• You may work with other students from class, but everyone should write up solutions independently.
• Homework is due by the end of class time, either as a hard copy or email attachment.
• Remember that no late homework is accepted, but you have two drops.

(1) (10 points) Exercise 9 (Velleman p. 82)
Let $I = \{1, 2\}$, and set

\[
A_1 = \{c\}, \quad A_2 = \{d\}, \quad B_1 = \{d\}, \quad B_2 = \{c\}.
\]

Then

\[
\bigcup_{i \in I} (A_i \cap B_i) = (A_1 \cap B_1) \cup (A_2 \cap B_2) = \emptyset \cup \emptyset = \emptyset.
\]

But

\[
\left( \bigcup_{i \in I} A_i \right) \cap \left( \bigcup_{i \in I} B_i \right) = (A_1 \cup A_2) \cap (B_1 \cup B_2) = \{c, d\} \cap \{c, d\} = \{c, d\}.
\]

So we have a counterexample.

(2) (10 points) Exercise 10 (Velleman p. 82)
We have

\[
\mathcal{P}(A \cap B) = \{x \mid x \subseteq (A \cap B)\}
\]

\[
= \{x \mid \forall y(y \in x \rightarrow (y \in A \land y \in B))\},
\]

and

\[
\mathcal{P}(A) \cap \mathcal{P}(B) = \{x \mid x \subseteq A\} \cap \{x \mid x \subseteq B\}
\]

\[
= \{x \mid x \subseteq A \land x \subseteq B\}
\]

\[
= \{x \mid \forall y(y \in x \rightarrow y \in A) \land \forall y(y \in x \rightarrow y \in B)\}.
\]

Here’s a formal proof that the two defining conditions are equivalent, hence that the sets have the same elements and are identical. First one direction:
(3) (15 points) Exercise 12 (Velleman p. 82)

(a) First we note that
\[ \bigcup_{i \in I} (A_i \cup B_i) = \{ x \mid \exists i (i \in I \land (x \in A_i \lor x \in B_i)) \}, \]
while
\[ \left( \bigcup_{i \in I} A_i \right) \cup \left( \bigcup_{i \in I} B_i \right) = \{ x \mid \exists i (i \in I \land x \in A_i) \lor \exists i (i \in I \land x \in B_i) \}. \]

One could give a formal proof that each defining condition implies the other, like I gave above for the previous question. Or one could argue informally in words (though mirroring our formal inference rules). The reasoning here is straightforward, so I’ll say no more.

(b) Here we have
\[ (\bigcap \mathcal{F}) \cap (\bigcap \mathcal{G}) = \{ x \mid \forall A (A \in \mathcal{F} \rightarrow x \in A) \land \forall A (A \in \mathcal{G} \rightarrow x \in A) \}, \]
and
\[ \bigcap (\mathcal{F} \cup \mathcal{G}) = \{ x \mid \forall A ((A \in \mathcal{F} \lor A \in \mathcal{G}) \rightarrow x \in A) \}. \]

Again I’ll write no more about deriving each defining condition from the other; if you have difficulty doing so, please ask me about the matter.

(c) Finally, unpacking definitions in this case yields
\[ \bigcap_{i \in I} (A_i \setminus B_i) = \{ x \mid \forall i (i \in I \rightarrow (x \in A_i \lor x \in B_i)) \}, \]
while

\[(\bigcap_{i \in I} A_i) \setminus \left( \bigcup_{i \in I} B_i \right) = \{ x \mid \forall i (i \in I \rightarrow x \in A_i) \land \forall i (i \in I \rightarrow x \notin B_i) \}\].

Same as parts (a) and (b).

(4) (10 points) Exercise 14 (Velleman p. 82)

(a) Unpacking the statement \(x \in \bigcup \mathcal{F}\) using the definitions, we get

\[\exists A (A \in \mathcal{F} \land x \in A)\].

