Sufficient Conditions and Necessary Conditions for the Sufficiency of Cut-Generating Functions

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Submitted on December 2015; Revised on October 2017

Abstract

Cut-generating functions (CGFs) have been studied since 1970s in the context of Mixed Integer Linear Programs (MILPs) and more general disjunctive programs and have drawn renewed attention recently. CGFs are critical in generating valid inequalities separating the origin from the convex hull descriptions of disjunctive sets. The sufficiency of CGFs to generate all cuts that separate the origin from the convex hull of disjunctive sets is an indispensable question for the justification of this research focus on CGFs. While this question has been answered affirmatively in a number of setups and under a variety of structural assumptions, it still remains open in the most general case. In this paper, we pursue this question by providing the most general sufficient conditions for the sufficiency of CGFs and establishing necessary conditions that demonstrate that our sufficient conditions are almost necessary. In addition, we identify and address a related sufficiency question: when is it possible to generate *all* of the necessary inequalities (not just the ones separating the origin) for the convex hull description of disjunctive sets by finite-valued functions? Our approach relies on studying the properties of a particular class of support functions that also was recently studied by Kılınç-Karzan and Steffy.

1 Introduction

In this paper, we study *disjunctive sets* of form

$$\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B}) := \{x \in \mathbb{R}^n : \ Ax \in \mathcal{B}, \ x \in \mathbb{R}^n_+\},$$

where A is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$, and $\mathcal{B}$ is a nonempty subset of $\mathbb{R}^m$. Usually, $\mathcal{B}$ is taken as a general nonempty, nonconvex set; thus $\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B})$ is nonconvex. We are interested in the valid inequalities describing the structure of $\overline{\text{conv}}(\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B}))$—the closed convex hull of $\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B})$. Since the cases $\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B}) = \emptyset$ and $\overline{\text{conv}}(\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B})) = \mathbb{R}^n_+$ are trivial, in this paper, we consider only the cases where $\overline{\text{conv}}(\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B}))$ is neither empty, nor is it equal to $\mathbb{R}^n_+$.

When $\mathcal{B}$ is a finite set, $\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B})$ is simply a disjunctive set such as those introduced and studied by Balas [2]. Furthermore, the set $\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B})$ with a closed set $\mathcal{B}$ satisfying $0 \notin \mathcal{B}$

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naturally arises in the context of separating a fractional solution from the feasible region of a Mixed Integer Linear Program (MILP) [16, 18]. In this context, Johnson [18] introduced and characterized minimal valid linear inequalities for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ where $\mathcal{B}$ is a finite set; Jeroslow [16] provided an explicit characterization of minimal inequalities based on the value functions of MILPs for MILPs with bounded feasible regions; and Blair [6] extended this characterization to MILPs with rational data. This body of work has strong connections to the subadditive strong duality theory for MILPs; see [15] for a survey of the earlier literature on the subadditive approach to MILP. The set $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ also arises in Gomory's corner relaxation after a standard transformation that embeds the integrality restrictions on the integer variables into the description of the set $\mathcal{B}$, see e.g., [8, Example 1.1]. Following up on Johnson's framework from [18], Kılınç-Karzan [19] introduced and studied disjunctive conic sets $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ where $\mathcal{B}$ is an arbitrary nonconvex (possibly infinite) set and the constraints $x \in \mathbb{R}^n_+$ in $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ is replaced with $x \in \mathcal{K}$ defined by a general regular (full-dimensional, closed, convex, and pointed) cone $\mathcal{K}$.

Given $\mathcal{B}$, an important class of papers study an infinite family of sets of the form $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ by varying A and n. This line of research is primarily motivated by the infinite group relaxations studied in the MILP context. In these infinite relaxations, the family of sets $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ are characterized solely by $\mathcal{B}$ and A is assumed to be composed of all possible column vectors from $\mathbb{R}^m$. The origin of these studies dates back to Gomory's foundational work on integer programming [13] where *cut-generating functions* (CGFs)—finite-valued functions that generate cut coefficients c_i for the cuts of form $\sum_{i=1}^n c_i x_i \geq 1$ based on solely the data A_i associated with a particular variable x_i—are introduced and examined for the first time. This was followed up by Gomory and Johnson [12] and others [17, 1] for infinite group relaxations associated with MILPs. Recent work along these lines has studied these infinite relaxations under a variety of structural assumptions on $\mathcal{B}$ and established strong connections between minimal inequalities and CGFs obtained from the gauge functions of maximal lattice-free sets for example when $\mathcal{B}$ is a general lattice [7] and when $\mathcal{B}$ is composed of lattice points contained in a rational polyhedron [11, 3]. We refer the readers to [5, 4] for recent surveys related to these infinite relaxations.

Motivated by the infinite relaxations used in the MILP context and to eliminate various structural assumptions imposed on $\mathcal{B}$ in the literature, Conforti et al. [8] studied the variant of $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ with varying n and $A \in \mathbb{R}^{m \times n}$ but a fixed nonempty closed set $\mathcal{B} \in \mathbb{R}^m$ under the assumption that $0 \notin \mathcal{B}$. This assumption immediately implies $0 \notin \overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ (see [8, Lemma 2.1]) and motivates the authors to focus on generating cuts of form $\sum_{i=1}^n c_i x_i \geq 1$ that separate the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. Within this context, Conforti et al. [8] extended the concept of CGFs for these sets parametrized with a general $\mathcal{B}$ and studied the structure of CGFs and their desirable properties, e.g., minimality, and their relation with the gauge functions of $\mathcal{B}$-free neighborhoods of the origin.

While the definition of CGFs places an emphasis on only the cuts separating the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ as opposed to generating all of the inequalities needed for the convex hull description, the sufficiency of CGFs for generating all cuts separating the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ has been vital to justify the recent research focus on CGFs. In the context of infinite relaxations associated with $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ where $m = 1$ and $\mathcal{B} = b + \mathbb{Z}$ for some $b \notin \mathbb{Z}$, such a sufficiency result can be traced back to Gomory and Johnson [12]. For the case of $m = 2$ and $\mathcal{B} = b + \mathbb{Z}^m$ for some $b \notin \mathbb{Z}^m$, Cornuéjols and Margot [9, Theorem 3.1] established that every inequality of form $\sum_{i=1}^n c_i x_i \geq 1$ valid for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ for all choices of A can be generated by a CGF. For the more general infinite relaxation where $\mathcal{B}$ is assumed to be a lattice of the form $\mathcal{B} = b + \mathbb{Z}^m$ for some

$b \notin \mathbb{Z}^m$, Zambelli [22, Theorem 1] showed that CGFs are sufficient to generate all cuts separating the origin. These results were further extended by Dey and Wolsey [11, Proposition 3.7] to the case where $\mathcal{B} = P \cap (b + \mathbb{Z}^m)$ such that $b \notin \mathbb{Z}^m$, a $P \subset \mathbb{R}^m$ is rational polyhedron and under the assumption that $\operatorname{conv}(\mathcal{B})$ is full dimensional. For general sets $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ with a varying matrix A and an arbitrary closed set $0 \notin \mathcal{B}$, the sufficiency of CGFs was shown in [8, Theorem 6.3] under the assumption that $\operatorname{cone}(A) = \mathbb{R}^m$ where $\operatorname{cone}(A)$ is the convex cone generated by the columns of A. On the other hand, in the more general setup of Conforti et al. [8] which involves a varying matrix A and an arbitrary closed set $0 \notin \mathcal{B}$ that is not necessarily a general lattice, such general sufficiency results for CGFs are no longer attainable without the assumption that $\operatorname{cone}(A) = \mathbb{R}^m$. Specifically, [8, Example 6.1] demonstrates a particular instance of A and $\mathcal{B}$ where not all necessary cuts separating the origin from $\overline{\operatorname{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ can be generated by CGFs. Later on, in the framework of [8], Cornuéjols, Wolsey and Yıldız [10, Theorem 1.1] established that CGFs are sufficient to give all of the cuts separating the origin from $\overline{\operatorname{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ under the structural assumption that $\mathcal{B} \subseteq \operatorname{cone}(A)$. Nevertheless, to the best of our knowledge, the complete sufficiency status of CGFs for generating all of the necessary inequalities separating the origin from $\overline{\operatorname{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$, with or without varying matrices A, still remains an open question. This is also stated as an open question recently in Basu et al. [4], and it is one of the main focuses of our paper.

Another focus of our paper is on identifying when finite-valued functions, in a way analogous to CGFs, are sufficient to generate all necessary inequalities for the complete description of $\overline{\operatorname{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. Such a sufficiency question encompasses the sufficiency question for CGFs to generate all necessary valid inequalities separating the origin from $\overline{\operatorname{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. Moreover, both the sufficiency of finite-valued functions for generating all necessary inequalities for $\overline{\operatorname{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ and the sufficiency of CGFs for generating inequalities that separate the origin are intrinsically related to the subadditive duality theory for MILPs. The feasible region of an MILP has a natural representation in the form of $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ where $\mathcal{B}$ possesses a specific structure, see [19, Example 2]. In addition, according to the subadditive strong duality theorem for MILPs, there exists a dual problem of the MILP based on functions that generate cut coefficients, and this dual achieves zero duality gap. In particular, the feasible region of the dual of an MILP is defined by all finite-valued, subadditive functions that are nondecreasing with respect to $\mathbb{R}^m_+$. In addition, such functions indeed produce the coefficients μ_i of any valid inequality $\mu^\top x \geq \mu_0$ by considering only the data A_i associated with each individual variable x_i. Therefore, these functions from MILP duals are closely related to CGFs whenever the inequality $\mu^\top x \geq \mu_0$ under consideration satisfies $\mu_0 > 0$. As a result, the strong MILP duality theorem implies the sufficiency of CGFs for generating all of the cuts of form $c^\top x \geq 1$ valid for $\overline{\operatorname{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ when $0 \notin \mathcal{B}$ and $\mathcal{B}$ has a specific structure. Furthermore, Morán et al. [21] has extended the strong duality theory for MILPs to MICPs of a specific form under a technical condition ([21, Theorem 2.4]). The feasible sets of MICPs studied in [21] can be represented in disjunctive form $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$, where the conic structure is embedded in the definition of the set $\mathcal{B}$ (see [19, Example 3]). We refer the readers to [19, Remark 12] and [20, Remark 2] for additional discussion relating the work of Morán et al. [21] to CGFs. Nevertheless, the sets $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ representing MILPs and these specific MICPs from [21] impose a specific structure on $\mathcal{B}$ and their sufficiency is established under some technical assumptions. Thus, these results on strong MILP (or MICP) duals do not fully answer the question on the sufficiency of finite-valued functions for generating all necessary inequalities for $\overline{\operatorname{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ or the sufficiency of CGFs for generating cuts separating the origin in the most general case.

The sufficiency questions on functions generating necessary valid inequalities for $\overline{\operatorname{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$

on CGFs for generating cuts separating the origin require an understanding of which inequalities are necessary in the description of $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. In this respect, for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ with a finite $\mathcal{B}$ Johnson [18] introduces and studies a hierarchy of classes of inequalities: extreme inequalities, minimal inequalities, and sublinear (subadditive) inequalities. In [19], these concepts are further generalized for $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ by considering arbitrary nonconvex sets $\mathcal{B}$ and looking into domination among inequalities with respect to a regular cone $\mathcal{K}$ as opposed to $\mathbb{R}^n_+$. In this hierarchy, extreme inequalities are the strongest possible and are necessary for the description of $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ in addition to the constraint $x \in \mathbb{R}^n_+$. On the other hand, $\mathbb{R}^n_+$-sublinear inequalities are defined based on some necessary conditions for $\mathbb{R}^n_+$-minimality, and thus $\mathbb{R}^n_+$-minimal inequalities are $\mathbb{R}^n_+$-sublinear. $\mathbb{R}^n_+$-sublinear inequalities have the desirable property that they admit easier algebraic characterizations. In particular, Kılınç-Karzan and Steffy [20] used these algebraic characterizations of $\mathcal{K}$-sublinear inequalities for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ in examining the connection between $\mathbb{R}^n_+$-sublinear inequalities for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ and the CGFs and then introducing the concept of *relaxed cut-generating functions* (relaxed CGFs) as the support functions of nonempty sets in the space of $\mathcal{B}$. It was shown in [20] that without any technical assumptions, the relaxed CGFs are sufficient to generate all necessary inequalities that separate the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ even when n and A are arbitrary and $\mathcal{B}$ is a general set. This is in contrast to the fact that establishing the sufficiency of regular CGFs requires additional structural assumptions. A major differentiating point between regular CGFs and relaxed CGFs is that regular CGFs are finite-valued everywhere while relaxed CGFs are not, and the finite-valuedness of CGFs is crucial for producing nontrivial valid inequalities for *all* instances of $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ with a fixed $\mathcal{B}$ but varying A and n.

In this paper, we pursue open questions surrounding the sufficiency of finite-valued sublinear functions to generate valid inequalities for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ in two different contexts:

(i) First, we examine the question of given a nonconvex set $\mathcal{B}$, whether we can generate *all* of the necessary inequalities for $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ by finite-valued sublinear functions.

(ii) Second, we look at the case of a given $\mathcal{B}$ satisfying $0 \notin \text{cl}(\mathcal{B})$ and ask: are all of the necessary valid inequalities of the form $c^\top x \geq 1$ that separate the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ generated by CGFs?

