Artificial Intelligence Methods for Social Good

M2-1 [Game Theory]:
Basics of Game Theory

08-537 (9-unit) and 08-737 (12-unit)
Instructor: Fei Fang
feifang@cmu.edu
Wean Hall 4126
Given coordinates of \( n \) residential areas in a city (assuming 2-D plane), denoted as \( x^1, \ldots, x^n \), the government wants to find a location that minimizes the sum of (Euclidean) distances to all residential areas to build a hospital. The optimization problem can be written as

- \( A: \min_x \sum_i |x^i - x| \)
- \( B: \min_x \sum_i \|x^i - x\|_2 \)
- \( C: \min_x \sum_i (x^i - x)^2 \)
- \( D: \) none of above

Quiz 1: Recap: Optimization Problem
The study of mathematical models of conflict and cooperation between intelligent decision makers

Used in economics, political science etc
Outline

- Basic Concepts in Games
- Basic Solution Concepts
- Compute Nash Equilibrium
- Compute Strong Stackelberg Equilibrium
Learning Objectives

- Understand the concept of
  - Game, Player, Action, Strategy, Payoff, Expected utility, Best response
  - Dominant Strategy, Maxmin Strategy, Minmax Strategy
  - Nash Equilibrium
  - Stackelberg Equilibrium, Strong Stackelberg Equilibrium

- Describe Minimax Theory

- Formulate the following problem as an optimization problem
  - Find NE in zero-sum games (LP)
  - Find SSE in two-player general-sum games (multiple LP and MILP)

- Know how to find the method/algorithm/solver/package you can use for solving the games

- Compute NE/SSE by hand or by calling a solver for small games
Let’s Play! Classical Games

- **Exp 1: Rock-Paper-Scissors (RPS)**
  - Rock beats Scissors
  - Scissors beats Paper
  - Paper beats Rock

- **Exp 2: Prisoner’s Dilemma (PD)**
  - If both Cooperate: 1 year in jail each
  - If one Defect, one Cooperate: 0 year for (D), 3 years for (C)
  - If both Defect: 2 years in jail each
Let’s Play! Classical Games

- **Exp 3: Battle of Sexes (BoS)**
  - If football together: Alex 😊😊, Berry 😊
  - If concert together: Alex 😊, Berry 😊😊
  - If not together: Alex 😞, Berry 😞

- **Tic-Tac-Toe (TTT)**
Basic Concepts in Games

- **Def 1: Game**
  - Players
  - Actions
  - Payoffs
Basic Concepts in Games

- **Representation**
  - Normal form (Matrix form, Strategic form, Standard form)
    - Move simultaneously
    - Bimatrix game (Two-player)
      - Exp 2: PD
  - Extensive form
    - Timing, Sequence of move
    - Game tree
    - Information
  - Natural description

---

**Normal form**

<table>
<thead>
<tr>
<th></th>
<th>Cooperate</th>
<th>Defect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cooperate</td>
<td>-1, -1</td>
<td>-3, 0</td>
</tr>
<tr>
<td>Defect</td>
<td>0, -3</td>
<td>-2, -2</td>
</tr>
</tbody>
</table>

---

**Extensive form**

Game tree with information sets:

- Player 1
  - Cooperate
  - Defect
- Player 2
  - Cooperate
  - Defect

---

Fei Fang
Basic Concepts in Games

- Pure Strategy
  - Choose one action deterministically

- Def 2: Mixed Strategy
  - Play randomly
  - Support: chosen with non-zero probability

- Def 3: Expected utility
  - Average utility weighted by probability
Quiz 2: Basic Concepts in Games

In Exp 1 (Rock-Paper-Scissors), if $s_1 = \left(\frac{1}{3}, \frac{2}{3}, 0\right)$, $s_2 = \left(0, \frac{1}{2}, \frac{1}{2}\right)$, what is $u$?

- $u = (0,0)$
- $u = (-\frac{1}{3}, \frac{1}{3})$
- $u = (-\frac{1}{2}, \frac{1}{3})$
- $u = (-\frac{1}{2}, \frac{1}{2})$

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Rock</th>
<th>Paper</th>
<th>Scissors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rock</td>
<td>0,0</td>
<td>-1,1</td>
<td>1,-1</td>
</tr>
<tr>
<td>Paper</td>
<td>1,-1</td>
<td>0,0</td>
<td>-1,1</td>
</tr>
<tr>
<td>Scissors</td>
<td>-1,1</td>
<td>1,-1</td>
<td>0,0</td>
</tr>
</tbody>
</table>
Basic Concepts in Games

- Def 4: Best Response
  - Set of actions or strategies leading to highest expected utility
- Thm 1 (Nash 1951): A mixed strategy is BR iff all actions in the support are BR
Learning Objectives

- Understand the concept of
  - Game, Player, Action, Strategy, Payoff, Expected utility, Best response
  - Dominant Strategy, Maxmin Strategy, Minmax Strategy
  - Nash Equilibrium
  - Stackelberg Equilibrium, Strong Stackelberg Equilibrium