When \(\mathcal{F} = \emptyset\), there can be no such \(A\); hence \(x \in \bigcup \mathcal{F}\) is false for any \(x\) in that case. That is, \(\bigcup \emptyset = \emptyset\).

(b) On the other hand, the statement \(x \in \bigcap \mathcal{F}\) unpacks as

\[\forall A (A \in \mathcal{F} \rightarrow x \in A)\].

Here when \(\mathcal{F} = \emptyset\), the antecedent of that conditional will be false for any \(A\), hence the conditional as a whole is true no matter what \(A\) is. Thus \(x \in \bigcap \mathcal{F}\) is true for all \(x\) in that case.

(5) Exercise 15 (Velleman p. 83)

(a) (5 points)

We have \(\forall A \in U (A \in R \leftrightarrow A \notin A)\) as our starting point, and we seek a contradiction. Instantiate the universal quantifier by plugging in \(R\) for \(A\), and we get

\[R \in R \leftrightarrow R \notin R\].

By the law of the excluded middle, we know that \(R \in R \lor R \notin R\); we get a contradiction in either case. Assuming \(R \in R\), the biconditional gives us \(R \notin R\) as well; and similarly in the other case.

(b) (20 points) In this part, provide a reasoned answer as to what conclusions might be forced upon us by Russell’s paradox. Give a well-written answer; a paragraph or two is sufficient.

(6) (30 points) Exercise 4 (Lemmon p. 159)

(a) We want to show that if we drop the EI and EE rules from our system, and only have existential quantifiers as shorthand – \(\exists x\) just means \(\neg \forall x \neg\) – then we still have the force of the rules EI and EE; that is, we can still derive anything we could before.

So what does a use of EI look like? Like this:

\[
\vdots
\]

\[a_1, \ldots, a_n \quad (k) \quad \varphi(a)\]

\[a_1, \ldots, a_n \quad (k + 1) \quad \exists x \varphi(x) \quad \text{EI}\]

As long as we can get from \((k)\) to what we have on \((k + 1)\) using the rules we have left, then we’ve got what we need. Here’s how we can do it:

\[
\vdots
\]

\[a_1, \ldots, a_n \quad (k) \quad \varphi(a)\]

\[a_1, \ldots, a_n \quad (k + 1) \quad \forall x \neg \varphi(x) \quad A\]

\[k + 1 \quad (k + 2) \quad \neg \varphi(a) \quad k + 1 \quad \text{UE}\]

\[a_1, \ldots, a_n, k + 1 \quad (k + 3) \quad \varphi(a) \land \neg \varphi(a) \quad k, k + 2 \quad \text{\&I}\]

\[a_1, \ldots, a_n \quad (k + 4) \quad \neg \forall x \neg \varphi(x) \quad k + 1, k + 3 \quad \text{RAA}\]

\[a_1, \ldots, a_n \quad (\ell) \quad \exists x \varphi(x) \quad k + 4 \quad \text{Df.} \exists\]
Now we need to do the same for EE. A generic use of that rule looks like:

\[
\begin{align*}
&\exists x \varphi(x) \quad (k) \\
&\vdots \\
&\varphi(a) \quad A \\
&\vdots \\
&\psi \quad (p) \\
&k,\ell, p \text{ EE} \\
\end{align*}
\]

Here EE demands that \( a \) not appear in \( \psi \) or any of the lines \( b_1 \) through \( b_m \). Now we show that in the “new” setting we could still get to where EE takes us:

\[
\begin{align*}
&\exists x \varphi(x) \quad (k) \\
&\vdots \\
&\varphi(a) \quad A \\
&\vdots \\
&\psi \quad (p) \\
&k,\ell, p \text{ EE} \\
\end{align*}
\]

And we’re done. Note that our use of UI on line \((p + 5)\) was only justified by the supposition that \( a \) doesn’t appear in \( \psi \) or on lines \( b_1 \) through \( b_m \).

(b) This is done just like part (a), though here UI is more difficult than UE. Neither is any more complicated than the treatment of EE above; if you have particular questions about this, please ask me.