The main distinction between these two cases is that the first one allows us to study all of the necessary valid inequalities for $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ (including but not necessarily limited to the ones separating the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$) while the second one focuses on only the ones that separate the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. To the best of our knowledge all of the prior literature has focused on only CGFs generating cuts that separate the origin. This is despite the fact that even when $0 \notin \overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ the description of $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ clearly may involve inequalities with right hand sides normalized to 0 or -1. Therefore, our results associated with our first question are new contributions. For our second question, we provide the most general sufficient conditions for CGFs to generate all cuts separating the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ when $0 \notin \text{cl}(\mathcal{B})$ (we will see later on that $0 \notin \text{cl}(\mathcal{B})$ implies $0 \notin \overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$). In particular, our sufficient conditions for CGFs in the context of the second question not only capture the existing conditions from the literature but also go further beyond. Our approach for both of these questions relies on constructing a specific class of support functions that are finite-valued everywhere and showing that under certain conditions, these functions are sufficient to generate *all* the inequalities of interest. For the first question, our sufficient conditions for the sufficiency of finite-valued support functions generating all necessary valid inequalities describing $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ include the case where $\mathcal{B} \setminus \text{cone}(A)$ is compact and the case where the closure of $\mathcal{B} \setminus \text{cone}(A)$ does not contain the origin and $\mathcal{B} \setminus \text{cone}(A)$ itself is contained in a closed cone intersecting $\text{cone}(A)$ only at the origin; see Propositions 3.6 and 3.8. For the second question, our sufficient conditions are slightly more general than the aforementioned cases; see Corollary 3.12 for a complete description of our sufficient conditions in the CGF context. To the best of our knowledge, the only sufficient condition studied in the previous literature in the CGF context was $\mathcal{B} \subseteq \text{cone}(A)$, see [8, 10]. Such a condition is not necessarily satisfied in the separation problems arising in the MILP context. On the other hand, our sufficient conditions for example cover the case of $\mathcal{B}$ being a compact set. This is immediately applicable in the MILP context when the integer variables are bounded as it leads to $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ with a finite set $\mathcal{B}$. Finally, in our developments, we also establish that if an extreme inequality can be generated by a CGF, then it can as well be generated by the support function of a specific bounded set; see Proposition 4.1. This observation plays a critical role in establishing our necessary conditions for the sufficiency of CGFs; see Corollary 4.4. Our sufficient conditions and necessary conditions for the sufficiency of CGFs are very close (see Corollaries 3.12 and 4.4); yet they do not match precisely. We conclude our study by providing examples to illustrate the gap between our sufficient conditions and necessary ones for the sufficiency of CGFs.

The remainder of the paper is organized as follows. Section 2 introduces our notation and describes previous results as they relate to minimal inequalities, sublinear inequalities, CGFs, and support function view on generation of valid inequalities. Sections 3 and 4 study respectively the sufficient conditions for both for the sufficiency of finite-valued sublinear functions and CGFs and the necessary conditions for the sufficiency of CGFs.

2 Notation and Preliminaries

We start by introducing our notation. For a set $S \subset \mathbb{R}^n$, we denote its topological interior, closure, boundary, convex hull, and closed convex hull by $\text{int}(S)$, $\text{cl}(S)$, $\text{bd}(S) := \text{cl}(S) \setminus \text{int}(S)$, $\text{conv}(S)$, and $\overline{\text{conv}}(S)$ respectively. We let $\text{cone}(S) := \{\alpha x + \beta y : x, y \in S, \alpha, \beta \geq 0\}$ denote the convex cone generated by S. We define the positive hull of S to be $\mathbb{R}_{++}(S) := \{ts : s \in S, t > 0\}$. While $\text{cone}(S)$ and $\mathbb{R}_{++}(S)$ are closely related, they differ in terms of their convexity properties. Note that even when S is nonconvex, $\mathbb{R}_{++}(S)$ can be nonconvex as well. Besides, 0 is not necessarily in $\mathbb{R}_{++}(S)$, and $\mathbb{R}_{++}(S)$ may not be closed. We denote the recession cone of S by $\text{Rec}(S) := \{y \in \mathbb{R}^n : x + \lambda y \in S \text{ for all } x \in S \text{ and } \lambda \geq 0\}$. The support function of S is defined as

$$\sigma_S(z) := \sup_{s \in \mathbb{R}^n} \{z^\top s : s \in S\}.$$

We define the kernel of a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ as $\text{Ker}(A) := \{u \in \mathbb{R}^n : Au = 0\}$ and its image as $\text{Im}(A) := \{Au : u \in \mathbb{R}^n\}$. For convenience, we also treat A as a real matrix and use $\text{cone}(A)$ to represent the convex cone generated by the columns of A. Given a cone $\mathcal{K} \subset \mathbb{R}^n$, we use $\mathcal{K}^* := \{y \in \mathbb{R}^n : x^\top y \geq 0 \, \forall x \in \mathcal{K}\}$ for its dual cone.

Throughout the paper, we use Matlab notation to denote vectors and matrices, and all vectors are to be understood in column form.

2.1 Classes of Valid Linear Inequalities

Given $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$, we are interested in the valid linear inequalities for $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. Consider the set of all vectors $0 \neq \mu \in \mathbb{R}^n$ such that $\vartheta(\mu)$ defined as

$$\vartheta(\mu) := \inf_x \left\{ \mu^\top x : x \in \mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}) \right\} \tag{1}$$

is finite. Then any nonzero vector $\mu \in \mathbb{R}^n$ and a number $\mu_0 \le \vartheta(\mu)$ defines a *valid linear inequality* of the form $\mu^\top x \ge \mu_0$ for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$. For shorthand notation, we denote the corresponding valid inequality by $(\mu; \mu_0)$. When $\vartheta(\mu) = -\infty$, we say that the inequality generated by μ is *trivial*. We refer to a valid inequality $(\mu; \mu_0)$ as *tight*[1)] if $\mu_0 = \vartheta(\mu)$.

Remark 2.1. *For any $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$, it is shown in [19, Proposition 6] that all nontrivial valid inequalities $(\mu; \mu_0)$ satisfy $\mu \in \mathbb{R}^n_+ + \mathrm{Im}(A^\top)$.* $\diamondsuit$

We define $C(A, \mathbb{R}^n_+, \mathcal{B}) = \{(\mu; \mu_0) \in \mathbb{R}^n \times \mathbb{R} : \ \mu_0 \le \vartheta(\mu)\}$ as the convex cone of all valid linear inequalities for the set $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$. Note that any convex cone K can be written as the sum of a linear subspace L and a pointed cone C. Here L represents the largest linear subspace contained in the cone K, also referred to as the *lineality space* of K. A unique representation of K in the form of $K = L + C$ can be obtained by requiring that C is contained in the orthogonal complement of L. A *generating set* (G_L, G_C) for a cone K is defined to be a minimal set of elements $G_L \subseteq L$, $G_C \subseteq C$ such that

$$K = \left\{ \sum_{w \in G_L} \alpha_w w + \sum_{v \in G_C} \lambda_v v : \ \lambda_v \ge 0 \right\}.$$

Given $A, \mathcal{B}$, an inequality $(\mu; \mu_0) \in C(A, \mathbb{R}^n_+, \mathcal{B})$ is called an *extreme inequality* if there exists a generating set for $C(A, \mathbb{R}^n_+, \mathcal{B})$ including $(\mu; \mu_0)$ as a generating inequality either in G_L or in G_C. When $\overline{\mathrm{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ is full dimensional, i.e., $G_L = \{0\}$, and polyhedral, a valid inequality is extreme if and only if it is a facet-defining inequality for $\overline{\mathrm{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. When $\overline{\mathrm{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ is nonpolyhedral, it may not have any facets. Therefore, in this paper we work with the concept of extreme inequalities as opposed to facet defining ones in comparing the strength of inequalities.

Understanding the structure of extreme valid linear inequalities is critical in terms of understanding the structure of $\overline{\mathrm{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. On the other hand, characterizing all extreme inequalities can be quite difficult for an arbitrary set $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$. A middle ground is obtained by studying the structure of slightly larger classes of inequalities. In particular, for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$, classes of $\mathbb{R}^n_+$-*minimal* and $\mathbb{R}^n_+$-*sublinear* ($\mathbb{R}^n_+$-subadditive) inequalities, where these notions are defined based on domination relations among inequalities with respect to $\mathbb{R}^n_+$, were introduced in [18] and further generalized to $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ with a regular cone $\mathcal{K}$ and studied in [19, 20]. A valid inequality $(\mu; \mu_0)$ *is dominated with respect to the cone* $\mathcal{K}$ *by* another valid inequality $(\rho; \rho_0)$ whenever $\mu - \rho \in \mathcal{K}^* \setminus \{0\}$ and $\rho_0 \ge \mu_0$, i.e., when $(\mu; \mu_0)$ is a consequence of the inequality $(\rho; \rho_0)$ and the constraint $x \in \mathcal{K}$. A valid inequality $(\mu; \mu_0)$ is $\mathcal{K}$-*minimal* if it is not dominated by any other valid inequality in this sense (see [19] for general regular cones $\mathcal{K}$ and [18] for $\mathcal{K} = \mathbb{R}^n_+$). Based on this domination notion, in the case of $\mathcal{K} = \mathbb{R}^n_+$, an inequality $(\mu; \mu_0)$ is $\mathbb{R}^n_+$-minimal if reducing any μ_i for $i \in \{1, \dots, n\}$ leads to a strict reduction in the right hand side value μ_0.[2)]

It is well-known [19, Proposition 2 and Corollary 2] that whenever $\mathcal{K}$-minimal inequalities exist, they are sufficient to describe $\overline{\mathrm{conv}}(\mathcal{S}(A, \mathcal{K}, \mathcal{B}))$ together with the original constraint $x \in \mathcal{K}$, and that $\mathcal{K}$-minimal inequalities exist when $\overline{\mathrm{conv}}(\mathcal{S}(A, \mathcal{K}, \mathcal{B}))$ is full dimensional. By isolating a number of algebraic necessary conditions for $\mathcal{K}$-minimality, [19] suggested the class of $\mathcal{K}$-*sublinear* inequalities that contain $\mathcal{K}$-minimal inequalities (see [19, Theorem 1]). When $\mathcal{K} = \mathbb{R}^n_+$, the $\mathbb{R}^n_+$-*sublinear inequalities* of [19] are indeed equivalent to the *subadditive inequalities* introduced in [18]

1) We note that our definition of *tightness* of an inequality does not require the corresponding hyperplane to have a nonempty intersection with the feasible region, as is sometimes the definition used in the literature.

2) The valid inequalities that are referred as minimal in [1, 6, 16] correspond to *tight* and $\mathbb{R}^n_+$-*minimal* inequalities with respect to the definitions in this paper as well as in [18, 19, 20].

(see e.g., [19, Remark 9]). The existence, sufficiency, and properties of $\mathcal{K}$-sublinear inequalities were further studied in [20] without making technical assumptions ensuring the existence of $\mathcal{K}$-minimal inequalities. Moreover, [20] also examined the connection between $\mathbb{R}^n_+$-sublinear inequalities and CGFs.

In this paper, we will focus on the concept of domination induced by the cone $\mathcal{K} = \mathbb{R}^n_+$. We will frequently use the notation and results from [19] and [20] related to $\mathbb{R}^n_+$-minimal and $\mathbb{R}^n_+$-sublinear inequalities. Because our focus in this paper is on the case of $\mathcal{K} = \mathbb{R}^n_+$, in order to simplify our terminology, we will refer to these inequalities simply as *minimal* and *sublinear* by dropping the $\mathbb{R}^n_+$- prefix. As far as this paper is concerned, we restate the definition of sublinear inequalities below and refer the reader to [19, 20] for related definitions and discussions in the case of general sets $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ involving a regular cone $\mathcal{K}$:

Definition 2.1. *Given $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$, a linear inequality $(\mu; \mu_0)$ with $\mu \neq 0$ and $\mu_0 \in \mathbb{R}$ is* sublinear *if it is valid for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ and for $i = 1, \ldots, n$, $\mu^\top u \geq 0$ holds for all u such that $Au = 0$ and $u + e_i \in \mathbb{R}^n_+$ where e_i denotes the i^{th} unit vector in $\mathbb{R}^n$.*

A number of entities and results from [19, 20] play critical roles in the characterization of sublinear inequalities and their connection with CGFs. Consider $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ and a nontrivial valid inequality $(\mu; \mu_0)$ for it. By Remark 2.1, we have $\mu \in \mathbb{R}^n_+ + \mathrm{Im}(A^\top)$. This allows us to associate with μ the following nonempty set

$$D_\mu := \{\lambda \in \mathbb{R}^m : \ A^\top \lambda \leq \mu\}, \tag{2}$$

and its support function $\sigma_{D_\mu}(\cdot)$. We next summarize a number of results from [19] that are functional in our analysis in the context of $\mathcal{K} = \mathbb{R}^n_+$.

Theorem 2.2. *Consider $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$. Then any nontrivial valid inequality $(\mu; \mu_0)$ satisfies*

(i) $\mu \in \mathbb{R}^n_+ + \mathrm{Im}(A^\top)$ (see [19, Proposition 6]),

(ii) $\vartheta(\mu) = \inf_{b \in \mathcal{B}} \sigma_{D_\mu}(b)$ (see [19, Proposition 8]), and

(iii) $\vartheta(\mu) \geq \mu_0$ (immediately follows from the validity of the inequality $(\mu; \mu_0)$).

Moreover, $(\mu; \mu_0)$ is a sublinear inequality if and only if it is valid ($\mu_0 \leq \vartheta(\mu)$) and $\sigma_{D_\mu}(A_i) = \mu_i$ for all $i = 1, \ldots, n$ where A_i denotes the i-th column of the matrix A (see [19, Theorem 4 and Proposition 10]).

We refer the interested reader to [18, Theorems 9-10] and [19, Remarks 9, 10, and 11] respectively for prior work and further comments related to the results summarized in Theorem 2.2.

It is shown [20, Proposition 2] that as long as $\overline{\mathrm{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})) \neq \mathbb{R}^n_+$, sublinear inequalities must exist. Moreover, one of the main results of [20] establishes that sublinear inequalities are always sufficient to describe $\overline{\mathrm{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. We restate [20, Proposition 3] below.

Proposition 2.3. *[20] Any nontrivial valid inequality $(\mu; \mu_0)$ for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ is equivalent to or dominated by a sublinear inequality given by $(\eta; \mu_0)$ where $\eta_i = \sigma_{D_\mu}(A_i)$ for all $i = 1, \ldots, n$ and the domination is defined with respect to the cone $\mathcal{K} = \mathbb{R}^n_+$.*

We highlight that unlike the existence and sufficiency of minimal inequalities, Proposition 2.3 does not make any assumptions on $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$. In addition, Proposition 2.3 establishes that for any $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ defined by a fixed A and $\mathcal{B}$, all of the extreme inequalities are sublinear, and thus when $\overline{\mathrm{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ is polyhedral, all of its facet-defining inequalities are also sublinear inequalities as well.

