Exercise 1. Prove that the following formulas are valid on any relational frame:

(K) $\Box(\phi \rightarrow \psi) \rightarrow (\Box \phi \rightarrow \Box \psi)$,
(Duality) $\Box \phi \leftrightarrow \neg \Diamond \neg \phi$.

Proof.

• (K) We want to show that for any model $M$ on a relational frame $\langle W, R \rangle$ and any $w \in W$,

$$M, w \models \Box(\phi \rightarrow \psi) \rightarrow (\Box \phi \rightarrow \Box \psi)$$

for any formulas $\phi, \psi$. From the inductive definition of truth, this amounts to showing that if (i) $M, w \models \Box(\phi \rightarrow \psi)$ and $M, w \models \Box \phi$, then (ii) $M, w \models \Box \psi$.

Supposing (i), we have that $M, u \models \phi \rightarrow \psi$ and $M, u \models \phi$ for any $u$ such that $wRu$. But it follows that $M, u \models \psi$ for any such $u$; thus $M, w \models \Box \psi$ as desired.

• (Duality) For any model $M$ and any $w \in W$,

$$M, w \models \neg \Diamond \neg \phi \iff M, w \not\models \Diamond \neg \phi \iff \neg \exists u \in W (wRu \land M, u \models \neg \phi) \iff \neg \exists u \in W (wRu \land M, u \not\models \phi) \iff \forall u \in W (wRu \rightarrow M, u \models \phi) \iff M, w \models \Box \phi.$$ 

This was for any $w$, so $M \models \Box \phi \leftrightarrow \neg \Diamond \neg \phi$.

Exercise 2. Prove that the following are valid on all neighborhood models:

(1) $\Box \phi \leftrightarrow \neg \Diamond \neg \phi$,
(2) From $\phi \leftrightarrow \psi$, infer $\Box \phi \leftrightarrow \Box \psi$.

*This is a selection of exercises from Week 3 of the 2006 Carnegie Mellon Summer School in Logic and Formal Epistemology.
Proof.

• (1) Let $\mathcal{M}$ be any model on an epistemic frame $\langle W, E \rangle$, and pick an arbitrary $w \in W$. We use the facts 1 – 5 which are quickly derivable from the definition of truth. Then

$$(\neg \diamond \neg \phi)^{\mathcal{M}} = W - (\diamond \neg \phi)^{\mathcal{M}} \quad \text{(by 2)}$$

$$= W - [W - m_{E}(W - (\neg \phi)^{\mathcal{M}})] \quad \text{(by 5)}$$

$$= W - [W - m_{E}(W - (W - (W - (\phi)^{\mathcal{M}})))] \quad \text{(by 2)}$$

$$= W - [W - m_{E}(\phi)^{\mathcal{M}}]$$

$$= m_{N}(\phi)^{\mathcal{M}}$$

$$= (\Box \phi)^{\mathcal{M}} \quad \text{(by 4)}.$$

• (2) If we have $\mathcal{M} \models \phi \leftrightarrow \psi$, that is just to say that $(\phi)^{\mathcal{M}} = (\psi)^{\mathcal{M}}$. But then

$$(\Box \phi)^{\mathcal{M}} = m_{E}(\phi)^{\mathcal{M}} = m_{E}(\psi)^{\mathcal{M}} = (\Box \psi)^{\mathcal{M}},$$

i.e. $\mathcal{M} \models \Box \phi \leftrightarrow \Box \psi$.

Exercise 3. Prove that the following are not valid in the class of all neighborhood models:

(K) $\Box(\phi \rightarrow \psi) \rightarrow (\Box \phi \rightarrow \Box \psi)$,

(T) $\Box \phi \rightarrow \phi$.

Proof.

• (K) The key point is that unless we specifically impose certain conditions, there are no restrictions on the makeup of the epistemic function $E$ of a frame $\langle W, E \rangle$. Note that in the previous exercise, the validity of the scheme and the inference rule depended solely on the definition of truth in a model, and not in any way on specific properties of the function $E$.

In order to falsify the present scheme, any model with a world $w$ for which

$$(\phi \rightarrow \psi)^{\mathcal{M}}, (\phi)^{\mathcal{M}} \in E(w),$$

$$(\psi)^{\mathcal{M}} \notin E(w),$$

will do the trick. As we'll see, in the present case the definition of truth does not rule this out, and the freedom we have in defining $E$ allows us to create the desired counterexample.

