MILPs states that when there is at least one feasible solution, that is, $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}) \neq \emptyset$, then one can write a dual optimization problem over all finite valued functions f that are nondecreasing with respect to $\mathbb{R}^m_+$ and subadditive (superadditive); and this dual attains the same objective value as the primal MILP problem (see [17, 19, 28]). Indeed, the functions appearing in the strong dual formulation of MILPs are CGFs because these functions act locally on each individual variable by only considering the associated data A_i and producing the cut coefficient c_i. As a result, strong MILP duality theorem implies the sufficiency of these functions for the corresponding specific sets $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ associated with the set of feasible solutions of MILPs.

Recently, under a technical condition, Morán et al. [27] has extended this strong duality theory to MICPs of a specific form (see [27, Theorem 2.4]). The structure of MICPs studied in [27] leads to specific disjunctive conic sets $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ where the domination is defined with respect to a nonnegative orthant and the cone involved in MICP appears in the definition of the set $\mathcal{B}$ (see [21, Example 3]). We refer the reader to [21, Remark 12] for additional discussion relating the work of [27] to CGFs. $\diamondsuit$

5 Characterization of $\mathcal{K}$-sublinear Inequalities for $\mathcal{K} = \mathcal{L}^n$

In this section, we refine the $\mathcal{K}$-sublinearity conditions (**A.0**)-(**A.2**) for the case of second-order cone $\mathcal{K} = \mathcal{L}^n$. In particular, for $\mathcal{L}^n$ with $n \geq 3$, we show that condition (**A.1**) is equivalent to condition (**A.0**); and hence one does not need to verify condition (**A.1**) explicitly. One of the consequences of this is the sufficiency of $\mathcal{L}^n$-sublinear inequalities along with $x \in \mathcal{L}^n$ constraint for describing $\overline{\text{conv}}(\mathcal{S}(A, \mathcal{L}^n, \mathcal{B}))$. Note that $\mathcal{K} = \mathcal{L}^2$ is just a rotation of the nonnegative orthant which can be covered by the analysis in the previous section.

In the case of $\mathcal{K} = \mathcal{L}^n$ for a fixed n and a given $\alpha \in \text{Ext}(\mathcal{L}^n)$, we define the following sets for convenience in our analysis:

$$V_\alpha = \left\{v \in \mathbb{R}^n : \alpha^T v = 1,\ v_n^2 = \sum_{i=1}^{n-1} v_i^2,\ v_n \geq 0\right\} \quad \text{and}$$
$$X_\alpha = \left\{x \in \mathbb{R}^n : x \in \mathcal{L}^n - v \ \ \forall v \in V_\alpha\right\}.$$

The following lemma gives an alternative statement of condition (**A.1**(α)).

Lemma 2 *Consider $\mathcal{S}(A,\mathcal{L}^n,\mathcal{B})$. Let $\alpha \in \operatorname{Ext}(\mathcal{L}^n)$. Then the linear inequality $(\mu;\mu_0)$ satisfies*

$$0 \le \langle \mu, u\rangle \quad \textit{for all } u \in \operatorname{Ker}(A)\cap X_\alpha, \tag{3}$$

*if only if it satisfies condition (**A.1**(α)).*

Proof Let u be any vector as in (**A.1**(α)), that is, $Au = 0$ and $\langle \alpha, v\rangle u + v \in \mathcal{L}^n \;\; \forall v \in \operatorname{Ext}(\mathcal{L}^n)$. We first show that $u \in \operatorname{Ker}(A)\cap X_\alpha$. Clearly $u \in \operatorname{Ker}(A)$. To see that $u \in X_\alpha$, let v' be an arbitrary vector in V_α. Since $v' \in \operatorname{Ext}(\mathcal{L}^n)$ and u is as in (**A.1**(α)), we have $\langle \alpha, v'\rangle u + v' \in \mathcal{L}^n$. Then $\langle \alpha, v'\rangle = 1$ implies $u \in \mathcal{L}^n - v'/\langle \alpha, v'\rangle = \mathcal{L}^n - v'$ implying that $u \in X_\alpha$. Now, consider $u \in \operatorname{Ker}(A)\cap X_\alpha$. Then $Au = 0$. Moreover, for any $v \in \operatorname{Ext}(\mathcal{L}^n)$ such that $\langle \alpha, v\rangle = 0$, we have $\langle \alpha, v\rangle u + v = v \in \mathcal{L}^n$. Also, for any $v \in \operatorname{Ext}(\mathcal{L}^n)$ such that $\langle \alpha, v\rangle > 0$, let $v' := \frac{v}{\langle \alpha, v\rangle}$. Then $v' \in \operatorname{Ext}(\mathcal{L}^n)$ and $\langle \alpha, v'\rangle = 1$; and thus $v' \in V_\alpha$. Because $u \in X_\alpha$, $u + v' \in \mathcal{L}^n$. From the definition of v' and since $\langle \alpha, v\rangle > 0$ and $\mathcal{L}^n$ is a cone, we have $\langle \alpha, v\rangle u + v \in \mathcal{L}^n$. Thus, for all $v \in \operatorname{Ext}(\mathcal{L}^n)$, we have $\langle \alpha, v\rangle u + v \in \mathcal{L}^n$ proving that u is a vector described in (**A.1**(α)).