2.2 Support Functions and Cut-Generating Functions

Proposition 2.3 signals the importance of certain support functions in generating valid inequalities. Note that support functions are sublinear. Next, we recall the following basic fact about sublinear functions which is also an important component of the subadditive duality theory for MIPs:

Lemma 2.4. *Suppose $\mathcal{B} \subset \mathbb{R}^m$ is given. Let $\sigma(\cdot)$ be any sublinear function. Then, the inequality $\sum_{i=1}^{n'} \sigma(A'_i)x_i \geq \inf_{b\in\mathcal{B}} \sigma(b)$ is valid for any $x \in \mathcal{S}(A', \mathbb{R}^{n'}_+, \mathcal{B})$ where the dimension n' and the matrix $A' \in \mathbb{R}^{m\times n'}$ are arbitrary, and A'_i denotes the i-th column of the matrix A'.*

Proof. Consider any $\bar{x} \in \mathcal{S}(A', \mathbb{R}^{n'}_+, \mathcal{B})$. Then there exists $\bar{b} \in \mathcal{B}$ such that $A'\bar{x} = \bar{b}$. Then we have the following relations

$$\sum_{i=1}^{n'} \sigma(A'_i)\bar{x}_i \geq \sigma(A'\bar{x}) = \sigma(\bar{b}) \geq \inf_{b\in\mathcal{B}} \sigma(b),$$

where the inequality holds because $\bar{x}$ is nonnegative and σ is sublinear, i.e., subadditive and positively homogeneous. □ □

Lemma 2.4 establishes that one can use sublinear functions in a structured way to generate valid linear inequalities for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$.

Yet Lemma 2.4 alone does not say anything about the sufficiency of sublinear functions to generate all necessary valid inequalities for the description of $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. On the other hand, Proposition 2.3 establishes the sufficiency of sublinear inequalities. Because every sublinear inequality $(\mu; \mu_0)$ is generated by a particular sublinear function, i.e., the support function of a nonempty set of form $D_\mu = \{\lambda \in \mathbb{R}^m : A^\top\lambda \leq \mu\}$ (see Theorem 2.2), these support functions are sufficient to generate all necessary valid inequalities for $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$.

Remark 2.5. *We infer from Theorem 2.2, Proposition 2.3 and Lemma 2.4 that the support functions, in particular the ones associated with the sets D_μ with $\mu \in \mathbb{R}^n_+ + \text{Im}(A^\top)$, are sufficient to generate all of the necessary nontrivial valid inequalities for $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ without any structural or technical assumptions, even when A and n are varying.* ◊

In addition to the discussion in Remark 2.5, note that the support functions σ_{D_μ} may not be finite-valued everywhere. In order to tackle the finite-valuedness condition on functions designed to generate valid inequalities for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$, in this paper, given a vector $\mu \in \mathbb{R}^n_+ + \text{Im}(A^\top)$ and $\rho > 0$, we will frequently study the support functions of specific bounded sets of the form $D_{\mu,\rho}$ where

$$D_{\mu,\rho} := \{\lambda \in D_\mu : \|\lambda\|_\infty \leq \rho\}.$$

We also note the following useful fact on the support functions of nonempty bounded sets.

Remark 2.6. *Let $D \subset \mathbb{R}^m$ be a nonempty, bounded set. Then, its support function σ_D is continuous everywhere. This is because support functions of nonempty sets are convex in general, and the support functions of nonempty bounded sets are finite-valued everywhere. Thus, the domain of σ_D is $\mathbb{R}^m$. Then using the fact that all convex functions are continuous in the interior of their domains (see for example [14, Lemma B.3.1.1]), we conclude that σ_D is continuous everywhere.* ◊

Conforti et al. [8] studied a variant of the set $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ with a fixed, closed, nonempty set $\mathcal{B} \in \mathbb{R}^m$, and varying n and $A \in \mathbb{R}^{m\times n}$ under the assumption that $0 \notin \mathcal{B}$. This assumption immediately implies $0 \notin \overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ (see [8, Lemma 2.1]) and motivates the authors to focus on generating cuts that separate the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. For this particular setup, Conforti et al. [8] introduced the concept of a *cut-generating function* as follows:

Definition 2.2. *Given a nonempty and closed set $\mathcal{B} \in \mathbb{R}^m$ satisfying $0 \notin \mathcal{B}$, a* cut-generating function *(CGF) for $\mathcal{B}$ is a function $f : \mathbb{R}^m \to \mathbb{R}$ such that for any natural number $n \in \mathbb{N}$ and any matrix $A \in \mathbb{R}^{m\times n}$, the linear inequality given by $\sum_{i=1}^n f(A_i)x_i \geq 1$ is valid for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ where A_i is the i-th column of the matrix A.*

The definition of CGFs immediately leads to the following simple yet useful lemma.

Lemma 2.7. *Given a nonempty set $\mathcal{B} \subset \mathbb{R}^m$, let $f(\cdot)$ be a CGF generating a valid inequality of the form $\sum_i f(A_i)x_i \geq 1$, then $\inf_{b\in\mathcal{B}} f(b) \geq 1$.*

Proof. Because $f(\cdot)$ is a CGF for the given set $\mathcal{B}$, for any dimension n' and any matrix $A' \in \mathbb{R}^{m\times n'}$, the inequality $\sum_{i=1}^{n'} f(A'_i)x'_i \geq 1$ generated by $f(\cdot)$ for the set $\mathcal{S}(A', \mathbb{R}^{n'}_+, \mathcal{B})$ needs to be valid, i.e., it is satisfied for all $x' \in \mathcal{S}(A', \mathbb{R}^{n'}_+, \mathcal{B})$ (see Definition 2.2). For any $b \in \mathcal{B}$, we construct an instance $\mathcal{S}(A', \mathbb{R}^{n'}_+, \mathcal{B})$ where $n' = 1$ and $A' = b$. Since $x' = 1 \in \mathcal{S}(A', \mathbb{R}^{n'}_+, \mathcal{B})$, $f(b) = \sum_{i=1}^{n'} f(A'_i)x'_i \geq 1$ holds, where the last inequality follows from $f(\cdot)$ being a CGF. Because this is true for all $b \in \mathcal{B}$, we arrive at $\inf_{b\in\mathcal{B}} f(b) \geq 1$. □ □

Motivated by the connection between sublinear inequalities and support functions used in generating such valid inequalities, Kılınç-Karzan and Steffy [20] introduced the following concept of *relaxed CGFs*:

Definition 2.3. *Given $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ and a set $\emptyset \neq D \subset \mathbb{R}^m$, the support function $\sigma_D : \mathbb{R}^m \to (\mathbb{R} \cup +\infty)$ of D is a* relaxed cut-generating function *for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ when $\inf_{b\in B} \sigma_D(b) \geq 1$.*

Relaxed CGFs are naturally related to regular CGFs. Along the lines of Remark 2.5, we note that an immediate corollary of Theorem 2.2, Proposition 2.3 and Lemmas 2.4 and 2.7 stated in the setup of Conforti et al. [8] is as follows:

Corollary 2.8. *[20] Let A_i be the i-th column of the matrix A for all $i = 1, \ldots, n$. Then any valid inequality $c^\top x \geq 1$ separating the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ is equivalent to or dominated by one of the form $\sum_{i=1}^n \sigma_{D_c}(A_i)x_i \geq 1$, obtained from a relaxed CGF $\sigma_{D_c} : \mathbb{R}^m \to (\mathbb{R} \cup +\infty)$.*

Corollary 2.8 implies that the relaxed CGFs are sufficient to generate all of the cuts separating the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ without any structural or technical assumptions, even when A and n are varying. In contrast to the sufficiency of relaxed CGFs, there are sets of the form $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ such that CGFs are not sufficient to generate all of the cuts separating the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ (see [8, Example 6.1]). In the framework of [8], the sufficiency of CGFs for generating all necessary cuts separating the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ was established in [10] under the additional structural assumption that $\mathcal{B} \subseteq \text{cone}(A)$. This result on sufficiency of CGFs was also reproven in [20, Proposition 5] by starting from the sufficiency of sublinear inequalities and their connection with relaxed CGFs and then showing that a specific class of finite-valued relaxed CGFs are sufficient under the same structural assumption $\mathcal{B} \subseteq \text{cone}(A)$. In particular, given an inequality $c^\top x \geq 1$ that is valid for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$, [20, Proposition 5] establishes that when

$\mathcal{B} \subseteq \text{cone}(A)$, we can always construct a set $D_{c,\rho}$ based on the vector c and some $\rho > 0$ such that the support function $\sigma_{D_{c,\rho}}(\cdot)$, generates a valid inequality which is equivalent to or dominates $c^\top x \geq 1$. Because the support functions of form $\sigma_{D_{c,\rho}}(\cdot)$ are finite-valued, they are indeed regular CGFs, and then this result implies that CGFs are also sufficient to generate all cuts separating the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ when $\mathcal{B} \subseteq \text{cone}(A)$.

There is a contrast between the sufficiency of relaxed CGFs and the insufficiency of regular CGFs. A major differentiating point between regular CGFs and relaxed CGFs is that regular CGFs are finite-valued everywhere while relaxed CGFs are not. In fact, in Lemma 2.4 and Corollary 2.8, the relaxed CGFs are simply support functions of some possibly unbounded sets and thus are not guaranteed to be finite-valued everywhere. For a specific instance $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ with a fixed matrix A, as long as a relaxed CGF is finite-valued for each column of A, it will generate nontrivial valid inequalities. As a result, a relaxed CGF being finite-valued is not necessary for this case. However, given a fixed $\mathcal{B}$, a CGF has to work, i.e., generate nontrivial valid inequalities, for *every instance* of $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ with varying n and A. Then, in these cases, it is critical to require the function to be finite-valued everywhere to serve as a regular CGF. This need for finite-valuedness of functions to be used for all instances of $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ with varying A and n naturally brings up the question of in what circumstances CGFs are sufficient to generate all of the necessary cuts of the form $c^\top x \geq 1$ that separate the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. In the next section, we explore such conditions. Furthermore, as a prelude to the preceding question, we also address the more general question of given $\mathcal{B}$, under what conditions we can generate *all* of the necessary inequalities needed for $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ by finite-valued support functions.

3 Sufficient Conditions for the Sufficiency of Support Functions and CGFs

The sufficiency of finite-valued support functions, as well as the sufficiency of CGFs, is primarily related to the question of whether every *extreme* inequality can be generated by such a function. Thus, we will keep our focus in this section as well as the next one on the extreme inequalities when needed.

We will first focus on the separation of *all* necessary valid inequalities defining $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. Our approach relies on establishing that under certain conditions, a particular class of polyhedral support functions that are finite-valued everywhere are sufficient to generate *all* necessary valid inequalities for the closed convex hull description. After presenting these general sufficient conditions for all necessary valid inequalities, we move on to further and more specialized sufficient conditions related to only the valid inequalities separating the origin. In the previous literature, sufficiency of CGFs (and also the sufficiency of the subset of relaxed CGFs that are finite-valued) to generate all valid inequalities separating the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ is established under a blanket assumption that $\mathcal{B} \subseteq \text{cone}(A)$. In this section, we will generalize these previous results to the cases where $\mathcal{B}$ is not contained in $\text{cone}(A)$. To this end, we partition the set $\mathcal{B}$ into two sets as

$$\mathcal{B}_1 := \mathcal{B} \cap \text{cone}(A) \quad \text{and} \quad \mathcal{B}_2 := \mathcal{B} \setminus \mathcal{B}_1. \tag{3}$$

We start with a lemma that allows us to combine together results established on partitioned sets.

Lemma 3.1. *Suppose $\mathcal{B} = \bigcup_{i=1}^k \mathcal{B}^i$ and for $i = 1, \ldots, k$, we have sets $\emptyset \neq D_i \subseteq \widehat{D}$ for some $\widehat{D}$. Then, for any $\eta \in \mathbb{R}$, $\inf_{b \in \mathcal{B}^i} \sigma_{D_i}(b) \geq \eta$ for $i = 1, \ldots, k$ implies $\inf_{b \in \mathcal{B}} \sigma_{\widehat{D}}(b) \geq \eta$.*

Proof. For any $i \in \{1,\dots,k\}$ and $b \in \mathcal{B}^i$, we have $\eta \le \sigma_{D_i}(b)$. Moreover, because $D_i \subseteq \widehat{D}$, we have $\sigma_{D_i}(z) \le \sigma_{\widehat{D}}(z)$ for all z. Thus, $\eta \le \sigma_{D_i}(b) \le \sigma_{\widehat{D}}(b)$ for all $b \in \mathcal{B}^i$ and for all i. As a result, $\eta \le \inf_{b\in\mathcal{B}} \sigma_{\widehat{D}}(b)$ since for any $b \in \mathcal{B}$, b is in $\mathcal{B}^i$ for some i. $\square$ $\square$

We will frequently use the following immediate corollary of this lemma stated in terms of sets of the form $D_{\mu,\rho}$.

Corollary 3.2. *Suppose $\mathcal{B} = \bigcup_{i=1}^k \mathcal{B}^i$ and $\inf_{b\in\mathcal{B}^i} \sigma_{D_{\mu,\rho_i}}(b) \ge \eta$ for $i = 1,\dots,k$. Let $\rho \ge \max_{i\in\{1,\dots,k\}}\{\rho_i\}$. Then $\inf_{b\in\mathcal{B}} \sigma_{D_{\mu,\rho}}(b) \ge \eta$.*

For a complete description of the cases where finite-valued support functions are sufficient, we next restate and reprove part (b) of [20, Proposition 5] which covers the case of $\mathcal{B}_2 = \emptyset$. We present it in three parts – Lemma 3.3, Proposition 3.4, and Corollary 3.5, which will be convenient for us in our further developments.

Lemma 3.3. *For any $\mu \in \mathbb{R}^n_+ + \mathrm{Im}(A^\top)$, we have $D_\mu \ne \emptyset$, and $\sigma_{D_\mu}(b)$ is finite if and only if $b \in \mathrm{cone}(A)$.*

Proof. The nonemptiness of D_μ is an immediate consequence of $\mu \in \mathbb{R}^n_+ + \mathrm{Im}(A^\top)$. The second statement is a direct consequence of Linear Programming strong duality theorem. $\square$ $\square$

Proposition 3.4. *Consider a nontrivial valid inequality $(\mu;\mu_0)$ for $\overline{\mathrm{conv}}(\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B}))$. Let $\mathcal{V}_\mu$ denote the set of extreme points of the polyhedral set D_μ, and $\rho_0 := \max\left\{\max_{v\in\mathcal{V}_\mu} \|v\|_\infty,\ 1 + \inf_{\lambda\in D_\mu} \|\lambda\|_\infty\right\}$. Then for any $\rho \ge \rho_0$,*

(i) $D_{\mu,\rho} := \{\lambda \in \mathbb{R}^m : A^\top\lambda \le \mu,\ \|\lambda\|_\infty \le \rho\}$ is nonempty. Moreover, $\sigma_{D_{\mu,\rho}}$, the support function of $D_{\mu,\rho}$, is finite-valued everywhere and piecewise linear;

(ii) for any $z \in \mathbb{R}^m$ such that $\sigma_{D_\mu}(z)$ is finite, we have $\sigma_{D_{\mu,\rho}}(z) = \sigma_{D_\mu}(z)$;

(iii) for all $i = 1,\dots,n$, $\sigma_{D_{\mu,\rho}}(A_i) \le \mu_i$ where A_i denote the i-th column of the matrix A, and $\sigma_{D_{\mu,\rho}}$ leads to a valid inequality that is equivalent to or dominates $\mu^\top x \ge \mu_0$ whenever $\inf_{b\in\mathcal{B}} \sigma_{D_{\mu,\rho}}(b) \ge \mu_0$.

