You may work with other students from class, but everyone should write up solutions independently.

Homework is due by the end of class time, either as a hard copy or email attachment.

Remember that no late homework is accepted, but you have two drops.

(1) (30 points) Exercise 1.c,d,e (Lemmon p. 128)

\[
\begin{array}{c|c|c}
\text{Line} & \text{Formula} & \text{Steps} \\
1 & (\exists x)\neg Fx \vdash \neg(\exists x)Fx & \text{A} \\
2 & (1) (\exists x)\neg Fx & \\
3 & (2) (x)Fx & \text{A} \\
4 & (3) \neg Fa & \text{A} \\
5 & (2, 4) Fa & \text{2 UE} \\
6 & (2, 3, 5) Fa \& \neg Fa & 4, 3 \& I \\
7 & (3, 6) \neg (x)Fx & 2, 5 \text{ RAA} \\
8 & (1, 6) \neg (x)Fx & 1, 3, 6 \text{ EE} \\
\end{array}
\]

\[
\begin{array}{c|c|c}
\text{Line} & \text{Formula} & \text{Steps} \\
1 & \neg (x)Fx \vdash (\exists x)\neg Fx & \\
2 & (1) \neg (x)Fx & \text{A} \\
3 & (2) \neg (\exists x)Fx & \text{A} \\
4 & (3) \neg Fa & \text{A} \\
5 & (3, 4) (\exists x)\neg Fx & 3 \text{ EI} \\
6 & (2, 3, 5) (\exists x)\neg Fx \& \neg (\exists x)Fx & 4, 3 \& I \\
7 & (2, 6) \neg \neg Fa & 3, 5 \text{ RAA} \\
8 & (2, 7) Fa & 6 \text{ DN} \\
9 & (2, 8) (x)Fx & 7 \text{ UI} \\
10 & (1, 2, 9) (x)Fx \& \neg (x)Fx & 8, 1 \& I \\
11 & (1, 10) \neg \neg (\exists x)Fx & 2, 9 \text{ RAA} \\
\end{array}
\]

\[
\begin{array}{c|c|c}
\text{Line} & \text{Formula} & \text{Steps} \\
1 & (x)\neg Fx \vdash \neg (\exists x)Fx & \text{A} \\
2 & (1) (x)\neg Fx & \text{A} \\
3 & (2) (\exists x)Fx & \text{A} \\
4 & (3) Fa & \text{A} \\
5 & (1, 4) \neg Fa & 1 \text{ UE} \\
6 & (1, 3, 5) Fa \& \neg Fa & 3, 4 \& I \\
7 & (3, 6) \neg (x)Fx & 1, 5 \text{ RAA} \\
8 & (2, 7) \neg (x)Fx & 2, 3, 6 \text{ EE} \\
9 & (1, 2, 8) (x)Fx \& \neg (x)Fx & 1, 7 \& I \\
10 & (1, 9) \neg (\exists x)Fx & 2, 8 \text{ RAA} \\
\end{array}
\]

Date: Due on Wednesday, March 17.
\[ -(\exists x)Fx \vdash (x)\neg Fx \]

1.  
\[ -(\exists x)Fx \quad A \]
2.  
\[ Fa \quad A \]
2.  
\[ (\exists x)Fx \quad 2 \text{ EI} \]
1,2  
\[ (\exists x)Fx \& -(\exists x)Fx \quad 3,1 \text{ &I} \]
1  
\[ \neg Fa \quad 2,4 \text{ RAA} \]
1  
\[ (x)(Fx \rightarrow \neg Gx) \vdash -(\exists x)(Fx \& Gx) \]

1.  
\[ (x)(Fx \rightarrow \neg Gx) \quad A \]
2  
\[ (\exists x)(Fx \& Gx) \quad A \]
3  
\[ Fa \rightarrow \neg Ga \quad 1 \text{ UE} \]
3  
\[ Fa \quad 3 \text{ &E} \]
3  
\[ Ga \quad 3 \text{ &E} \]
1,3  
\[ \neg Ga \quad 4,5 \text{ MPP} \]
1,3  
\[ Ga \& \neg Ga \quad 6,7 \text{ &I} \]
3  
\[ -(x)(Fx \rightarrow \neg Gx) \quad 1,8 \text{ RAA} \]
2  
\[ -(x)(Fx \rightarrow \neg Gx) \quad 2,3,9 \text{ EE} \]
1,2  
\[ (x)(Fx \rightarrow \neg Gx) \& -(x)(Fx \rightarrow \neg Gx) \quad 1,10 \text{ &I} \]
1  
\[ -(\exists x)(Fx \& Gx) \quad 2,11 \text{ RAA} \]

-(\exists x)(Fx \& Gx) \vdash (x)(Fx \rightarrow \neg Gx)

1.  
\[ -(\exists x)(Fx \& Gx) \quad A \]
2  
\[ -(Fa \rightarrow \neg Ga) \quad A \]
3  
\[ \neg Fa \quad A \]
3  
\[ Fa \rightarrow \neg Ga \quad 3 \text{ SI(S) 51} \]
2,3  
\[ (Fa \rightarrow \neg Ga) \& -(Fa \rightarrow \neg Ga) \quad 4,2 \text{ &I} \]
2  
\[ \neg \neg Fa \quad 3,5 \text{ RAA} \]
2  
\[ Fa \quad 6 \text{ DN} \]
8  
\[ \neg Ga \quad A \]
8  
\[ Fa \rightarrow \neg Ga \quad 8 \text{ SI(S) 50} \]
2,8  
\[ (Fa \rightarrow \neg Ga) \& -(Fa \rightarrow \neg Ga) \quad 9,2 \text{ &I} \]
2  
\[ \neg \neg Ga \quad 8,10 \text{ RAA} \]
2  
\[ Ga \quad 11 \text{ DN} \]
2  
\[ Fa \& Ga \quad 7,12 \text{ &I} \]
2  
\[ (\exists x)(Fx \& Gx) \quad 13 \text{ EI} \]
1,2  
\[ (\exists x)(Fx \& Gx) \& -(\exists x)(Fx \& Gx) \quad 14,1 \text{ &I} \]
1  
\[ -(Fa \rightarrow \neg Ga) \quad 2,15 \text{ RAA} \]
1  
\[ Fa \rightarrow \neg Ga \quad 16 \text{ DN} \]
1  
\[ (x)(Fx \rightarrow \neg Gx) \quad 17 \text{ UI} \]