□

We next show that in fact $X_\alpha \subseteq \mathcal{L}^n$; and thus condition (3) is already covered by condition (**A.0**). Because condition (**A.0**) is a consequence of condition (**A.1**), this proves their equivalence. As a result, the definition of $\mathcal{L}^n$-sublinearity is significantly simplified.

Proposition 6 *When $\mathcal{K} = \mathcal{L}^n$ where $n \ge 3$, any valid inequality that satisfies conditions (**A.0**) and (**A.2**) is $\mathcal{K}$-sublinear.*

Proof To establish this result, given any $\alpha \in \operatorname{Ext}(\mathcal{L}^n)$, we will prove that $X_\alpha \subseteq \mathcal{L}^n$.

Claim If $x \in X_\alpha$, then $x_n \ge 0$.

Proof of Claim: Consider the $n-2$ dimensional subspace $\{x \in \mathbb{R}^n : x^T\alpha = 0,\, x_n = 0\}$, and let $\hat{u}$ be any nonzero vector from this subspace. Without loss of generality we may assume that $\hat{u}$ is scaled such that $\sum_{i=1}^{n-1} \hat{u}_i^2 = 1/\alpha_n$. Next, we construct two vectors u, w by setting $u_i = \hat{u}_i$ and $w_i = -\hat{u}_i$ for $i \in \{1,\dots,n-1\}$ and $u_n = w_n = \sum_{i=1}^{n-1} \hat{u}_i^2$. Then by construction $u, w \in \operatorname{Ext}(\mathcal{L}^n)$, $\alpha^T u = \alpha^T w = 1$, i.e., $u, w \in V_\alpha$, and $u_i = -w_i$ for $i \in \{1,\dots,n-1\}$, $u_n = w_n$. Because $u, w \in V_\alpha$, $X_\alpha \subseteq (\mathcal{L}^n - u)\cap(\mathcal{L}^n - w)$. Next, we show that if $x \in X_\alpha \subseteq (\mathcal{L}^n - u)\cap(\mathcal{L}^n - w)$, we must have $x_n \ge 0$. Note that

$$\mathcal{L}^n - v = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n-1} (x_i + v_i)^2 \le (x_n + v_n)^2,\; x_n + v_n \ge 0 \right\}.$$

Therefore, $x \in X_\alpha$ satisfies $x_n + u_n = x_n + w_n \ge 0$ as well as the following two conditions:

$$\sum_{i=1}^{n-1} (x_i + u_i)^2 \le (x_n + u_n)^2,$$
$$\sum_{i=1}^{n-1} (x_i - u_i)^2 = \sum_{i=1}^{n-1} (x_i + w_i)^2 \le (x_n + w_n)^2 = (x_n + u_n)^2.$$

These two inequalities lead to:

$$\begin{aligned}(x_n + u_n)^2 &\ge \max\left\{ \sum_{i=1}^{n-1} (x_i + u_i)^2,\; \sum_{i=1}^{n-1} (x_i - u_i)^2 \right\} \\ &= \sum_{i=1}^{n-1} x_i^2 + \sum_{i=1}^{n-1} u_i^2 + \underbrace{\max\left\{ \sum_{i=1}^{n-1} 2x_iu_i,\; -\sum_{i=1}^{n-1} 2x_iu_i \right\}}_{\ge 0}.\end{aligned}$$