Let $p, q$ be atomic proposition letters. We’ll define a frame $\langle W, E \rangle$ with $W = \{w, u\}$ and $E(w) = \{\emptyset, W\}$ ($E(u)$ is not important, as we only need one world in one model to falsify the scheme). Then we’ll define a model $\mathcal{M}$ on the frame using a valuation such that $V(p) = \emptyset$ and $V(q) = \{u\}$. Given our valuation, $(p)^{\mathcal{M}} = \emptyset$, $(q)^{\mathcal{M}} = \{u\}$ and $(p \rightarrow q)^{\mathcal{M}} = W$. Thus, given our definition of $E(w)$, we have

$$\mathcal{M}, w \models \Box(p \rightarrow q) \wedge \Box p \wedge \neg \Box q,$$

which is precisely what we wanted.
• (T) In order to falsify $\square \phi \rightarrow \phi$, we proceed similarly. Just define a model such that the neighborhood function and valuation guarantee that there is a formula $\phi$ and a world $w$ for which $w \not\in (\phi)^M$ (hence $M, w \models \neg \phi$), yet $(\phi)^M \in E(w)$ (hence $M, w \models \square \phi$). Again, the complete freedom we have in deciding what sets of worlds to use in defining valuations and neighborhood functions allows us to create the appropriate model.

Exercise 4. Prove that a neighborhood frame $\mathcal{F}$ is supplemented iff, if $X \cap Y \in \mathcal{F}$ then $X, Y \in \mathcal{F}$.

Proof. $(\Rightarrow)$. Suppose $\mathcal{F}$ is supplemented, and let $X \cap Y \in \mathcal{F}$. Well, $X, Y \supseteq X \cap Y$, and so $X, Y \in \mathcal{F}$.

$(\Leftarrow)$. Let $X \in \mathcal{F}$. If $Y \supseteq X$, then $X \cap Y = X \in \mathcal{F}$. By the supposed condition, then, $Y \in \mathcal{F}$ as well. Thus $\mathcal{F}$ is closed under supersets, i.e. is supplemented.

Exercise 5. For a relational frame $(W, R)$, we observed that the reflexivity of $R$ could be expressed by

$$w \in \bigcap N_w, \text{ for all } w \in W.$$ 

Make analogous observations for the situations when $R$ is (a) serial, (b) Euclidean, and (c) symmetric.

Analogues. The thing to note is that $wRv \iff v \in R^{-}(w) \iff v \in \bigcap N_w$.

Thus we saw that reflexivity could be expressed as:

$$wRw \iff w \in \bigcap N_w.$$ 

Each of the conditions—seriality, Euclideanosity(?), symmetry—makes mention only of immediate $R$-descendants of given worlds, and so can similarly be briefly expressed in terms of $\bigcap N_w$’s.

(a) serial - $\forall w \exists v (wRv)$:

$$\forall w \exists v (v \in \bigcap N_w).$$

(b) Euclidean - $\forall w, v, u (wRv \land wRu \rightarrow vRu)$:

$$\forall w, v, u (v, u \in \bigcap N_w \rightarrow u \in \bigcap N_v).$$

(c) symmetric - $\forall w, v (wRv \rightarrow vRw)$:

$$\forall w, v (v \in \bigcap N_w \rightarrow w \in \bigcap N_v).$$

$\square$
Exercise 6. Find properties on neighborhood frames that are defined by the following formulas:
(1) \(\Diamond \top\),
(2) \(\neg \Box \phi \rightarrow \Box \neg \Box \phi\),
(3) \(\Box \phi \rightarrow \Diamond \phi\).

Properties. We denote our frame by \(\langle W, E \rangle\), and we can read off the frame properties from the formulas using the definition of truth in a model.
(1) \(\forall w \in W(\emptyset \notin E(w))\).
(2) \(\forall w \in W(X \notin E(w) \rightarrow \{v \mid X \notin E(v)\} \in E(w))\).
(3) \(\forall w \in W(X \in E(w) \rightarrow (W - X) \notin E(w))\).

Exercise 7. Prove that \(EN\) is sound and strongly complete w.r.t neighborhood frames that contain the unit.

Proof. Given the fact that \(E\) is sound and strongly complete w.r.t. the class of all neighborhood frames, the claim of the exercise follows from the fact that \(N\) is valid in a frame iff it contains the unit (which is a lemma left to the reader in the notes). Why does this suffice?

Let \(F\) be the class of all neighborhood frames; let \(N\) be the class of all neighborhood frames which contain the unit. Then:
\[
\Gamma \models_N \phi \iff \forall \mathcal{F} \in \mathcal{N}(\mathcal{F} \models \Gamma \Rightarrow \mathcal{F} \models \phi) \quad \text{(definition)}
\iff \forall \mathcal{F} \in \mathcal{F}(\mathcal{F} \models N \Rightarrow \mathcal{F} \models \Gamma \Rightarrow \mathcal{F} \models \phi) \quad (\text{logic})
\iff \Gamma \models_{\mathcal{F}} N \rightarrow \phi \quad \text{(soundness/completeness of \(E\))}
\iff \Gamma \models_{\mathcal{F}} \mathcal{N} \rightarrow \phi \quad \text{(definition)}.
\]

Exercise 8. Prove that \(K\) is sound and strongly complete w.r.t. the class of augmented neighborhood frames.