(2) (20 points) Exercise 3.a,f (Lemmon p. 128)

\[ (x)(P \rightarrow Fx) \vdash P \rightarrow (x)Fx \]

1.  
\[ (x)(P \rightarrow Fx) \quad A \]
2  
\[ P \quad A \]
1  
\[ P \rightarrow Fa \quad 1 \text{ UE} \]
1,2  
\[ Fa \quad 3,2 \text{ MPP} \]
1  
\[ P \rightarrow Fa \quad 2,4 \text{ CP} \]
1  
\[ (x)(P \rightarrow Fx) \quad 5 \text{ UI} \]
(3) (10 points) Give a proof of this sequent:

\[(\exists x)(\exists y)(z)Pxyz \vdash (z)(\exists y)(\exists x)Pxyz.\]

\[
\begin{array}{ll}
1\, & (1)\, (\exists x)(\exists y)(z)Pxyz \quad A \\
2\, & (2)\, (\exists y)(z)Pxyz \quad A \\
3\, & (3)\, (z)Pabz \quad A \\
3\, & (4)\, Pabc \quad 3\ UI \\
3\, & (5)\, (\exists x)Pxbc \quad 4\ EI \\
3\, & (6)\, (\exists y)(\exists x)Pxye \quad 5\ EI \\
3\, & (7)\, (z)(\exists y)(\exists x)Pxyz \quad 6\ UI \\
2\, & (8)\, (z)(\exists y)(\exists x)Pxyz \quad 2,3,7\ EE \\
1\, & (9)\, (z)(\exists y)(\exists x)Pxyz \quad 1,2,8\ EE \\
\end{array}
\]

(4) (40 points) A predicate calculus wff is said to be in prenex normal form if all of its quantifiers are up front. For example,

\[(x)(\exists y)(Qx \to Pxy)\]
is in prenex normal form, whereas

$$(x)(Qx \rightarrow (\exists y)Px_y)$$

is not. Appealing to facts that appear in Chapter 3 of Lemmon (including exercises), explain as fully as you can why the following is true:

$(\star)$ For every wff $\varphi$, there is a wff $\psi$ such that

- $\psi$ is in prenex normal form.
- $\varphi \vdash \psi$.

Note: Don’t just throw some facts down on the paper. Treat your answer like a short essay that is meant to argue for the truth of $(\star)$; it will be graded as such.

Here, I wanted you to think about why this result holds for every wff, and how you might demonstrate that fact. If an arbitrary wff $\varphi$ isn’t already in prenex normal form, why isn’t it? Well, by the definition, it’s because there is at least one quantifier that doesn’t appear at the beginning of the wff. If you think about the parse tree of a wff, it is in prenex normal form precisely when all the quantifiers are the last things added in its construction.

So if a wff isn’t in prenex normal form, it’s because a quantifier step appears low down in its parse tree; for instance, our non-prenex wff above has the parse tree

\[
\langle x \rangle \langle Qx \rightarrow (\exists y)Pxy \rangle \\
\begin{array}{c}
Qa \\
Qa \rightarrow (\exists y)Pay \\
Qa \langle \exists y \rangle Pay \\
Pab
\end{array}
\]

The existential quantifier is attached before the conditional. We need to know that such quantifiers can be “pushed up the parse tree” until they are not below any propositional connectives. As long as each individual quantifier can always be pushed up the tree in such a way, we can repeat the process for however many there are in the wff, and the result will be a prenex wff.

Luckily, results we saw in lecture and in some of the previous questions above point the way. Sequents 118 and 119 (one of which we proved in lecture), as well as Exercises 1.c,d and 3.a-f on p. 128 (note that you were assigned 3.a,f above) all illustrate that a single universal or existential quantifier that is buried in a wff, can be brought one step closer to the top of the parse tree, and in such a way that the resulting wff is interderivable with the one we started with. So, by repeated use of these various results, we see that any single buried quantifier can be pushed up as desired. We can then repeat that whole process as needed for any other buried quantifiers, and the result is a wff in prenex normal form that is still equivalent to our original wff.
Here is an example, where I have indicated which of the cited results would be useful at each stage:

\[(x)P_x \rightarrow (\exists y)Q_{xy} \lor (z)F_z \quad \vdash \quad (\exists y)((x)P_x \rightarrow Q_{xy}) \lor (z)F_z \quad (119)\]

\[\vdash (\exists y)(\exists x)(P_x \rightarrow Q_{xy}) \lor (z)F_z \quad (3f)\]

\[\vdash (z)((\exists y)(\exists x)(P_x \rightarrow Q_{xy}) \lor F_z) \quad (3d)\]

\[\vdash (z)(\exists y)((\exists x)(P_x \rightarrow Q_{xy}) \lor F_z) \quad (3d)\]

\[\vdash (z)(\exists y)(\exists x)((P_x \rightarrow Q_{xy}) \lor F_z) \quad (3d)\]