Because $u \in \operatorname{Ext}(\mathcal{L}^n)$, the above inequality implies that $x_n^2 + 2x_n u_n = x_n(x_n + 2u_n) \geq \sum_{i=1}^{n-1} x_i^2 \geq 0$. Also, $0 \leq x_n + u_n < x_n + 2u_n$ since $u_n > 0$. Thus, $x_n(x_n + 2u_n) \geq 0$ can hold only if $x_n \geq 0$, which establishes our claim that $x_n \geq 0$. □

The nonnegativity of x_n implies $x_n + v_n \geq 0$ for any $v \in V_\alpha$. Therefore, for x with $x_n \geq 0$, $x \in X_\alpha$ if and only if:

$$\sum_{i=1}^{n-1} x_i^2 - x_n^2 \leq \underbrace{v_n^2 - \sum_{i=1}^{n-1} v_i^2}_{=0 \quad \forall v \in V_\alpha} - 2\left(\sum_{j=1}^{n-1} x_j v_j - x_n v_n\right) \quad \forall v \in V_\alpha. \tag{4}$$

Moreover, using the definition of V_α, (4) can be rewritten as

$$\sum_{i=1}^{n-1} x_i^2 - x_n^2 \leq 2 \inf_{v \in \mathbb{R}^n} \left\{ x_n v_n - \sum_{j=1}^{n-1} x_j v_j : \alpha^T v = 1,\, v_n^2 = \sum_{i=1}^{n-1} v_i^2,\ v_n \geq 0 \right\}.$$

The right-hand side of this inequality involves optimizing a linear function over a nonconvex set. Nevertheless, note that

$$\begin{aligned}\overline{\operatorname{conv}} \left\{ v \in \mathbb{R}^n : \alpha^T v = 1,\, v_n^2 = \sum_{i=1}^{n-1} v_i^2,\ v_n \geq 0 \right\} &= \left\{ v \in \mathbb{R}^n : \alpha^T v = 1,\, v_n^2 \geq \sum_{i=1}^{n-1} v_i^2,\ v_n \geq 0 \right\} \\ &= \left\{ v \in \mathbb{R}^n : \alpha^T v = 1,\, v \in \mathcal{L}^n \right\}.\end{aligned}$$

Moreover, optimizing a linear function over a set is equivalent to optimizing it over its closed convex hull. Therefore, we arrive at the following inequality with a conic optimization problem

$$\sum_{i=1}^{n-1} x_i^2 - x_n^2 \leq 2 \inf_{v \in \mathbb{R}^n} \left\{ x_n v_n - \sum_{j=1}^{n-1} x_j v_j : \alpha^T v = 1,\, v \in \mathcal{L}^n \right\}.$$

Note that the above conic optimization problem is bounded below; and $v = \frac{\alpha}{\|\alpha\|_2}$ is a feasible solution to it. Next, we show that this problem is strictly feasible. Because α is an extreme ray of $\mathcal{L}^n$, there exists a unique ray $\gamma \in \mathcal{L}^n$ such that $\alpha^T \gamma = 0$, that is, $\gamma_i = -\alpha_i$ for $i = 1, \ldots, n-1$ and $\gamma_n = \alpha_n$. Let v be any feasible solution to this optimization problem. Then $v + \gamma$ is also feasible because v and γ are both in the cone and are not collinear and $\alpha^T(v + \gamma) = \alpha^T v = 1$. Moreover, since $\mathcal{L}^n$ is strictly convex, summing up any two distinct extreme rays from $\mathcal{L}^n$ results in an interior point of $\mathcal{L}^n$. Thus, $v + \gamma$ is a strictly feasible solution to the above optimization problem. Therefore, the strong conic duality theorem [8] together with $(\mathcal{L}^n)^* = \mathcal{L}^n$ implies