Cheap Proof. If we wanted to use facts known from relational semantics, then a result mentioned in the notes yields a very quick proof. Namely, it was shown that every augmented frame has an equivalent relational frame, and vice versa. Since \(K\) is the minimal normal modal logic, the one that is sound and strongly complete in the class of all relational frames, the foregoing result immediately implies that \(K\) is also sound and strongly complete in the class of augmented neighborhood frames.

But of course that uses a rather healthy presupposition from relational semantics. We’d rather actually have a direct proof, and we can use the completeness results from the notes in order to get the one for this exercise, without detouring through known facts from relational semantics.
Proof (following Chellas). It is not hard to verify that $K$ is sound for augmented neighborhood frames. We will use the canonicity approach to prove completeness. All we need to do is show that $K$ has a canonical model on an augmented frame. Consider the minimal canonical model for $K$:

$$M := \langle M_K, N_{\text{min}}^K, V_K \rangle.$$ 

We form a new model $M' := \langle M_K, N, V_K \rangle$, where

$$N(\Gamma) := \{ X \subseteq M_K \mid \cap N_{\text{min}}^K(\Gamma) \subseteq X \}.$$ 

So $N(\Gamma)$ is obtained by throwing in the core, and closing under supersets; thus the frame of $M'$ is augmented (and in fact, this construction is known as the augmentation of the frame). Now we just need to show that $M'$ is a canonical for $K$.

We still have $M_K$ as the set of worlds, and $V_K$ as the valuation; all that needs to be checked is that

$$\square \phi \in \Gamma \iff \models_K \phi \in N(\Gamma).$$ 

By our definition, $\models_K \phi$ means precisely that $\cap N_{\text{min}}^K(\Gamma) \subseteq \models_K \phi$. But because it was minimal, $N_{\text{min}}^K(\Gamma) = \{ \models_K \psi \mid \square \psi \in \Gamma \}$. So it amounts to

$$\cap \{ \models_K \psi \mid \square \psi \in \Gamma \} \subseteq \models_K \phi.$$ 

Unpacking some of the definitions here (it could be useful to check these details), this just says that for all $\Delta \in M_K$

$$\{ \psi \mid \square \psi \in \Gamma \} \subseteq \Delta \implies \phi \in \Delta.$$ 

We claim that this condition is in fact equivalent to the condition $\square \phi \in \Gamma$, which finishes the proof that $M'$ is canonical.

Proof of Claim: One direction is immediate: supposing $\square \phi \in \Gamma$, then clearly $\{ \psi \mid \square \psi \in \Gamma \} \subseteq \Delta$ implies $\phi \in \Delta$. Going the other way is trickier.

Suppose that $\phi$ belongs to every $K$-maximal set $\Delta \supseteq \{ \psi \mid \square \psi \in \Gamma \}$. It follows from Lindenbaum’s lemma that this is equivalent to

$$\{ \psi \mid \square \psi \in \Gamma \} \vdash_k \phi.$$ 

By compactness an the deduction theorem, there are $\psi_1, \ldots, \psi_n$ from this set such that

$$\vdash_k \bigwedge_i \psi_i \rightarrow \phi.$$ 

Then by the inference rule $RK$ of $K$,

$$\vdash_k \bigwedge_i \square \psi_i \rightarrow \square \phi.$$ 

But each $\square \psi_i$ is in $\Gamma$; thus $\Gamma \vdash_k \square \phi$, i.e. $\square \phi \in \Gamma$. \qed
Exercise 9. Which properties continue to be admissible when □ is interpreted as ‘it is highly probable that ...’? Construct a counterexample to (C).

Answer. Certainly everything in E will still be acceptable when we think of □ as indicating high probability. And $N = □\top$ should be fine as well, since any tautology is of course highly probable. Furthermore, if $A \land B$ is highly probable, then since $A \land B$ implies both $A$ and $B$, each of these will be highly probable; thus the scheme

$$M \equiv □(A \land B) \rightarrow □(A) \land □(B)$$

should continue to hold.

But $C$ will not. It is certainly possible to have two propositions $A, B$, the probability of each being greater than our threshold $t$, for which the probability of $A \land B$ is $< t$. If we think of Venn diagrams, say, propositions that partially overlap are the ones that could potentially be of the sort we have in mind. As a concrete example, perhaps it’s highly probable that you will be taller than a certain height ($A$) and highly probable that you’ll have a certain hair color ($B$), yet not highly probable that both will pan out.

Thus, among the systems we have looked at, EMN seems right as a way to capture the notion of high probability via □.

Exercise 10. Show that probability cores are nested.