Proof. Note that Remark 2.1 and Lemma 3.3 imply D_μ is nonempty and polyhedral. Therefore, $\rho_0 \ge 1 + \inf_{\lambda\in D_\mu} \|\lambda\|_\infty = 1 + \min_{\lambda\in D_\mu} \|\lambda\|_\infty$. Also, if $\mathcal{V}_\mu = \emptyset$, then $\max_{v\in\mathcal{V}_\mu} \|v\|_\infty = -\infty$, and thus the definition of ρ_0 implies $\rho_0 = 1 + \inf_{\lambda\in D_\mu} \|\lambda\|_\infty = 1 + \min_{\lambda\in D_\mu} \|\lambda\|_\infty$. Then there exists $\bar\lambda \in D_\mu$ such that $\|\bar\lambda\|_\infty \le \rho_0$, and hence $D_{\mu,\rho_0} \ne \emptyset$. On the other hand, if $\mathcal{V}_\mu \ne \emptyset$, we have $\|v\|_\infty \le \rho_0$ for each $v \in \mathcal{V}_\mu$ by definition. Therefore, $v \in D_{\mu,\rho_0}$ for each $v \in \mathcal{V}_\mu$ and D_{μ,ρ_0} is nonempty. In both cases, as a super set of D_{μ,ρ_0}, $D_{\mu,\rho}$ is also nonempty. As $D_{\mu,\rho}$ is a nonempty and bounded set, its support function is finite-valued everywhere and piecewise linear.

Moreover, $D_{\mu,\rho} \subseteq D_\mu$ implies $\sigma_{D_{\mu,\rho}}(z) \le \sigma_{D_\mu}(z)$ for every $z \in \mathbb{R}^n$. For any $z \in \mathbb{R}^n$ such that $\sigma_{D_\mu}(z)$ is finite, we have

$$\sigma_{D_\mu}(z) = \max_{v\in\mathcal{V}_\mu}\{z^\top v\} \le \sigma_{D_{\mu,\rho_0}}(z) \le \sigma_{D_{\mu,\rho}}(z) \le \sigma_{D_\mu}(z),$$

where the equation follows from the fact that for the given z $\sigma_{D_\mu}(z)$ is finite and thus its optimal value is achieved at an extreme point, and the inequalities follow respectively from by the definition of ρ_0 and the relations $\rho_0 \le \rho$ and $\sigma_{D_{\mu,\rho}}(z) \le \sigma_{D_\mu}(z)$ for any z. Therefore, based on this relation we deduce $\sigma_{D_\mu}(z) = \sigma_{D_{\mu,\rho}}(z)$ for every $z \in \mathbb{R}^n$.

For part (iii), once again, $\sigma_{D_{\mu,\rho}}(A_i) \le \sigma_{D_\mu}(A_i) \le \mu_i$ for all $i = 1, \ldots, n$ where the last inequality follows from Proposition 2.3. When $\inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b) \ge \mu_0$, Lemma 2.4 indicates that the support function $\sigma_{D_{\mu,\rho}}(\cdot)$ leads to the valid inequality $\sum_{i=1}^n \sigma_{D_{\mu,\rho}}(A_i)x_i \ge \mu_0$. Taken together with $\sigma_{D_{\mu,\rho}}(A_i) \le \mu_i$ for all i, we conclude that $\sigma_{D_{\mu,\rho}}(\cdot)$ generates an inequality which is equivalent to or dominates $(\mu;\mu_0)$. □ □

Proposition 3.4 together with Lemma 3.3 leads to the following corollary which handles the case of $\mathcal{B}_2 = \emptyset$ when $\mathcal{B}$ is partitioned as in (3). Then, this recovers [20, Proposition 5].

Corollary 3.5. *Suppose $\mathcal{B} \subseteq \operatorname{cone}(A)$. Consider a nontrivial inequality $(\mu;\mu_0)$ valid for $\overline{\operatorname{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. Let $\mathcal{V}_\mu$ denote the set of extreme points of the polyhedral set D_μ, and $\rho_0 := \max\left\{\max_{v\in\mathcal{V}_\mu} \|v\|_\infty,\ 1 + \inf_{\lambda\in D_\mu} \|\lambda\|_\infty\right\}$. Then for any $\rho \ge \rho_0$, $\inf_{b\in\mathcal{B}} \sigma_{D_{\mu,\rho}}(b) \ge \mu_0$ and $\sigma_{D_{\mu,\rho}}$ leads to a valid inequality that is equivalent to or dominates $\mu^\top x \ge \mu_0$.*

Proof. Since $\mathcal{B} \subseteq \operatorname{cone}(A)$, Lemma 3.3 indicates that $\sigma_{D_\mu}(b)$ is finite for all $b \in \mathcal{B}$. Thus, we have $\inf_{b\in\mathcal{B}} \sigma_{D_{\mu,\rho}}(b) = \inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b) = \vartheta(\mu) \ge \mu_0$, where the first equality follows from Proposition 3.4(ii), the second equality follows from Theorem 2.2 and the fact that $(\mu;\mu_0)$ is nontrivial, and the inequality follows from the validity of the inequality $(\mu;\mu_0)$ for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$. The proof then follows from Proposition 3.4(iii). □ □

From now on, we will consider the cases where $\mathcal{B}_2$ may be nonempty. We start from the case where $\mathcal{B}_2$ is a compact set and generalize $\mathcal{B}_2$ step by step. In all of the cases we cover next, we will consider the support functions of bounded, nonempty, polyhedral sets of form $D_{\mu,\rho}$. Hence, the resulting support functions will be finite-valued everywhere and piecewise linear. Moreover, when the underlying inequality $(\mu;\mu_0)$ is such that $\mu_0 > 0$, these functions will satisfy the requirements of being a CGF due to their construction and finite-valuedness. Our most general conclusion for the sufficiency of CGFs generating all inequalities separating the origin is stated as Corollary 3.12.

Proposition 3.6. *Suppose $\mathcal{B}$ is partitioned as described in (3) and $\mathcal{B}_2$ is a compact set. Consider a nontrivial valid inequality $(\mu;\mu_0)$ for $\overline{\operatorname{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. Then there exists $\rho_1 \in (0,\infty)$ such that for any $\rho \ge \rho_1$, $\inf_{b\in\mathcal{B}} \sigma_{D_{\mu,\rho}}(b) \ge \mu_0$, and $\sigma_{D_{\mu,\rho}}$ leads to a valid inequality that is equivalent to or dominates $\mu^\top x \ge \mu_0$.*

Proof. By Corollary 3.5, without loss of generality we assume $\mathcal{B}_2 \ne \emptyset$. Let ρ_0 be defined as in Proposition 3.4. For any $\rho \ge \rho_0$, Corollary 3.5 indicates that $\sigma_{D_{\mu,\rho}}(b) \ge \mu_0$ for all $b \in \mathcal{B}_1$. Next, we show that there exists $\rho_1 \ge \rho_0$ such that $\inf_{b\in\mathcal{B}_2} \sigma_{D_{\mu,\rho_1}}(b) \ge \mu_0$.

Given the recession cone of D_μ, i.e., $\operatorname{Rec}(D_\mu) = \{d \in \mathbb{R}^m : A^\top d \le 0\}$, let $d_b := \operatorname{Proj}_{\operatorname{Rec}(D_\mu)}(b)$ be the projection of b onto $\operatorname{Rec}(D_\mu)$. Then the definition of d_b implies $\langle b - d_b, d - d_b\rangle \le 0$ for all $d \in \operatorname{Rec}(D_\mu)$ (see [14, Theorem A.3.1.1]). We claim that $d_b \ne 0$ for all $b \in \mathcal{B}_2$. In fact, if $d_b = 0$ for some $b \in \mathcal{B}_2$, then $b^\top d = \langle b - 0, d - 0\rangle \le 0$ for all $d \in \operatorname{Rec}(D_\mu)$. Then, from Farkas' Lemma, $b \in \operatorname{cone}(A)$, which contradicts to the assumption $\mathcal{B}_2 \cap \operatorname{cone}(A) = \emptyset$. Note $0 \in \operatorname{Rec}(D_\mu)$, and hence $\langle b - d_b, 0 - d_b\rangle \le 0$. Because $\langle b - d_b, 0 - d_b\rangle \le 0$, we have $b^\top d_b \ge \|d_b\|_2^2 > 0$ for all $b \in \mathcal{B}_2$. Let $\hat\lambda$ be a point in D_μ, and let $t_b := \max\left\{\frac{\mu_0 - b^\top\hat\lambda}{b^\top d_b}, 0\right\}$. Then by definition of t_b, we have $b^\top(\hat\lambda + t_b d_b) \ge \mu_0$. By selecting $\rho_b := \|\hat\lambda + t_b d_b\|_\infty$, we get ρ_b, which continuously depends on b. Note also that $\hat\lambda + t_b d_b \in D_{\mu,\rho_b}$, and we have $\sigma_{D_{\mu,\rho_b}}(b) \ge b^\top(\hat\lambda + t_b d_b) \ge \mu_0$. As $\mathcal{B}_2$ is compact, $\rho_1 := \sup_{b\in\mathcal{B}_2}\{\rho_b\}$

is finite and $\inf_{b\in\mathcal{B}_2} \sigma_{D_{\mu,\rho_1}}(b) \geq \mu_0$. Then by Corollary 3.2 and Proposition 3.4(*iii*), the result follows. $\square$ $\square$

Remark 3.7. *In Proposition 3.6, we do not assume that $0 \notin \overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. Moreover, μ_0 is not necessarily assumed to be 1 in Proposition 3.6. Thus, the inequalities $(\mu; \mu_0)$ considered in Proposition 3.6 covers all nontrivial valid inequalities including the ones that may or may not separate the origin even if $0 \notin \overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. Then, when $\mathcal{B}_2$ is a compact set, Proposition 3.6 establishes that* every *valid inequality $(\mu; \mu_0)$ which is necessary for the description of $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ can be generated by the support function of a bounded set of form $D_{\mu,\rho}$. Hence, when $\mathcal{B}_2$ is a compact set, $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ can in fact be generated by finite-valued support functions. Because valid inequalities with $\mu_0 > 0$ are also included in this list, and in the case of $\mu_0 > 0$ these functions are simply finite-valued relaxed CGFs which are indeed regular CGFs. Consequently, when $0 \notin \overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ and $\mathcal{B}$ is a compact set, we deduce from Proposition 3.6 the sufficiency of CGFs for generating every cut separating the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ as well.* $\diamondsuit$

In the following, we let

$$\mathcal{N}(z_0; \delta) := \{z : \|z - z_0\|_\infty < \delta\} \tag{4}$$

be the δ-neighborhood of x_0 under ℓ_∞-norm, and also define

$$\mathcal{CN}(z_0; \delta) := \{tz : z \in \mathcal{N}(z_0; \delta),\ t \geq 1\}. \tag{5}$$

Proposition 3.8. *Suppose $\mathcal{B}$ is partitioned as described in (3), $0 \notin \text{cl}(\mathcal{B}_2)$, and $\text{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \text{cone}(A) \subseteq \{0\}$.*[3] *Consider any nontrivial valid inequality $(\mu; \mu_0)$ for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$. Then there exists $\rho_2 \in (0, \infty)$ such that for any $\rho \geq \rho_2$, $\inf_{b\in\mathcal{B}} \sigma_{D_{c,\rho}}(b) \geq \mu_0$, and $\sigma_{D_{c,\rho}}$ leads to a valid inequality that is equivalent to or dominates $\mu^\top x \geq \mu_0$.*

Proof. By Corollary 3.5, we assume $\mathcal{B}_2 \neq \emptyset$ without loss of generality. Since $0 \notin \text{cl}(\mathcal{B}_2)$, there exists $\delta > 0$ such that $\mathcal{N}(0; \delta) \cap \mathcal{B}_2 = \emptyset$. Consider the compact set $G := \text{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \text{cl}(\mathcal{N}(0; 2\delta) \setminus \mathcal{N}(0; \delta))$. Note that $G \neq \emptyset$ because for any $b \in \mathcal{B}_2$, by construction, there exists $\hat{b} \in G$ and $t > 0$ such that $b = t\hat{b}$. Furthermore, since $\mathcal{N}(0; \delta) \cap \mathcal{B}_2 = \emptyset$, we indeed have $t \geq 1$ in such a representation of $b = t\hat{b}$ with $\hat{b} \in G$ for any $b \in \mathcal{B}_2$. Applying Proposition 3.6 to $\mathcal{B}_1 \cup G$, there exists $\rho_2 > 0$ such that for any $\rho \geq \rho_2$, we have $\sigma_{D_{c,\rho}}(\hat{b}) \geq \mu_0$ for all $\hat{b} \in \mathcal{B}_1 \cup G$. Also, for any $b \in \mathcal{B}_2$, using the existence of $\hat{b} \in G$ and $t \geq 1$ such that $b = t\hat{b}$ and the fact that support functions are positively homogeneous of degree 1, we arrive at $\sigma_{D_{c,\rho}}(b) = t\sigma_{D_{c,\rho}}(\hat{b}) \geq t\mu_0 \geq \mu_0$. This completes the proof. $\square$ $\square$

Note that the conditions of Propositions 3.6 and 3.8, e.g., $\mathcal{B}_2$ is a compact set, are independent of the individual valid inequalities $\mu^\top x \geq \mu_0$ (yet the resulting ρ_1 and ρ_2 values might depend on μ). Thus, they apply uniformly to all nontrivial valid inequalities for $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. Then from the point of view of the sufficiency of CGFs, these propositions indicate that under the corresponding conditions *every* valid inequality separating the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ is equivalent to or dominated by an inequality generated by a support function of the form $\sigma_{D_{c,\rho}}$ that is finite-valued everywhere. Recall also that when $0 \notin \text{cl}(\mathcal{B})$, we have $0 \notin \overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$; see [8, Lemma 2.1]. Hence, we arrive at the following corollary:

[3] $\text{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \text{cone}(A) \subsetneq \{0\}$ if and only if $B_2 = \emptyset$.