- Describe Minimax Theory

- Formulate the following problem as an optimization problem
  - Find NE in zero-sum games (LP)
  - Find SSE in two-player general-sum games (multiple LP and MILP)

- Know how to find the method/algorithm/solver/package you can use for solving the games

- Compute NE/SSE by hand or by calling a solver for small games
Outline

- Basic Concepts in Games
- Basic Solution Concepts
- Compute Nash Equilibrium
- Compute Strong Stackelberg Equilibrium
Basic Solution Concepts

- How should a player play?
- **Def 5: Dominant Strategy**
  - One strategy is always better/never worse/never worse and sometimes better than any other strategy
  - Focus on single player’s strategy
- **Exp 2: PD**

- **Dominant strategy equilibrium/solution**
  - Every player plays a dominant strategy
  - Focus on strategy profile for all players

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cooperate</td>
<td>Cooperate</td>
</tr>
<tr>
<td>-1,-1</td>
<td>-3,-0</td>
</tr>
</tbody>
</table>
Def 6: Nash Equilibrium

- Every player’s strategy is a best response to others’ strategy profile
- Focus on strategy profile for all players
- One cannot gain by unilateral deviation
- Pure Strategy Nash Equilibrium (PSNE)
- Mixed Strategy Nash Equilibrium (MSNE)
- Exp 2: PD
Basic Solution Concepts

- Thm 2 (Nash 1951): NE always exists in finite games
  - Brouwer's fixed point theorem
Basic Solution Concepts

- **Def 7: Maxmin Strategy** (applicable to multiplayer games)
  - Maximize worst case expected utility
  - Focus on single player’s strategy

- **Def 8: Minmax Strategy** (two-player games only)
  - Minimize best case expected utility for the other player (just want to harm your opponent)
  - Focus on single player’s strategy
Basic Solution Concepts

- Thm 3: (Minimax Theorem, von Neumann 1928)
  Minmax = Maxmin in 2-player zero-sum games
- Further, Minmax = Maxmin = NE (Nash 1951)
Exp 4: Power of Commitment

- NE utility = (2, 1)
- If leader (player 1) commits to playing $b$, then player has to play $d$, leading to a utility of 3 for leader.
- If leader (player 1) commits to playing $a$ and $b$ uniformly randomly, then player still has to play $d$, leading to a utility of 3.5 for leader.

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>2, 1</td>
</tr>
<tr>
<td>b</td>
<td>1, 0</td>
</tr>
</tbody>
</table>
Basic Solution Concepts

- **Def 9: Best Response Function**
  - A mapping from a strategy of one player to a strategy of another player in the best response set

- **Def 10: Stackelberg Equilibrium**
  - Leader vs follower game
  - Leader commits to a strategy
  - Follower responds according a best response function
  - Focus on strategy profile for all players
Def 11: Strong Stackelberg Equilibrium (SSE)
- Follower breaks tie in favor of the leader
Basic Solution Concepts

- Def 11: Strong Stackelberg Equilibrium (SSE)
  - Follower breaks tie in favor of the leader
  - Leader can induce the follower to do so by perturbing the strategy in the right direction
  - SSE always exist in two-player finite games
Quiz 3: Basic Solution Concepts

What is the relationship of leader’s expected utility in SSE and NE in two-player games?

- $u^{SSE} \geq u^{NE}$
- $u^{SSE} = u^{NE}$
- $u^{SSE} \leq u^{NE}$
- none of the above
Learning Objectives

- Understand the concept of
  - Game, Player, Action, Strategy, Payoff, Expected utility, Best response
  - Dominant Strategy, Maxmin Strategy, Minmax Strategy
  - Nash Equilibrium
  - Stackelberg Equilibrium, Strong Stackelberg Equilibrium

- Describe Minimax Theory

- Formulate the following problem as an optimization problem
  - Find NE in zero-sum games (LP)
  - Find SSE in two-player general-sum games (multiple LP and MILP)

- Know how to find the method/algorithm/solver/package you can use for solving the games

- Compute NE/SSE by hand or by calling a solver for small games
Outline

- Basic Concepts in Games
- Basic Solution Concepts
- Compute Nash Equilibrium
- Compute Stackelberg Equilibrium
Compute Nash Equilibrium

- Find pure strategy Nash Equilibrium (PSNE)
  - Enumerate all action profile
  - Check if no incentive to deviate
Compute Nash Equilibrium

- Find all Nash Equilibrium (two-player)
  - Special case: Zero-sum game
    - Polynomial time solvable (minmax or maxmin LP)
  - General case
    - PPAD-Complete (Chen & Deng, 2006)
      - Unlikely to have polynomial time algorithm
      - Conjecture: slightly easier than NP-Complete problems
Compute Nash Equilibrium