Proof. For any two cores $C_1, C_2$ we want to show that either $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$. So suppose for contradiction that $C_1 \not\subseteq C_2$ and $C_2 \not\subseteq C_1$; i.e. there’s an $x \in C_1 - C_2$ and a $y \in C_2 - C_1$.

Now $\{x\}$ is disjoint from $C_2$, and $\{y\}$ is a nonempty subset of $C_2$; so the strong superiority condition yields $P(\{x\} | \{x, y\}) = 0$. Arguing symmetrically with the core $C_1$, we have $P(\{y\} | \{x, y\}) = 0$ as well. But now by additivity we get

$$P(\{x, y\} | \{x, y\}) = P(\{x\} | \{x, y\}) + P(\{y\} | \{x, y\}) = 0 + 0 = 0,$$

contradicting the condition that $P(\cdot | \{x, y\})$ must be a probability measure (or constantly 1).

Exercise 11. Show that all non-empty subsets of a probabilistic core are normal.

Proof. Let $C$ be a core, and let $\emptyset \neq A \subseteq C$. To show that $A$ is normal, we need to show that $P(\cdot | A)$ is a probability measure. Given that we have condition (I) (which says that for any $A$, either $P(\cdot | A)$ is a probability measure or $P(\cdot | A)$ is constantly 1) it suffices to show that $P(X | A) \neq 1$ for some set $X$.

The empty set $\emptyset$ is certainly disjoint from $C$, and so the strong superiority condition for cores says that $P(\emptyset | A \cup \emptyset) = 0$. But $A \cup \emptyset = A$; thus we have $P(\emptyset | A) = 0$, and we are done.

Exercise 12. Show via an example how we can dissolve the finite lottery paradox.

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Proof. The abandonment of the scheme $C$ (that is, $\Box A \land \Box B \rightarrow \Box(A \land B)$) lets us avoid the finite lottery paradox. Suppose there is a lottery with, say, 10,000,000 tickets. It’s enough tickets that, for each one, I believe it’s a loser; letting $W_i$ assert that the $i$-th ticket wins, my belief that each one is a loser can be expressed as the enormous conjunction

$$(\dagger) \quad \bigwedge_{i=1}^{10,000,000} \Box \neg W_i.$$ 

Now I also believe the lottery is a fair one; specifically, I do believe that one of the tickets will win; how do we express this? By

$$(\star) \quad \Box \left( \bigvee_{i=1}^{10,000,000} W_i \right).$$ 

We can see that $(\dagger)$ and $(\star)$ are pulling in opposite directions, but they are not contradictory and so we have no paradox yet. If we could appeal to $C$, then from $(\dagger)$ we could conclude

$$\Box \left( \bigwedge_{i=1}^{10,000,000} \neg W_i \right),$$

which together with $(\star)$ means we have inconsistent beliefs. We would then be in the thick of the paradox, but without recourse to $C$, we remain in the clear.

Exercise 13. Prove that all axioms of $S5$ are positively valid.

Proof.

- (K) Suppose $\Box(\phi \rightarrow \psi)$ and $\Box(\phi)$ are in a saturated theory $\sigma$. Then $(\phi \rightarrow \psi) \in \sigma$ and $\phi \in \sigma$ by (A1). By logical consequence, $\psi \in \sigma$; by (A1) again, $\Box \psi \in \sigma$.
- (T) If $\Box \phi$, then $\phi$ results immediately from (A1).
- (4) Instantiate (A1) with $\Box \phi$.
- (5) Instantiate (A1) with $\neg \Box \phi$.

Exercise 14. Are all positively valid theses also negatively valid?

Answer. Yes. Let $A$ be positively valid. Then $A \in \sigma$ for any saturated theory $\sigma$. But our saturated theories are consistent; thus $\neg A \notin \sigma$ for any saturated theory. That is, $A$ is negatively valid.
Exercise 15. Is Moore’s paradox consistently statable in any consistent saturated theory?

Answer. No. The statement is of the form $H \land \neg B(H)$. But $H \in \sigma$ if and only if $B(H) \in \sigma$; thus Moore’s paradox is part of no consistent saturated theory. □

Exercise 16. Prove that if $\leq$ is an entrenchment relation then the function $\div$ obtained via

$$K \div A := \begin{cases} K \cap \{B \mid A < B\} & \text{if } A \in K \text{ and } A \notin Cn(LK), \\ K & \text{OW.} \end{cases}$$

is a mild contraction.

Proof. See my handout “A Brief Introduction to Belief Revision” for the solution to an analogous problem. □

Exercise 17. Prove that if $\div$ is a mild contraction then the relation $\leq$ obtained via

$$A \leq B \equiv (A \notin (K \div B) \text{ or } B \in Cn(LK))$$

is an entrenchment relation.

Proof. Again, see “A Brief Introduction to Belief Revision” for something similar. □