Corollary 3.9. *Suppose $\mathcal{B}$ is partitioned as described in* (3)*. Whenever $0 \notin \mathrm{cl}(\mathcal{B})$ and $\mathrm{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \mathrm{cone}(A) \subseteq \{0\}$, the CGFs are sufficient to generate all valid inequalities separating the origin from $\overline{\mathrm{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$.*

In the rest of this section, instead of focusing on generating *every* necessary valid inequality for $\overline{\mathrm{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$, we will keep our focus on the sufficiency of CGFs to generate valid inequalities of the form $c^\top x \geq 1$ that separate the origin from $\overline{\mathrm{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. Therefore, from now on, we assume $0 \notin \overline{\mathrm{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. Note that $0 \notin \overline{\mathrm{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ implies $0 \notin \mathcal{B}$ as well.

Remark 3.10. *[8, Lemma 2.1] states that when $0 \notin \mathrm{cl}(\mathcal{B})$, we have $0 \notin \overline{\mathrm{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. The condition of Proposition 3.6, i.e., $\mathcal{B}_2$ is a compact set, together with the definition of $\mathcal{B}_2$ in* (3) *immediately implies that $0 \notin \mathcal{B}_2 = \mathrm{cl}(\mathcal{B}_2)$.*

Nevertheless, it is possible to have $0 \notin \overline{\mathrm{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ yet $0 \in \mathrm{cl}(\mathcal{B})$. This happens when for example there is a sequence in $\mathcal{B}$ converging to 0 but every point in this sequence does not belong to $\mathrm{cone}(A)$, i.e., they are from $\mathcal{B}_2$. In this case, either there is no extreme inequality separating the origin from $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ or CGFs cannot be sufficient. In fact, suppose $0 \in \mathrm{cl}(\mathcal{B}_2)$, and let $b^i \in \mathcal{B}_2$ be a nonzero sequence of points converging to 0. Then $\|b^i\|_2 \to 0$ as $i \to \infty$. Suppose that there exists an extreme inequality $c^\top x \geq 1$ separating the origin from the set $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$. Let $\sigma(\cdot)$ be a CGF generating $c^\top x \geq 1$. Without loss of generality, we can assume $\sigma(\cdot)$ to be sublinear (see [8, Remark 1.4 and Theorem 2.3]). Also, by Lemma 2.7, we have $\inf_{b \in \mathcal{B}} \sigma(b) \geq 1$, which implies $\sigma(b^i) \geq 1$ for all i. Since CGFs are finite-valued, sublinear and thus convex functions, $\sigma(\cdot)$ is a continuous function (see [14, Lemma B.3.1.1]) and thus is bounded on any compact set in its domain. But then $\lim_{i \to \infty} \sigma(\frac{b^i}{\|b^i\|_2}) = \lim_{i \to \infty} \frac{\sigma(b^i)}{\|b^i\|_2} = +\infty$ contradicts the fact that $\sigma(\cdot)$ is bounded in the unit disk $\{b : \|b\|_2 \leq 1\}$. $\diamondsuit$

So far in this section, we have studied the cases where $\mathcal{B}_2$ is bounded away from $\mathrm{cone}(A)$ by a closed cone, i.e., $\mathrm{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \mathrm{cone}(A) \subseteq \{0\}$. In our next proposition, we allow nontrivial intersection of $\mathrm{cl}(\mathbb{R}_{++}(\mathcal{B}_2))$ and $\mathrm{cone}(A)$. Although $\mathcal{B}_2 \cap \mathrm{cone}(A) = \emptyset$ by construction, there are at least two ways for a ray $\{td : t \geq 0\}$ to be contained in $\mathrm{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \mathrm{cone}(A)$. First, there may exist $\bar{t}d \in \mathrm{cl}(\mathcal{B}_2) \cap \mathrm{cone}(A)$ for some $\bar{t} > 0$. That is, $\bar{t}d$ is a limit point of a sequence Q_1 in $\mathcal{B}_2$. Second, when $\mathcal{B}_2$ is unbounded, it is possible to have a sequence Q_2 in $\mathcal{B}_2$ whose closure does not intersect with $\mathrm{cone}(A)$ but $\mathrm{cl}(\mathbb{R}_{++}(Q_2)) \cap \mathrm{cone}(A) \supsetneq \{0\}$. We demonstrate these cases in Example 3.1 and Figure 1.

Example 3.1 (Figure 1)**.** Suppose A is the 2×2 identity matrix and $\mathcal{B} = \{[1;0],[0;1]\} \cup Q_1 \cup Q_2$, where $Q_1 := \{[2;-1/n] : n \in \mathbb{Z}_{++}\}$ and $Q_2 = \{[-\frac{1}{2};n] : n \in \mathbb{Z}_{++}\}$. Then $\overline{\mathrm{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})) = \overline{\mathrm{conv}}(\{[1;0],[0;1]\})$, and $c^\top x \geq 1$ is valid for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ if and only if $c := [c_1; c_2]$ satisfy $c_1, c_2 \geq 1$. Following the partition of $\mathcal{B}$ given in (3), we have $\mathcal{B}_1 = \{[1;0],[0;1]\}$ and $\mathcal{B}_2 = Q_1 \cup Q_2$.

The sequences Q_1 and Q_2 have different characteristics. Q_1 is not closed, and $[2;0]$ is its limit point. On the other hand, Q_2 is closed while $\mathbb{R}_{++}(Q_2)$ is not, and $\{[b_1;b_2] : b_1 = 0, b_2 \geq 0\}$ is the limit ray of $\mathbb{R}_{++}(Q_2)$. See Figure 1 plotted in the $\mathcal{B}$ space. $\diamondsuit$

Our next result generalizes Proposition 3.8 in the CGF context. That is, given a valid inequality of the form $c^\top x \geq 1$, Proposition 3.11 gives a more general sufficient condition for generating a valid inequality equivalent to or dominating $c^\top x \geq 1$ by a support function of the form $\sigma_{D_{c,\rho}}$. However, Proposition 3.11 involves a number of nontrivial conditions, some of which have to be checked for

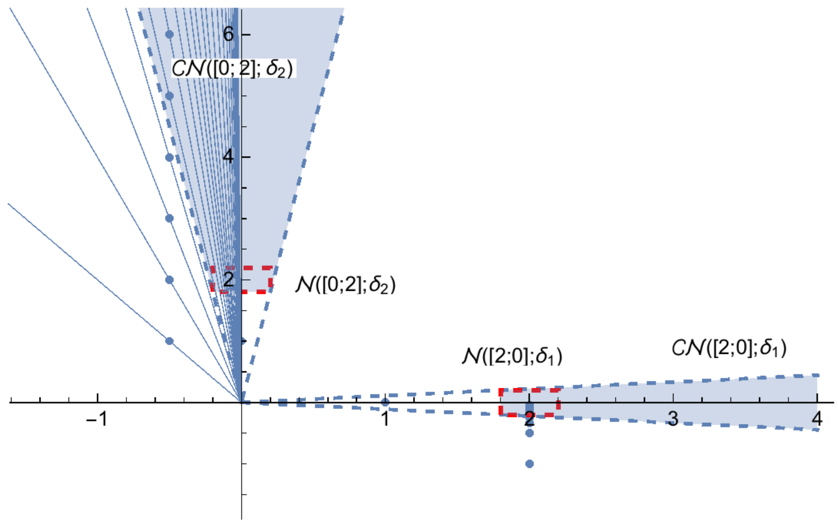


Figure 1: $\mathcal{N}(d;\delta)$ and $\mathcal{CN}(d;\delta)$ in Example 3.1

each valid inequality $(c;1)$ separately. After stating the proposition but before giving its proof, we explain the conditions involved in it and discuss these conditions on an example.

Proposition 3.11. *Suppose $\mathcal{B}$ is partitioned as described in* (3)*. Let $c^\top x \geq 1$ be a valid inequality separating the origin from $\overline{\text{conv}}(\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B}))$. If there exists a set $\mathcal{D} \subseteq \text{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \text{cone}(A)$ such that:*

(i) $\mathbb{R}_{++}(\mathcal{D}) \cup \{0\} \supseteq \text{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \text{cone}(A)$,

(ii) $td \notin \text{cl}(\mathcal{B}_2) \cap \text{cone}(A)$ *for any* $d \in \mathcal{D}$ *and* $0 \leq t < 1$,

(iii) $\sigma_{D_c}(d) > 1$ *for all* $d \in \mathcal{D}$.

Then there exists $\rho_3 \in (0,\infty)$ such that for any $\rho \geq \rho_3$, $\inf_{b\in\mathcal{B}} \sigma_{D_{c,\rho}}(b) \geq 1$, and $\sigma_{D_{c,\rho}}$ leads to a valid inequality that is equivalent to or dominates $c^\top x \geq 1$.

The intuition behind the conditions of Proposition 3.11 is roughly as follows. When $\text{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \text{cone}(A) \supsetneq \{0\}$, for each ray $\{td : t \geq 0\}$ in $\text{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \text{cone}(A)$, a representative $\bar{t}d$ with $\bar{t} > 0$ can be chosen to form a basis $\mathcal{D}$, see condition $[(i)]$. For a relaxed CGF $\sigma_{D_c}(\cdot)$ to generate $c^\top x \geq 1$, it is essential to require $\sigma_{D_c}(b) \geq 1$ for all $b \in \mathcal{B}$. Therefore, if $\bar{t}d$ is a limit point of $\mathcal{B}_2$, we care about the relation between $\sigma_{D_c}(\bar{t}d)$ and 1; this amounts to condition $[(iii)]$. Whenever $\sigma_{D_c}(d) > 0$, if $t_1 d$ and $t_2 d$ are both limit points of $\mathcal{B}_2$, from the sublinearity of $\sigma_{D_c}(\cdot)$, we have $\sigma_{D_c}(t_1 d) > \sigma_{D_c}(t_2 d)$ for all $t_1 > t_2$. Therefore, when choosing the representatives for $\mathcal{D}$ in Proposition 3.11, we pick the one with the smaller norm in condition $[(ii)]$.

The conditions of Proposition 3.11 admit an interpretation in the space of x variables, i.e., $\mathbb{R}^n$, as well: Figure 2 depicts two examples where A is a 2×2 invertible matrix. In this case, each point $b \in \mathcal{B}$ corresponds to a unique point $\bar{x}_b = A^{-1}b \in \mathbb{R}^2$. The shaded area in these pictures corresponds to all of the points $\bar{x}_b$ for some $b \in \mathcal{B}$. We denote this set by $A^{-1}(\mathcal{B}) := \{x : Ax \in \mathcal{B}\}$. Note that $\mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B}) = \mathcal{S}(A,\mathbb{R}^n_+,\mathcal{B}_1) = A^{-1}(\mathcal{B}_1) = A^{-1}(\mathcal{B}) \cap \mathbb{R}^n_+$. In particular, $\bar{x}_b \geq 0$ if and only if $b \in \mathcal{B}_1$. Also, because A is invertible, $\bar{x}_b$ is on the boundary of $\mathbb{R}^2_+$ if and only if $b \in \text{bd}(\text{cone}(A))$.

Therefore, the intersection of $\mathbb{R}^2_+$ and the shaded area, i.e., $\mathcal{S}(A, \mathbb{R}^2_+, \mathcal{B})$, corresponds to $\mathcal{B}_1$ in the space of $\mathcal{B}$, and the rest of the shaded area is the counterpart of $\mathcal{B}_2$. We will next examine the point marked as $\bar{x}_d$. Note that $\bar{x}_d$ is in the shaded area on the left figure, but it is not in the shaded area on the right one. Using the fact that A is invertible, we deduce in both pictures that the nonnegative x_1-axis $\{x : x_1 \geq 0\} = \mathrm{cl}(\mathbb{R}_{++}(\bar{x}_d))$ corresponds to $\mathrm{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \mathrm{cone}(A)$ in the space of $\mathcal{B}$, and the fact that $\bar{x}_d$ is a limit point of the lower part of the shaded area represents that d is a limit point of $\mathcal{B}_2$. In addition, in both pictures, $D = \{d\}$ satisfies Proposition 3.11(ii) because no point between $\bar{x}_d$ and the origin is in the closure of the lower part of the shade area – the part under x_1-axis. Recall that $\sigma_{D_c}(d) = \max\{d^\top \lambda : A^\top \lambda \leq c\} = \min\{c^\top x : Ax = d, x \geq 0\} = c^\top \bar{x}_d$ since $\bar{x}_d \in \mathrm{cone}(A)$ and A is an invertible matrix. For $l_0 := \{x : c^\top x = 0\}$ and $l_1 := \{x : c^\top x = 1\}$, the left picture shows the case where $\sigma_{D_c}(d) = c^\top \bar{x}_d > 1$ and the right picture shows the case where $\sigma_{D_c}(d) = c^\top \bar{x}_d < 1$. Then, in the context of these particular examples, we observe that when one of the conditions of Proposition 3.11, e.g., condition (iii), is violated, the inequality given by $c^\top x \geq 1$ cuts off a part of $\mathcal{S}(A, \mathbb{R}^n_+, \mathrm{cl}(\mathcal{B}))$ in the space of x variables.

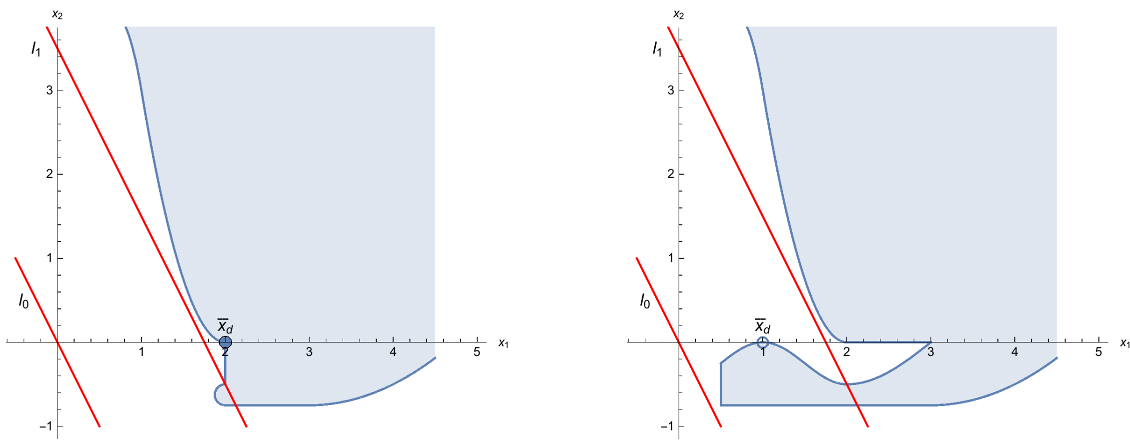


Figure 2: Interpretation of conditions in Proposition 3.11 in the space of x variables

Proof. Let ρ_0 be as defined in Corollary 3.5. Then $\inf_{b \in \mathcal{B}_1} \sigma_{D_{c,\rho_0}}(b) \geq 1$.