$$\begin{aligned}\sum_{i=1}^{n-1} x_i^2 - x_n^2 &\leq 2 \sup_{\tau \in \mathbb{R}} \left\{ \tau : \begin{bmatrix} -x_1 - \alpha_1 \tau \\ \vdots \\ -x_{n-1} - \alpha_{n-1}\tau \\ x_n - \alpha_n \tau \end{bmatrix} \in \mathcal{L}^n \right\} \\ &= 2 \sup_{\tau \in \mathbb{R}} \left\{ \tau : \sum_{i=1}^{n-1} (-x_i - \alpha_i \tau)^2 \leq (x_n - \alpha_n \tau)^2,\ x_n - \alpha_n \tau \geq 0 \right\} \\ &= 2 \sup_{\tau \in \mathbb{R}} \left\{ \tau : \sum_{i=1}^{n-1} x_i^2 - x_n^2 + 2(\alpha^T x)\tau + \underbrace{\left(\sum_{i=1}^{n-1} \alpha_i^2 - \alpha_n^2\right)}_{=0 \text{ as } \alpha \in \operatorname{Ext}(\mathcal{L}^n)} \tau^2 \leq 0,\ \tau \leq \frac{x_n}{\alpha_n} \right\} \\ &= 2 \sup_{\tau \in \mathbb{R}} \left\{ \tau : 2(\alpha^T x)\tau \leq x_n^2 - \sum_{i=1}^{n-1} x_i^2,\ \tau \leq \frac{x_n}{\alpha_n} \right\}.\end{aligned}$$

In order to finish the proof, it is sufficient to show that the above requirement is equivalent to $x \in \pm\mathcal{L}^n$. We will consider the following two cases separately:

Case 1: $\alpha^T x > 0$.
In this case we have

$$\sum_{i=1}^{n-1} x_i^2 - x_n^2 \leq \min\left\{\frac{2x_n}{\alpha_n}, \frac{\sum_{i=1}^{n-1} x_i^2 - x_n^2}{-(\alpha^T x)}\right\} \leq \frac{\sum_{i=1}^{n-1} x_i^2 - x_n^2}{-(\alpha^T x)},$$

equivalently $\left(\sum_{i=1}^{n-1} x_i^2 - x_n^2\right)\underbrace{\left(1+\frac{1}{(\alpha^T x)}\right)}_{>0} \leq 0$, which holds if and only if $x \in \pm\mathcal{L}^n$. Because $x_n \geq 0$, this is possible only if $x \in \mathcal{L}^n$.

Case 2: $\alpha^T x \leq 0$.
In this case we establish the result by showing that the optimization problem above is feasible only when $x \in \mathcal{L}^n$. Using the nonpositivity of $\alpha^T x$, we rewrite the constraints of the optimization problem as

$$\sum_{i=1}^{n-1} x_i^2 - x_n^2 \leq -2(\alpha^T x)\tau \leq -2(\alpha^T x)\frac{x_n}{\alpha_n}.$$

Then this implies

$$\sum_{i=1}^{n-1} x_i^2 - x_n^2 + 2x_n^2 + \frac{2x_n}{\alpha_n}\sum_{i=1}^{n-1} \alpha_i x_i \leq 0.$$

Because $\alpha \in \text{Ext}(\mathcal{L}^n)$, we have $\alpha_n^2 = \sum_{i=1}^{n-1} \alpha_i^2$, leading to:

$$0 \geq \sum_{i=1}^{n-1}\left(x_i^2 + \frac{2x_n}{\alpha_n}\alpha_i x_i + \frac{\alpha_i^2}{\alpha_n^2}x_n^2\right) = \sum_{i=1}^{n-1}\left(x_i + \frac{\alpha_i}{\alpha_n}x_n\right)^2.$$

This is possible only if $x_i = \frac{-\alpha_i}{\alpha_n}x_n \; \forall i = 1, ..n-1$ and thus:

$$\sum_{i=1}^{n-1} x_i^2 = \left(\frac{x_n^2}{\alpha_n^2}\right)\sum_{i=1}^{n-1}\alpha_i^2 = x_n^2,$$

which holds if and only if $x \in \pm\partial\mathcal{L}^n$.

Finally, combining $x \in \pm\partial\mathcal{L}^n$ with $x_n \geq 0$, we conclude that $x \in \mathcal{L}^n$. □

Lemma 2 and Proposition 6 imply that for $\mathcal{K} = \mathcal{L}^n$ with $n \geq 3$, conditions (**A.0**) and (**A.1**) are equivalent; and hence any valid inequality satisfying conditions (**A.0**) and (**A.2**) is $\mathcal{L}^n$-sublinear. In particular, by [21, Proposition 6], we arrive at the following corollary.

Corollary 3 *When $n \geq 3$, any valid inequality for $\mathcal{S}(A, \mathcal{L}^n, \mathcal{B})$ is $\mathcal{L}^n$-sublinear.*

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