- Find all Nash Equilibrium (two-player)
  - Support Enumeration Method
    - Enumerate support pair
    - For each possible support pair
      - Compute the probability so as to keep the other player indifferent among actions in the support
      - Check if no incentive to deviate
      - Or combine the two steps: solve an LP
Compute Nash Equilibrium

- **Exp 3 (Battle of Sexes)**
  - Enumerate support pair
    - Support size=1: PSNE!
    - Support size=2: Alex: (Football, Concert), Berry: (Football, Concert)
    - Different support size: no NE exist (#constraints>#variables)
  - Compute the probability so as to keep the other player indifferent among actions in the support
  - Why? (check Thm 1)

|       | Berry
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Alex</td>
<td>Football</td>
</tr>
<tr>
<td>Football</td>
<td>2,1</td>
</tr>
<tr>
<td>Concert</td>
<td>0,0</td>
</tr>
</tbody>
</table>
Quiz 4: Compute Nash Equilibrium

- What is the probability of Berry choosing Football in NE of Exp 3 (Battle of Sexes) with support size=2?
  - 0
  - 1
  - \( \frac{1}{3} \)
  - 2
  - \( \frac{2}{3} \)

<table>
<thead>
<tr>
<th></th>
<th>Football</th>
<th>Concert</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alex</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Football</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>Concert</td>
<td>0,0</td>
<td>1,2</td>
</tr>
</tbody>
</table>
Compute Nash Equilibrium

- Find all Nash Equilibrium (two-player)
  - Support Enumeration Method
    - Enumerate support pair
    - For each possible support pair
      - Compute the probability so as to keep the other player indifferent among actions in the support
      - Check if no incentive to deviate
      - Or combine the two steps: solve an LP
  - Lemke-Howson Algorithm
    - Linear Complementarity (LCP) formulation (another special class of optimization problem)
    - Solve by pivoting on support (similar to Simplex algorithm)
  - In practice, available solvers/packages: nashpy (python), gambit project (http://www.gambit-project.org/)
Learning Objectives

- Understand the concept of
  - Game, Player, Action, Strategy, Payoff, Expected utility, Best response
  - Dominant Strategy, Maxmin Strategy, Minmax Strategy
  - Nash Equilibrium
  - Stackelberg Equilibrium, Strong Stackelberg Equilibrium

- Describe Minimax Theory

- Formulate the following problem as an optimization problem
  - Find NE in zero-sum games (LP)
  - Find SSE in two-player general-sum games (MILP)

- Know how to find the method/algorithm/solver/package you can use for solving the games

- Compute NE/SSE by hand or by calling a solver for small games
Outline

- Basic Concepts in Games
- Basic Solution Concepts
- Compute Nash Equilibrium
- Compute Strong Stackelberg Equilibrium
Find Strong Stackelberg Equilibrium (not restricted to pure strategy)

- Special case (zero-sum): \( \text{SSE}=\text{NE}=\text{Minmax}=\text{Maxmin} \)
- General case: Polynomial time solvable
  - Multiple Linear Programming
  - Mixed Integer Linear Programming
    - Exponential in theory
    - Efficient in practice in some cases

Solvers available? Course project idea (9 units only): develop a SSE solver package in Python
Learning Objectives

- Understand the concept of
  - Game, Player, Action, Strategy, Payoff, Expected utility, Best response
  - Dominant Strategy, Maxmin Strategy, Minmax Strategy
  - Nash Equilibrium
  - Stackelberg Equilibrium, Strong Stackelberg Equilibrium

- Describe Minimax Theory

- Formulate the following problem as an optimization problem
  - Find NE in zero-sum games (LP)
  - Find SSE in two-player general-sum games (MILP)

- Know how to find the method/algorithmsolver/package you can use for solving the games

- Compute NE/SSE by hand or by calling a solver for small games
Summary

- Basic Concepts in Games
- Basic Solution Concepts
- Compute Nash Equilibrium
- Compute Strong Stackelberg Equilibrium

Key take-away:
- There are various solution concepts in games
- In two-player zero-sum games, many solution concepts lead to same strategy and utility
- Finding NE or SSE can be formulated as one or more optimization problem
## Summary

<table>
<thead>
<tr>
<th>Solution Concepts</th>
<th>Key Algorithm In Class</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minmax/Maxmin</td>
<td>LP</td>
</tr>
<tr>
<td>Nash Equilibrium</td>
<td>LP for zero-sum, Support enumeration for general-sum</td>
</tr>
<tr>
<td>Strong Stackelberg Equilibrium</td>
<td>LP for zero-sum, multiple LP or MILP for general-sum</td>
</tr>
</tbody>
</table>
Game Theory: Additional Resources

- **Text book**
  - *Algorithmic Game Theory 1st Edition, Chapters 1-3*
  - *Noam Nisan (Editor), Tim Roughgarden (Editor), Eva Tardos (Editor), Vijay V. Vazirani (Editor)*

- **Online course**
  - [https://www.youtube.com/user/gametheoryonline](https://www.youtube.com/user/gametheoryonline)