From Lemma 3.3, we have $\sigma_{D_c}(d)$ is finite for all $d \in \mathrm{cone}(A)$, in particular for all $d \in \mathcal{D}$. Then from Proposition 3.4(ii) and using the premise (iii) of the proposition, we conclude $\sigma_{D_{c,\rho_0}}(d) = \sigma_{D_c}(d) > 1$ for all $d \in \mathcal{D}$. For each $d \in \mathcal{D} \subseteq \mathrm{cone}(A)$, because $\sigma_{D_{c,\rho_0}}(\cdot)$ is a continuous function (see Remark 2.6), there exists $\delta_d > 0$ such that $\sigma_{D_{c,\rho_0}}(b) \geq 1$ for all $b \in \mathcal{N}(d; \delta_d)$. Without loss of generality, we will assume $\delta_d \leq 1$ for all $d \in \mathcal{D}$. Let

$$E_1 = \bigcup_{d \in \mathcal{D}} \mathcal{CN}(d; \delta_d),$$

where $\mathcal{CN}(d; \delta_d)$ is as defined in (5). Since support functions are positively homogeneous of degree 1 and $\sigma_{D_{c,\rho_0}}(b) \geq 1$ for all $b \in \mathcal{N}(d; \delta_d)$ and $d \in \mathcal{D}$, we have $\sigma_{D_{c,\rho_0}}(b) \geq 1$ for all $b \in E_1$, i.e., $\inf_{b \in E_1} \sigma_{D_{c,\rho_0}}(b) \geq 1$.

Next, we define

$$E_2 := \mathcal{B}_2 \setminus E_1 = \mathcal{B}_2 \setminus \left(\bigcup_{d \in \mathcal{D}} \mathcal{CN}(d; \delta_d) \right).$$

We first show $\mathrm{cl}(E_2) \cap \mathrm{cone}(A) = \emptyset$. If not, there exists $d \in \mathrm{cone}(A)$ and $\{b_n\} \subseteq E_2$ such that $b_n \to d$ as $n \to \infty$. Because $\mathbb{R}_{++}(d) \subseteq \mathbb{R}_{++}(\mathrm{cl}(E_2)) \subseteq \mathbb{R}_{++}(\mathrm{cl}(\mathbb{R}_{++}(E_2))) = \mathrm{cl}(\mathbb{R}_{++}(E_2)) \subseteq \mathrm{cl}(\mathbb{R}_{++}(\mathcal{B}_2))$ and $\mathbb{R}_{++}(d) \subseteq \mathrm{cone}(A)$,

$$\mathbb{R}_{++}(d) \subseteq \mathrm{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \mathrm{cone}(A) \subseteq \mathbb{R}_{++}(\mathcal{D}) \cup \{0\},$$

which implies $\mathbb{R}_{++}(d) \subseteq \mathbb{R}_{++}(\mathcal{D})$. Therefore, there exists $t > 0$ such that $\bar{d} = d/t \in \mathcal{D}$. If $t \geq 1$, then $d \in \mathcal{CN}(\bar{d}; \delta_{\bar{d}})$. Then, because $\mathcal{CN}(\bar{d}; \delta_{\bar{d}})$ is an open set and thus $d \in \mathrm{int}(\mathcal{CN}(\bar{d}; \delta_{\bar{d}}))$, this contradicts to the assumption that $\{b_n\} \subseteq E_2 = \mathcal{B}_2 \setminus \left(\bigcup_{d \in \mathcal{D}} \mathcal{CN}(d; \delta_d)\right)$ and $b_n \to d$. On the other hand, if $t < 1$, then

$$d = t\bar{d} \in \mathrm{cl}(E_2) \cap \mathrm{cone}(A) \subseteq \mathrm{cl}(\mathcal{B}_2) \cap \mathrm{cone}(A),$$

which contradicts to the premise (ii).

Now we show $\mathrm{cl}(\mathbb{R}_{++}(E_2)) \cap \mathrm{cone}(A) \subseteq \{0\}$. In this case, Proposition 3.8 implies that there exists $\rho_2 \in (0, \infty)$ such that for any $\rho \geq \rho_2$, $\inf_{b \in E_2} \sigma_{D_{c,\rho}}(b) \geq 1$. In fact, if $\mathrm{cl}(\mathbb{R}_{++}(E_2)) \cap \mathrm{cone}(A) \supsetneq \{0\}$, there exists $d \in \mathrm{cl}(\mathbb{R}_{++}(E_2)) \cap \mathrm{cone}(A)$ and $\{b_n\} \subseteq E_2$ such that $\frac{b_n}{\|b_n\|_\infty} \to \frac{d}{\|d\|_\infty}$ as $n \to \infty$. Since $\mathbb{R}_{++}(d) \subseteq \mathbb{R}_{++}(\mathcal{D})$, we can assume $d \in \mathcal{D}$ without loss of generality. If $\{b_n\}$ is bounded, then there exists a subsequence $\{b_{n_k}\}$ of $\{b_n\}$ and $K > 0$ such that $\|b_{n_k}\|_\infty \to K$ as $k \to \infty$. Therefore,

$$b_{n_k} = \frac{b_{n_k}}{\|b_{n_k}\|_\infty} \cdot \|b_{n_k}\|_\infty \to \frac{K}{\|d\|_\infty} d$$

as $k \to \infty$. However, this contradicts with our conclusion in the previous paragraph that $\mathrm{cl}(E_2) \cap \mathrm{cone}(A) = \emptyset$. As a result, we conclude $\|b_n\|_\infty \to \infty$. For the pre-defined $\delta_d > 0$, as $\frac{b_n}{\|b_n\|_\infty} \to \frac{d}{\|d\|_\infty}$, there exists $N > 0$ such that $\|b_N\|_\infty > \|d\|_\infty$ and $\left\|\frac{b_N}{\|b_N\|_\infty} - \frac{d}{\|d\|_\infty}\right\|_\infty < \frac{\delta_d}{\|d\|_\infty}$. Therefore,

$$\left\|b_N - \frac{\|b_N\|_\infty}{\|d\|_\infty} d\right\|_\infty < \frac{\|b_N\|_\infty}{\|d\|_\infty} \delta_d \tag{6}$$

Note that $\mathcal{CN}(d; \delta_d) = \left\{b \in \bigcup_{t \geq 1} \mathcal{N}(td; t\delta_d)\right\}$. Moreover, $\frac{\|b_N\|_\infty}{\|d\|_\infty} > 1$. Hence, inequality (6) implies $b_N \in \mathcal{CN}(d; \delta_d)$. Then this contradicts the assumption $b_N \in E_2$.

As $\mathcal{B} = \mathcal{B}_1 \cup E_1 \cup E_2$, Corollary 3.2 implies that $\inf_{b \in \mathcal{B}} \sigma_{D_{c,\rho}}(b) \geq 1$ for any $\rho \geq \rho_3 := \max\{\rho_0, \rho_2\}$. It follows from Proposition 3.4(iii) that $\sigma_{D_{c,\rho}}$ leads to a valid inequality that is equivalent to or dominates $c^\top x \geq 1$. □ □

Note that Proposition 3.11 with $\mathcal{D} = \emptyset$ recovers the implications of Proposition 3.8 for CGFs as a trivial case. The condition that $\sigma_{D_c}(d) > 1$ for all $d \in \mathcal{D}$ in Proposition 3.11 can be further refined by separating $\mathcal{D}$ into two parts. The following corollary slightly generalizes Proposition 3.11 in this sense.

Corollary 3.12. *Suppose $\mathcal{B}$ is partitioned as described in* (3). *Let $c^\top x \geq 1$ be a valid inequality separating the origin from $\overline{\mathrm{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. If there exist sets $\mathcal{D}_1 \subseteq \mathrm{cl}(\mathcal{B}_2) \cap \mathrm{cone}(A)$ and $\mathcal{D}_2 \subseteq (\mathrm{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \setminus \mathrm{cl}(\mathcal{B}_2)) \cap \mathrm{cone}(A)$ such that:*

(i) $\mathbb{R}_{++}(\mathcal{D}_1 \cup \mathcal{D}_2) \cup \{0\} \supseteq \mathrm{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \mathrm{cone}(A)$,

(ii) $td \notin \mathrm{cl}(\mathcal{B}_2) \cap \mathrm{cone}(A)$ *for any* $d \in \mathcal{D}_1$ *and* $0 \leq t < 1$,

(iii) $\sigma_{D_c}(d) > 1$ *for all* $d \in \mathcal{D}_1$ *and* $\sigma_{D_c}(d) > 0$ *for all* $d \in \mathcal{D}_2$.

Then there exists $\rho_4 \in (0, \infty)$ *such that for any* $\rho \geq \rho_4$, $\inf_{b \in \mathcal{B}} \sigma_{D_{c,\rho}}(b) \geq 1$, *and* $\sigma_{D_{c,\rho}}$ *leads to a valid inequality that is equivalent to or dominates* $c^\top x \geq 1$.

Proof. Let $\mathcal{D}_3 := \left\{ \frac{d}{\sigma_{D_c}(d)/2} : d \in \mathcal{D}_2 \setminus \mathbb{R}_{++}(\mathcal{D}_1) \right\}$. Then the corollary follows from applying Proposition 3.11 to $\mathcal{D}_1 \cup \mathcal{D}_3$. □ □

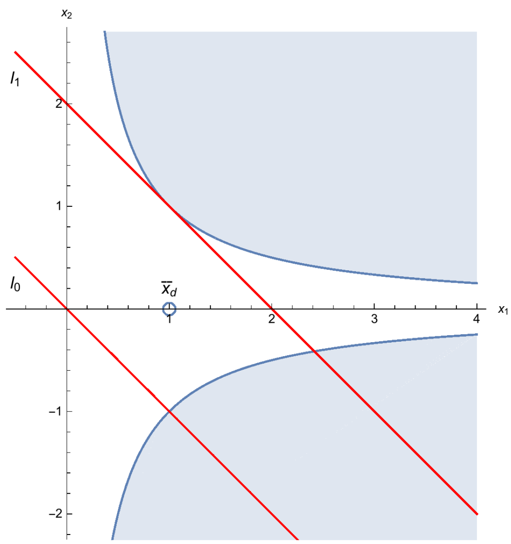
The collection of conditions in Corollary 3.12 is equivalent to the ones in Proposition 3.11. If a set $\mathcal{D}$ satisfying the requirements of Proposition 3.11 exists, one can simply set $\mathcal{D}_1 = \mathcal{D}$ and $\mathcal{D}_2 = \emptyset$, and the conditions in Corollary 3.12 will be satisfied. On the other hand, as shown in the proof of Corollary 3.12, $\mathcal{D}$ in Proposition 3.11 can be constructed from $\mathcal{D}_1$ and $\mathcal{D}_2$ in Corollary 3.12.

Similar to Figure 2, Figure 3 shows an interpretation of the conditions in Corollary 3.12 in the space of x variables. We still assume that A is a 2×2 invertible matrix. In both of the pictures below, we use the shaded area to represent $A^{-1}(\mathcal{B}) := \{x : Ax \in \mathcal{B}\}$. In particular, $A^{-1}(\mathcal{B}_1) = \mathcal{S}(A, R^2_+, \mathcal{B})$ is the upper part of the shaded area, and $A^{-1}(\mathcal{B}_2)$ is the lower part in these pictures. Moreover, in these pictures, $A^{-1}(\mathbb{R}_{++}(\mathcal{B}_2))$ is the fourth quadrant and $A^{-1}(\mathrm{cl}(\mathbb{R}_{++}(\mathcal{B}_2)))$ is the fourth quadrant with its boundary. Note that, in both pictures, $\bar{x}_d$ is not a limit point of the lower part of the shaded area, and correspondingly, d is not a limit point of $\mathcal{B}_2$. However, $\mathbb{R}_{++}(\bar{x}_d) = \{x : x_1 \geq 0\} \subseteq A^{-1}(\mathrm{cl}(\mathbb{R}_{++}(\mathcal{B}_2))) \cap \mathbb{R}^2_+$. Because A is invertible, this relation corresponds to $\mathbb{R}_{++}(d) \subseteq \mathrm{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \mathrm{cone}(A)$ in the space of $\mathcal{B}$. By letting $\mathcal{D}_1 = \emptyset$ and $\mathcal{D}_2 = \{d\}$, conditions (i) and (ii) in Corollary 3.12 are satisfied. The left picture shows the case where $\sigma_{D_c}(d) = c^\top \bar{x}_d > 0$, and thus (iii) is also satisfied. The right one shows the case where $\sigma_{D_c}(d) = c^\top \bar{x}_d < 0$, and thus condition (iii) fails. In the case when Corollary 3.12(iii) fails, we observe that $c^\top x \geq 0$ cuts off $\mathbb{R}_{++}(\bar{x}_d)$, which is a part of $\mathcal{S}(A, \mathbb{R}^2_+, \mathrm{cl}(\mathbb{R}_{++}(\mathcal{B})))$ in the space of x variables. On the other hand, such a situation cannot be observed for any valid inequality in the left picture because the distance between $\mathbb{R}_{++}(\bar{x}_d)$ and $\mathcal{S}(A, \mathbb{R}^2_+, \mathcal{B})$ is zero, and hence these sets cannot be separated by any valid inequality.

