You may work with other students from class, but everyone should write up solutions independently.

Homework is due by the end of class time, either as a hard copy or email attachment.

Remember that no late homework is accepted, but you have two drops.

(1) (15 points) Show that there are uncountably many real numbers in the interval $[0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$.

**Hint:** Every $x \in [0, 1]$ has a decimal (or base 10) expansion like 0.1859..., but it also has a binary (base 2) expansion. Use that fact to relate this to Cantor’s theorem.

Suppose, for the sake of contradiction, that $[0, 1]$ is countable. As we have seen, that would mean there is some surjection $f : \mathbb{Z}^+ \to [0, 1]$. Using the hint, we know that any $x \in [0, 1]$ can be represented as an infinite sequence of 1’s and 0’s, i.e. by its binary expansion. That means we could lay out a table as in the proof of Cantor’s theorem:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(1)$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>$f(2)$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>$f(3)$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>$f(4)$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>...</td>
</tr>
<tr>
<td>$f(5)$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>...</td>
</tr>
</tbody>
</table>

By our supposition, every $x \in [0, 1]$ must appear as the $f(k)$ row for some $k$. But consider the real $d$ which corresponds to the expansion whose digits come from the diagonal of our table (as illustrated, this is (1, 1, 0, 1, 1, ...)). Now let $c \in [0, 1]$ be that real whose binary expansion is the opposite of $d$’s (i.e. flip all the 1’s and 0’s). By our supposition, $c = f(k)$ for some $k$. But by the construction of $c$, it cannot possibly agree with the row $f(k)$ at the $k$-th digit; so we have a contradiction as desired. Thus $[0, 1]$ is indeed uncountable.

(Note: *technically*, we need to be a little more careful than the argument above. The reason is that some reals in $[0, 1]$ could have more than one binary expansion associated with them. For instance, .011111... and .100000... are both expansions for the real number 1/2. If this kind of thing happened often enough, then the fact that there are uncountably many expansions wouldn’t necessarily mean there are uncountably many reals in $[0, 1]$. But as it turns out, there are only countably many reals that have more than one expansion associated with them. **In grading, I didn’t take into account whether this wrinkle occurred to you or**

Date: Due on Friday, April 30.
not; I only wanted you to get the main idea. I only mention it here to be mathematically honest.)

(2) (24 points) Finds ways to express each of these statements in the language \(\mathcal{L}_A\):

(a) “\(a\) is greater than \(b\).”
(b) “\(a\) is a square number.”
(c) “\(a\) is an odd number.”

Note: \(\mathcal{L}_A\) has the symbols +, ·, 0, 1 and =. Your answers can use only those symbols, along with our logical symbols (connectives, quantifiers).

(a) \(\exists x (a = b + x \land \neg (x = 0))\)
(b) \(\exists x (x \cdot x = a)\)
(c) \(\neg \exists x (x + x = a)\)

(3) (30 points) Show that PA proves the associativity of +. That is,

\[ \text{PA} \vdash \forall x \forall y \forall z (x + (y + z) = (x + y) + z) \]

Do this by mathematical induction on the variable \(z\). Namely, first show that

\[ \text{PA} \vdash \forall x \forall y (x + (y + 0) = (x + y) + 0) \]

Then show the following conditional fact: if

\[ \text{PA} \vdash \forall x \forall y (x + (y + a) = (x + y) + a) \]

then also

\[ \text{PA} \vdash \forall x \forall y (x + (y + (a + 1)) = (x + y) + (a + 1)) \]

Note: Don’t give fully formal line-by-line proofs. Argue informally, but do be clear how you are using the axioms of PA to draw the conclusions that you do.

First we establish

\[ \text{PA} \vdash \forall x \forall y \forall z (x + (y + z) = (x + y) + z) \]

Setting \(z = 0\), we have, for any \(x\) and \(y\):

\[ x + (y + 0) = x + y \text{ (since } y + 0 = y \text{ by Ax.3)} \]
\[ = (x + y) + 0 \text{ (again using Ax.3)} \]

That’s all we need.

Alright, so now assume that

\[ \text{PA} \vdash \forall x \forall y (x + (y + a) = (x + y) + a) \]

On the basis of the PA axioms and that additional assumption, we need to show that

\[ \text{PA} \vdash \forall x \forall y (x + (y + (a + 1)) = (x + y) + (a + 1)) \]

We proceed thus:

\[ x + (y + (a + 1)) = x + ((y + a) + 1) \text{ (Ax.4)} \]
\[ = (x + (y + a)) + 1 \text{ (Ax.4)} \]
\[ = ((x + y) + a) + 1 \text{ (Assumption)} \]
\[ = (x + y) + (a + 1) \text{ (Ax.4)} \]
And that’s it. From the two things we’ve shown, mathematical induction
tells us that
\[ \text{PA} \vdash \forall x \forall y \forall z (x + (y + z) = (x + y) + z). \]

(4) (16 points) In the same vein as the previous question, show that
\[ \text{PA} \vdash \forall x (0 + x = x). \]

**Note:** If you look at the axioms of PA and find one that looks like it does all the work so there’s nothing for you to do, think again.

First note that 0 + 0 = 0 by Axiom 3. Now **assume** that
\[ \text{PA} \vdash 0 + a = a. \]
We need to show that, under that assumption, \( \text{PA} \vdash 0 + (a + 1) = a + 1 \) as well. Note:
\[
0 + (a + 1) = (0 + a) + 1 \text{ (Ax.4)}
= a + 1 \text{ (Assumption)}
\]
That’s just what we needed. So finally, by mathematical induction, we have
\[ \text{PA} \vdash \forall x (0 + x = x). \]

(5) (15 points) Suppose we have a domain \( D \) of known objects. Show that
\( A \subseteq D \) is decidable if and only if both \( A \) and \( D \setminus A \) are semi-decidable.

For the first direction of the biconditional, suppose that \( A \) is decidable. Then certainly \( A \) is semi-decidable, which should be clear from the definitions. But note that \( A \) being decidable means we have a program \( P \) which always correctly answers YES or NO depending on whether a given \( d \in D \) is in \( A \) or not. Now simply modify that program by switching the answers it gives; clearly that new program witnesses the fact that \( D \setminus A \) is decidable, hence semi-decidable as well.

Now for the other direction, suppose that both \( A \) and \( D \setminus A \) are semi-decidable. What does this mean? We have some program \( P \) such that:
- Given any \( d \in D \) as input, if \( d \in A \) then \( P \) will eventually output a YES answer. And if \( d \notin A \), \( P \) will just run forever and not give an answer.

Similarly, we have a program \( Q \) such that:
- Given any \( d \in D \) as input, if \( d \notin A \) then \( Q \) will eventually output a YES answer. And if \( d \in A \), \( Q \) will just run forever and not give an answer.

So now let the program \( R \) consist of just running \( P \) and \( Q \) at the same time. (You can think of this as alternately running a step of each program, or just think of this as running program \( P \) on one machine and program \( Q \) on a second machine.) For any input \( d \in D \), either \( d \in A \) or not. In the first case, \( P \) will give the answer YES; if that happens, \( R \) says YES. In the other case, \( Q \) will give the answer YES; if that happens, \( R \) says NO.

The program \( R \) thus witnesses the fact that \( A \) is decidable.