You may work with other students from class, but everyone should write up solutions independently.

Homework is due by the end of class time, either as a hard copy or email attachment.

Remember that no late homework is accepted, but you have two drops.

(1) (10 points) Give a formal proof of this sequent:

\[ \vdash \exists x (x = a). \]

\begin{align*}
1 & \quad \neg \exists x (x = a) \quad \text{A} \\
1 & \quad \forall x \neg (x = a) \quad 1 \text{ SI(S)} \\
1 & \quad \neg (a = a) \quad 2 \text{ UE} \\
1 & \quad a = a \quad =I \\
1 & \quad a = a \land \neg (a = a) \quad 4,3 \land I \\
1 & \quad \neg \exists x (x = a) \quad 1,5 \text{ RAA} \\
1 & \quad \exists x (x = a) \quad 6 \text{ DN}
\end{align*}

Note that the following is NOT a correct proof:

\begin{align*}
(1) & \quad a = a \quad =I \\
(2) & \quad \exists x (x = a) \quad 1 \text{ EI}
\end{align*}

If you are unclear about why it is incorrect, check with Lemmon on the correct use of the EI rule. (Bottom line: we cannot replace just one of the occurrences of \( a \) with \( x \).)

(2) (16 points) Let \( \varphi \) be the wff

\[ \exists x \exists y \exists z (x \neq y \land y \neq z \land x \neq z \land P(x, y) \land P(y, z)), \]

let \( \psi \) be the wff

\[ \forall x \exists y \forall z (P(x, z) \rightarrow (P(y, x) \land P(z, y))), \]

and let \( \theta \) be the wff

\[ \exists x \exists y \neg P(x, y). \]

Construct an interpretation \( \mathcal{M} \) whose universe is \( U = \{1, 2, 3, 4, 5\} \), and which is such that

\[ \mathcal{M} \models \varphi \land \psi \land \theta. \]

You are asked for an interpretation \( \mathcal{M} \). Since I dictated that its universe be \( U = \{1, 2, 3, 4, 5\} \), all that is left to specify is the truth set of the binary predicate \( P \). I claim that setting the truth set to be

\[ \{(1, 2), (2, 3), (3, 1)\} \]

works just fine. I will not elaborate as to why that works; if you find the matter unclear, you should consult the notes on interpretations, or myself.
(3) (28 points) Exercise 4 (Velleman p. 186)
   (a) Neither reflexive, nor symmetric, nor transitive.
   (b) Neither reflexive, nor symmetric, nor transitive.
   (c) Reflexive, symmetric, and transitive.
   (d) Transitive, but neither reflexive nor symmetric.

(4) (16 points) Exercise 11 (Velleman p. 187)
Supposing that the relation $R$ on $A$ is reflexive, we will show that $R \subseteq R \circ R$. Let $(a, b) \in R$. Since $R$ is reflexive, we in fact have
\[(a, b) \in R \land (b, b) \in R.\]
Thus $b$ itself witnesses the existential quantifier in
\[\exists c((a, c) \in R \land (c, b) \in R).\]
But that is precisely the condition for $(a, b)$ to be in $R \circ R$, so we are done.

(5) (30 points) Exercise 14 (Velleman p. 187)
   (a) If $R_1$ and $R_2$ are both reflexive, then $(a, a) \in R_1$ and $(a, a) \in R_2$ for any $a \in A$. But then $(a, a) \in (R_1 \cap R_2)$ as well; so $R_1 \cap R_2$ is reflexive.
   (b) Suppose $R_1$ and $R_2$ are symmetric. Let $(a, b) \in (R_1 \cap R_2)$. Then $(a, b) \in R_1$ and $(a, b) \in R_2$. Since each of these is symmetric, $(b, a) \in R_1$ and $(b, a) \in R_2$. That is, $(b, a) \in (R_1 \cap R_2)$. Thus $R_1 \cap R_2$ is symmetric.
   (c) Suppose $R_1$ and $R_2$ are transitive. Suppose $(a, b)$ and $(b, c)$ are in $R_1 \cap R_2$. Then $(a, b)$ and $(b, c)$ are in each of $R_1$, $R_2$. Since each is transitive, we have $(a, c) \in R_1$ and $(a, c) \in R_2$, i.e. $(a, c) \in (R_1 \cap R_2)$. So $R_1 \cap R_2$ is transitive as well.