Remark 3.13. *We would like to highlight the fact that the conditions in Proposition 3.11 and Corollary 3.12 do depend on specific valid inequalities* $c^\top x \geq 1$ *via the support function* σ_{D_c}. *In order to conclude the sufficiency of CGFs with Proposition 3.11 or Corollary 3.12, one needs to verify that the associated conditions involving the function* σ_{D_c} *are satisfied by* every extreme valid inequality. *This is in contrast to the earlier results such as Proposition 3.8 and Corollary 3.9. For example, in the case where* $\mathcal{B}_2 = \emptyset$ *(resp.* $\mathrm{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \mathrm{cone}(A) \subseteq \{0\}$*), Corollary 3.5 (resp. Proposition 3.8) can be uniformly applied to every valid inequality. So, in those cases, the sufficiency of CGFs can be concluded independent of examining each c vector separately.* ♢

In general verifying the conditions of Proposition 3.11 and Corollary 3.12 for all extreme valid inequalities separating the origin can be difficult. Below, we demonstrate how these conditions can be verified for Example 3.1.

Example 3.1 (Continued)**.** Let $\mathcal{D}_1 = \{[2; 0]\}$ and $\mathcal{D}_2 = \{[0; 1]\}$. Then Conditions (i) and (ii) in Corollary 3.12 are satisfied. Moreover, recall that the validity of any inequality of form $c^\top x \geq 1$ for $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$ requires $\min\{c_1, c_2\} \geq 1$. Then it is clear that $\sigma_{D_c}([2; 0]) = 2c_1 > 1$ and $\sigma_{D_c}([0; 1]) = c_2 > 0$. Therefore, based on Corollary 3.12, for each valid inequality $c^\top x \geq 1$, there exists $\rho_c > 0$ such that $\inf_{b \in \mathcal{B}} \sigma_{D_{c,\rho}}(b) \geq 1$ for any $\rho \geq \rho_c$. Thus, in this example, CGFs are sufficient to generate



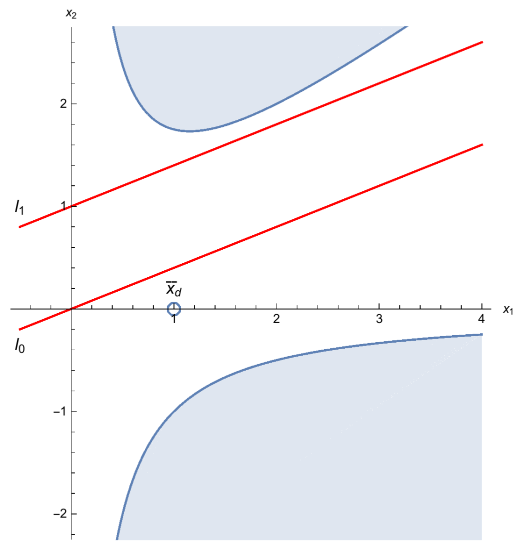


Figure 3: Interpretation of conditions in Corollary 3.12 in the space of x variables

all valid inequalities separating the origin from $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$. In fact, we can get the same conclusion without using Corollary 3.12: For any valid inequality $c^\top x \geq 1$, by setting $\rho \geq \max\{c_1, c_2\} \geq \min\{c_1, c_2\} \geq 1$, we have for any $n \in \mathbb{Z}_{++}$ the following relations:

$$\sigma_{D_{c,\rho}}\left([2; -\frac{1}{n}]\right) = \max\left\{2\lambda_1 - \frac{\lambda_2}{n} : \; -\rho \leq \lambda_1 \leq c_1, \; -\rho \leq \lambda_2 \leq c_2\right\} = 2c_1 + \frac{\rho}{n} \geq 1, \text{ and}$$
$$\sigma_{D_{c,\rho}}\left([-\frac{1}{2}; n]\right) = \max\left\{-\frac{\lambda_1}{2} + n\lambda_2 : \; -\rho \leq \lambda_1 \leq c_1, \; -\rho \leq \lambda_2 \leq c_2\right\} = \frac{\rho}{2} + c_2 n \geq 1.$$

Thus, $\inf_{b \in \mathcal{B}} \sigma_{D_{c,\rho}}(b) \geq 1$ whenever $\rho \geq \max\{c_1, c_2\} \geq 1$. ♢

4 Necessary Conditions for the Sufficiency of CGFs

In this section, we first show that if an extreme inequality can be generated by a cut-generating function, then it can as well be generated by the support function of a bounded set of the form $D_{c,\rho}$. Then, inspired by the conditions given in Corollary 3.12, we provide two necessary conditions for the sufficiency of CGFs that almost match with our sufficient conditions given in Corollary 3.12. We close by providing examples that highlight the gap between our sufficient conditions from Section 3 for the sufficiency of CGFs for generating all necessary valid inequalities separating the origin and our necessary conditions from this section.

Proposition 4.1. *Consider any extreme inequality $c^\top x \geq 1$ separating the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. Assume that there exists a CGF $\sigma(\cdot)$ generating a valid inequality that is equivalent to $c^\top x \geq 1$. Then there exists a finite $\rho > 0$ such that the set $D_{c,\rho}$ is nonempty, and its support function $\sigma_{D_{c,\rho}}(\cdot)$ generates a valid inequality that is equivalent to $c^\top x \geq 1$.*

Proof. Because $c^\top x \geq 1$ is extreme and all undominated extreme inequalities are tight and sublinear, it is also sublinear and $\vartheta(c) = 1$. Suppose that there exists a CGF $\sigma(\cdot)$ generating an inequality equivalent to $c^\top x \geq 1$. Thus, $\sigma(\cdot)$ is finite-valued and $\sigma(A_i) = c_i$ for all i.

Moreover, $\sigma(\cdot)$ is a CGF generating an extreme inequality, in view of [8, Remark 1.4 and Theorem 2.3], without loss of generality, we can assume that $\sigma(\cdot)$ is a sublinear function.

Let $D_\sigma := \{\lambda \in \mathbb{R}^m : z^\top \lambda \le \sigma(z)\ \forall z \in \mathbb{R}^m\}$. Then by [14, Theorem C.3.1.1] (see also [14, Corollary C.3.1.2]), we have $\sigma(\cdot)$ is the support function of D_σ. Because $\sigma(\cdot)$ is a CGF and hence is finite-valued, by [14, Proposition C.2.1.3] D_σ is a bounded set. D_σ is also nonempty. If $D_\sigma = \emptyset$, then $\sigma(z) = \sigma_{D_\sigma}(z) = -\infty$ for all z and it would not be possible to have $\sum_i \sigma(A_i)x_i \ge 1$ as a valid inequality for any nonempty set $\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})$. Using the definition of D_σ and the fact that $\sigma(A_i) = c_i$, we conclude that the inequalities $A_i^\top \lambda \le \sigma(A_i) \le c_i$ are valid for D_σ. Thus, $D_\sigma \subseteq D_c$.

Let $\rho := 1 + \sup_{\lambda \in D_\sigma} \|\lambda\|_\infty$. Because D_σ is nonempty and bounded, $\rho \in (0, \infty)$. Also, by construction, $D_\sigma \subseteq D_{c,\rho} \subseteq D_c$ implying $\sigma_{D_c}(z) \ge \sigma_{D_{c,\rho}}(z) \ge \sigma_{D_\sigma}(z)$ for all z. From the definition of D_c, we immediately have $c_i \ge \sigma_{D_c}(A_i)$ for all i. Furthermore, $\sigma_{D_\sigma}(A_i) = c_i$ since $\sigma(\cdot)$ generates $c^\top x \ge 1$. Therefore, $c_i \ge \sigma_{D_c}(A_i) \ge \sigma_{D_{c,\rho}}(A_i) \ge \sigma_{D_\sigma}(A_i) = c_i$ for all i. In addition, from Lemma 2.7, we have $1 \le \inf_{b\in B} \sigma(b)$, which then implies that $1 \le \inf_{b\in B} \sigma(b) \le \inf_{b\in B} \sigma_{D_c}(b)$. Finally, because $\sigma_{D_\sigma}(\cdot) = \sigma(\cdot)$ and $\vartheta(c) = 1 \le \inf_{b\in B} \sigma(b)$, we have $\inf_{b\in\mathcal{B}} \sigma_{D_{c,\rho}}(b) \ge \inf_{b\in B} \sigma_{D_\sigma}(b) = \inf_{b\in B} \sigma(b) \ge \vartheta(c)$. Thus, the function $\sigma_{D_{c,\rho}}$ generates $c^\top x \ge 1$ as well. □ □

In particular, Proposition 4.1 implies the following corollary:

Corollary 4.2. *Whenever CGFs are sufficient to generate all valid inequalities that separate the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$, then the CGFs obtained from the support functions of sets of form $D_{c,\rho}$ are also sufficient.*

Our necessary conditions given in the following two propositions are inspired by the two sets $\mathcal{D}_1$ and $\mathcal{D}_2$ described in Corollary 3.12.

Proposition 4.3. *Let $\mathcal{B}$ be partitioned as described in* (3)*. Suppose there exists a valid inequality $c^\top x \ge 1$ separating the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ and either there exists a nonzero vector $d \in \text{cl}(\mathcal{B}_2) \cap \text{cone}(A)$ satisfying $\sigma_{D_c}(d) < 1$ or there exists a vector $d \in \text{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \text{cone}(A)$ satisfying $\sigma_{D_c}(d) < 0$. Then, for any finite ρ such that the set $D_{c,\rho}$ is nonempty, the support function $\sigma_{D_{c,\rho}}(\cdot)$ cannot generate a valid inequality that is equivalent to or dominates $c^\top x \ge 1$.*

Proof. Consider any $\rho \in (0, \infty)$ such that $D_{c,\rho} \ne \emptyset$. Then $\sigma_{D_{c,\rho}}(z) \le \sigma_{D_c}(z)$ for all z because $D_{c,\rho} \subseteq D_c$.

Suppose there exists a nonzero vector $d \in \text{cl}(\mathcal{B}_2) \cap \text{cone}(A)$ such that $\sigma_{D_c}(d) < 1$, then $\sigma_{D_{c,\rho}}(d) \le \sigma_{D_c}(d) < 1$. Moreover, from Remark 2.6, the function $\sigma_{D_{c,\rho}}(\cdot)$ is continuous, and thus there exists $\delta > 0$ such that for all $b \in \mathcal{N}(d;\delta)$, we have $\sigma_{D_{c,\rho}}(b) < 1$. Because $d \in \text{cl}(\mathcal{B}_2)$, there exists a sequence $\{b^i\}$ in $\mathcal{B}_2$ converging to d. Hence, there exists $\bar{b} \in \mathcal{B}_2 \cap \mathcal{N}(d;\delta)$, implying $\inf_{b\in\mathcal{B}} \sigma_{D_{c,\rho}}(b) \le \sigma_{D_{c,\rho}}(\bar{b}) < 1$.

On the other hand, if there exists a vector $d \in \text{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \text{cone}(A)$ such that $\sigma_{D_c}(d) < 0$, then $d \ne 0$ because σ_{D_c} is the support function of a nonempty set (see [14, Section C.2] and [19, Section 4]). Moreover, $\sigma_{D_{c,\rho}}(d) \le \sigma_{D_c}(d) < 0$ and there exists $\delta > 0$ such that for all $b \in \mathcal{N}(d;\delta)$, we have $\sigma_{D_{c,\rho}}(b) < 0$. Because $d \in \text{cl}(\mathbb{R}_{++}(\mathcal{B}_2))$, there exist a sequence $\{b^i\}$ in $\mathcal{B}_2$ and a sequence of positive scalars $\{t^i\}$ such that $t^i b^i$ converges to d. Hence, there exists $\bar{t} > 0$ and $\bar{b} \in \mathcal{B}_2$ such that $\bar{t}\bar{b} \in \mathcal{N}(d;\delta)$, and thus $0 > \frac{1}{\bar{t}}\sigma_{D_{c,\rho}}(\bar{t}\bar{b}) = \sigma_{D_{c,\rho}}(\bar{b})$. This implies $\inf_{b\in\mathcal{B}} \sigma_{D_{c,\rho}}(b) \le \sigma_{D_{c,\rho}}(\bar{b}) < 0$.

Therefore, by Lemma 2.7, we cannot generate an inequality that is equivalent to or dominates $c^\top x \ge 1$ using the support function of $D_{c,\rho}$. □ □

Proposition 4.3 together with Proposition 4.1 lead to the following result.

Corollary 4.4. *Let $\mathcal{B}$ be partitioned as described in* (3)*. Suppose there exists an extreme inequality $c^\top x \geq 1$ separating the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$ and either a nonzero vector $d \in \text{cl}(\mathcal{B}_2) \cap \text{cone}(A)$ satisfying $\sigma_{D_c}(d) < 1$ or a vector $d \in \text{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \text{cone}(A)$ satisfying $\sigma_{D_c}(d) < 0$. Then there is no CGF that can generate the inequality $c^\top x \geq 1$, and hence for such sets CGFs are not sufficient to generate all valid inequalities separating the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$.*

Proof. Assume for contradiction that there exists a CGF $\sigma(\cdot)$ that generates the extreme inequality $c^\top x \geq 1$. Then by Proposition 4.1, there exists a finite ρ such that the support function of the set $D_{c,\rho}$ also generates the inequality $c^\top x \geq 1$. But, this contradicts Proposition 4.3. □ □

Conforti et al. [8] introduced the following example (see [8, Example 6.1]) to show that CGFs may not always be sufficient to generate all valid inequalities separating the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B}))$. In the following, we revisit this example and its slight variant studied in [19]; see Section 4.3, Example 10 and remarks afterwards in [19].

Example 4.1. Let A be the 2×2 identity matrix and $\mathcal{B} = \{[0; 1]\} \cup \{[n; -1] : n \in \mathbb{Z}\}$. Then $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^2_+, \mathcal{B})) = \mathcal{S}(A, \mathbb{R}^2_+, \mathcal{B}) = \{[0; 1]\}$. The valid inequality $c^\top x \geq 1$ with $c = [-1; 1]$ separates the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^2_+, \mathcal{B}))$. Let $d = [1; 0]$. Then $d \in \text{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \text{cone}(A) = \{b : b_1 \geq 0,\, b_2 = 0\}$ and $\sigma_{D_c}(d) = \max\{\lambda^\top d : A^\top \lambda \leq c\} = \max\{\lambda_1 : \lambda_1 \leq -1\} = -1 < 0$. By Corollary 4.4, there is no CGF that can generate this inequality, and thus CGFs are not sufficient to generate all cuts separating the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^2_+, \mathcal{B}))$.

On the other hand, [19] has examined the variant of this example by setting $\mathcal{B} = \{[0; 1]\} \cup \{[n; -1] : n \in \mathbb{Z}_-\}$. In this case, $\text{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \text{cone}(A) = \{0\}$; and thus by Corollary 3.9, CGFs are sufficient. ♢

We conclude this section with two pairs of examples. These examples illustrate the gap between our sufficient conditions for CGFs from Section 3 and our necessary conditions presented in this section. In particular, Examples 4.2 and 4.4 show that our sufficient condition stated in Corollary 3.12 has room for improvement. That is, it is possible to have a CGF generating an extreme inequality $c^\top x \geq 1$ even when $\sigma_{D_c}(d) = 1$ for the only point $d \neq 0$ in $\text{cl}(\mathcal{B}_2) \cap \text{cone}(A)$ or $\sigma_{D_c}(d) = 0$ for all points in $\text{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \text{cone}(A)$. In contrast to these, Examples 4.3 and 4.5 demonstrate cases of an extreme inequality of the form $c^\top x \geq 1$ that cannot be generated by any CGF when there exists a $0 \neq d \in \text{cl}(\mathcal{B}_2) \cap \text{cone}(A)$ such that $\sigma_{D_c}(d) = 1$ or $0 \neq d \in \text{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \text{cone}(A)$ such that $\sigma_{D_c}(d) = 0$. The main difference in these examples is in the way the sequence of points in $\mathcal{B}_2$ approach to a point in $\text{cone}(A)$ (Examples 4.2 and 4.3) or the way they go to infinity (Examples 4.4 and 4.5).

Example 4.2. Suppose A is the 2×2 identity matrix and $\mathcal{B} = \{[1; 0], [0; 1]\} \cup \{[1; -1/n] : n \in \mathbb{Z}_{++}\}$. Then $\mathcal{B}_1 = \{[1; 0], [0; 1]\}$, $\mathcal{B}_2 = \{[1; -1/n] : n \in \mathbb{Z}_{++}\}$, $\text{cl}(\mathcal{B}_2) \cap \text{cone}(A) = \{[1; 0]\}$ and $\text{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \text{cone}(A) = \text{cone}([1; 0])$. Consider a valid inequality $c^\top x \geq 1$ separating the origin from $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^2_+, \mathcal{B}))$. Because $\overline{\text{conv}}(\mathcal{S}(A, \mathbb{R}^2_+, \mathcal{B})) = \overline{\text{conv}}(\{[1; 0], [0; 1]\})$, $c^\top x \geq 1$ is valid if and only if $c := [c_1; c_2]$ satisfies $c_1, c_2 \geq 1$. Note that $\sigma_{D_c}([1; 0]) = \max\{\lambda_1 : \lambda \leq c\} = c_1$. When $c_1 > 1$, we have $\sigma_{D_c}([1; 0]) > 1$. Then from Corollary 3.12, by taking $\mathcal{D}_1 = \{[1; 0]\}$ and $\mathcal{D}_2 = \emptyset$, we obtain $\inf_{b \in \mathcal{B}} \sigma_{D_{c,\rho}}(b) \geq 1$ for some $0 < \rho < +\infty$. On the other hand, the conditions in Corollary 3.12 are not satisfied when $c_1 = 1$ because $\text{cl}(\mathcal{B}_2) \cap \text{cone}(A) = \{[1; 0]\}$ and $\sigma_{D_c}([1; 0]) = 1$. However, in this case, for any $\rho \geq 1$ and $n \in \mathbb{Z}_{++}$, we have

$$\sigma_{D_{c,\rho}}\left([1; -\frac{1}{n}]\right) = \max\left\{\lambda_1 - \frac{\lambda_2}{n} :\ -\rho \leq \lambda_1 \leq 1,\ -\rho \leq \lambda_2 \leq \min\{\rho, c_2\}\right\} = 1 + \frac{\rho}{n} \geq 1.$$

Hence, $\inf_{b\in\mathcal{B}} \sigma_{D_{c,\rho}}(b) \geq 1$ even when $c_1 = 1$. This establishes the sufficiency of CGFs in this example even though the conditions in Corollary 3.12 are not satisfied for the extreme inequality $x_1 + x_2 \geq 1$. $\diamondsuit$

Example 4.3. Suppose A is the 2×2 identity matrix, $\mathcal{B} = \{[1;0],[0;1]\} \cup \{[1-1/\sqrt{n}; -1/n] : n \in \mathbb{Z}_{++}\}$. Then $\mathcal{B}_1 = \{[1;0],[0;1]\}$, $\mathcal{B}_2 = \{[1-1/\sqrt{n}; -1/n] : n \in \mathbb{Z}_{++}\}$, $\mathrm{cl}(\mathcal{B}_2) \cap \mathrm{cone}(A) = \{[1;0]\}$ and $\mathrm{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \mathrm{cone}(A) = \mathrm{cone}([1;0])$. Consider the extreme inequality $c^\top x \geq 1$ with $c = [1;1]$ separating the origin from $\overline{\mathrm{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})) = \overline{\mathrm{conv}}(\{[1;0],[0;1]\})$. Note that the conditions in Corollary 4.4 are not satisfied because $\sigma_{D_c}([1;0]) = c_1 = 1$ and $\sigma_{D_c}(b) > 0$ for any $b \in \mathbb{R}_{++}([1;0])$. On the other hand, for any $\rho > 0$ and $n \in \mathbb{Z}_{++}$, we have

$$\begin{aligned}\sigma_{D_{c,\rho}}\left([1-\frac{1}{\sqrt{n}}; -\frac{1}{n}]\right) &= \max\left\{\left(1-\frac{1}{\sqrt{n}}\right)\lambda_1 - \frac{\lambda_2}{n} :\ -\rho \leq \lambda_1 \leq \min\{\rho,1\},\ -\rho \leq \lambda_2 \leq \min\{\rho,1\}\right\} \\ &= \left(1-\frac{1}{\sqrt{n}}\right)\min\{\rho,1\} + \frac{\rho}{n} = \min\left\{\rho - \frac{\rho}{\sqrt{n}} + \frac{\rho}{n},\ 1 - \frac{1}{\sqrt{n}} + \frac{\rho}{n}\right\}.\end{aligned}$$

For any fixed $\rho > 0$, when $n > \rho^2$, we immediately have $1 - \frac{1}{\sqrt{n}} + \frac{\rho}{n} < 1$. Hence, $\sigma_{D_{c,\rho}}([1 - \frac{1}{\sqrt{n}}; -\frac{1}{n}]) < 1$, which implies $\inf_{b\in\mathcal{B}} \sigma_{D_{c,\rho}}(b) < 1$. Therefore, for any finite ρ such that the set $D_{c,\rho} := \{\lambda \in \mathbb{R}^m :\ A^\top\lambda \leq c,\ \|\lambda\|_\infty \leq \rho\}$ is nonempty, the support function $\sigma_{D_{c,\rho}}(\cdot)$ cannot generate a valid inequality that is equivalent to or dominates $c^\top x \geq 1$. Then by Proposition 4.1, there is no CGF that generates this inequality or another one that dominates it. This demonstrates a case where even though the conditions in Corollary 4.4 are not satisfied, there is an extreme inequality which cannot be generated by any CGF. $\diamondsuit$

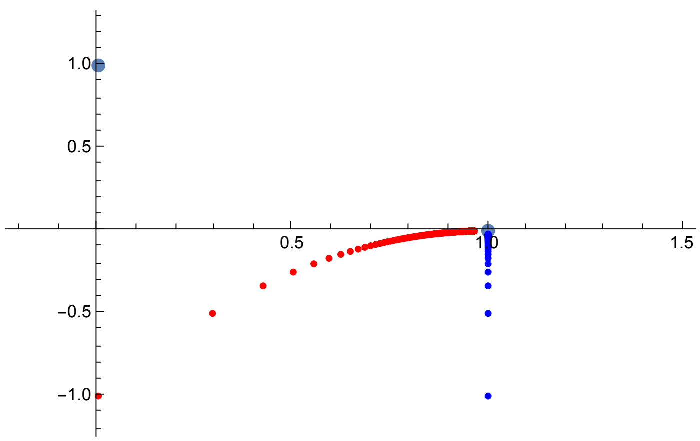


Figure 4: Two ways to approach $[1;0]$ as in Examples 4.2 and 4.3.

Example 4.4. Suppose A is the 2×2 identity matrix, $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ where $\mathcal{B}_1 = \{b : b_1 \geq 1,\ b_2 \geq 1\}$ and $\mathcal{B}_2 = \{[n;-1] : n \in \mathbb{Z}_{++}\}$. Then $\mathrm{cl}(\mathcal{B}_2) \cap \mathrm{cone}(A) = \emptyset$ and $\mathrm{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \mathrm{cone}(A) = \mathrm{cone}([1;0])$. Consider a valid inequality $c^\top x \geq 1$ separating the origin from $\overline{\mathrm{conv}}(\mathcal{S}(A, \mathbb{R}^2_+, \mathcal{B}))$. Because the recession cone of $\overline{\mathrm{conv}}(\mathcal{S}(A, \mathbb{R}^2_+, \mathcal{B})) = \{x : x_1 \geq 1,\ x_2 \geq 1\}$ is $\mathbb{R}^2_+$, $c^\top x \geq 1$ is valid only if $c := [c_1; c_2]$ satisfies $c_1, c_2 \geq 0$. For any $d = [d_1; d_2] \in \mathbb{R}_{++}([1;0])$, $\sigma_{D_c}(d) = \max\{\lambda_1 d_1 : \lambda \leq c\} = c_1 d_1$. When

$c_1 > 0$, we have $\sigma_{D_c}(d) > 0$ for any $d \in \mathbb{R}_{++}([1;0])$. By taking $\mathcal{D}_1 = \emptyset$ and $\mathcal{D}_2 = \{[1;0]\}$ in Corollary 3.12, we obtain that there exists $0 < \rho < +\infty$ such that $\inf_{b\in\mathcal{B}} \sigma_{D_{c,\rho}}(b) \geq 1$. On the other hand, the conditions in Corollary 3.12 are not satisfied when $c_1 = 0$ because $\mathrm{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \mathrm{cone}(A) = \mathbb{R}_{++}([1;0]) \cup \{0\}$ and $\sigma_{D_c}(d) = 0$ for all $d \in \mathbb{R}_{++}([1;0])$. However, even in this case, for any $\rho \geq 1$ and $n \in \mathbb{Z}_{++}$, we have

$$\sigma_{D_{c,\rho}}([n;-1]) = \max\{n\lambda_1 - \lambda_2 : \ -\rho \leq \lambda_1 \leq 0, \ -\rho \leq \lambda_2 \leq \min\{\rho, c_2\}\} = 0 + \rho \geq 1.$$

Hence, $\inf_{b\in\mathcal{B}} \sigma_{D_{c,\rho}}(b) \geq 1$ even when $c_1 = 0$. This establishes the sufficiency of CGFs in this example even though the conditions in Corollary 3.12 are not satisfied for the extreme inequality $x_2 \geq 1$. ♢

Example 4.5. Suppose A is the 2×2 identity matrix, $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ where $\mathcal{B}_1 = \{b : b_1 \geq 1, b_2 \geq 1\}$ and $\mathcal{B}_2 = \{[n;-1/n] : n \in \mathbb{Z}_{++}\}$. Then $\mathrm{cl}(\mathcal{B}_2) \cap \mathrm{cone}(A) = \emptyset$ and $\mathrm{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \mathrm{cone}(A) = \mathrm{cone}([1;0])$. Consider the extreme inequality $c^\top x \geq 1$ where $c = [0;1]$ that separates the origin from $\overline{\mathrm{conv}}(\mathcal{S}(A, \mathbb{R}^n_+, \mathcal{B})) = \{x : x_1 \geq 1, x_2 \geq 1\}$. Note that the conditions in Corollary 4.4 are not satisfied because $\mathrm{cl}(\mathcal{B}_2) \cap \mathrm{cone}(A) = \emptyset$ and $\sigma_{D_c}(d) = 0$ for any $d \in \mathrm{cl}(\mathbb{R}_{++}(\mathcal{B}_2)) \cap \mathrm{cone}(A)$. On the other hand, for any fixed $\rho > 0$ and $n \in \mathbb{Z}_{++}$, we have

$$\sigma_{D_{c,\rho}}\left([n;-\frac{1}{n}]\right) = \max\left\{n\lambda_1 - \frac{\lambda_2}{n} : \ -\rho \leq \lambda_1 \leq 0, \ -\rho \leq \lambda_2 \leq \min\{\rho, 1\}\right\} = 0 + \frac{\rho}{n}.$$

For any fixed $\rho > 0$, we have $\sigma_{D_{c,\rho}}([n;-\frac{1}{n}]) < 1$ when $n > \rho$. Thus, $\inf_{b\in\mathcal{B}} \sigma_{D_{c,\rho}}(b) < 1$ for any fixed $\rho > 0$. Therefore, for any finite ρ such that the set $D_{c,\rho} := \{\lambda \in \mathbb{R}^m : A^\top\lambda \leq c, \ \|\lambda\|_\infty \leq \rho\}$ is nonempty, the support function $\sigma_{D_{c,\rho}}(\cdot)$ cannot generate a valid inequality that is equivalent to or dominates $c^\top x \geq 1$, i.e, $x_2 \geq 1$. Then by Proposition 4.1, there is no CGF that generates this inequality or another one that dominates it. This demonstrates a case where even though the conditions in Corollary 4.4 are not satisfied, there is an extreme inequality which cannot be generated by any CGF. ♢

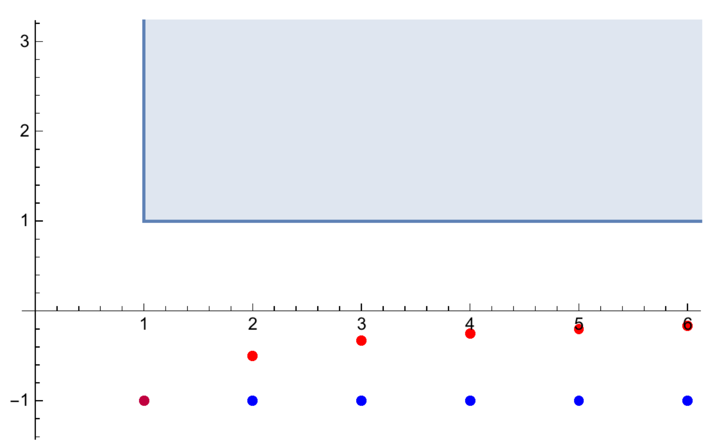


Figure 5: Two unbounded sequences as in Examples 4.4 and 4.5.

Acknowledgments

The authors wish to thank the review team for their constructive feedback that improved the presentation of the material in this paper. This research is supported in part by NSF grant CMMI 1454548.

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