Notes on Classical Scattering (from Taylor’s Classical Mechanics)

(All figures from Taylor’s book)

Danny Darvish
June 7, 2017

The scattering angle and impact parameter

In scattering, we assume the projectile starts far enough away from the target that it moves freely with only kinetic energy. The impact parameter $b$ and the scattering angle $\theta$ are shown in Figure 1 for a fixed target exerting a non-contact (e.g. Coulomb) force on a projectile, and in Figure 2 for a contact-only force (e.g. billiard balls). By “target” we can also mean the center of the target.

There should be a unique $\theta$ for every $b$, and one of the main goals of classical collision theory is to determine $\theta = \theta(b)$. But in experiment, it is easy to measure $\theta$ while it is impossible to measure $b$. Measuring only $\theta$ doesn’t tell us much, but measuring $b$ can tell us about the size of the target and the range of the force. The impossibility of measuring $b$ leads to the notion of the collision cross section.

The cross section

Since we usually can’t measure $b$ for a single collision, we take a statistical approach. Consider Figure 3. The cross section here, $\sigma$, is equal to $\pi R^2$. Since we don’t know $b$, we don’t know for sure if a single projectile will strike a target.

Define a number density per unit area, $n_{\text{tar}}$, and the total area of the the assembly, $A$. The total area of the targets is $n_{\text{tar}} A \sigma$, so the probability that a projectile will hit a target is

$$P_{\text{hit}} = \frac{n_{\text{tar}} A \sigma}{A} = n_{\text{tar}} \sigma. \quad (1)$$

If we send a large number, $N_{\text{inc}}$, particles, then we should expect

$$N_{\text{sc}} = N_{\text{inc}} n_{\text{tar}} \sigma \quad (2)$$

Figure 1: Fixed target exerting force on projectile
of them to scatter. Since we can measure $N_{sc}$, $N_{inc}$, and $n_{tar}$, we can determine $\sigma$ experimentally. If we instead consider a rate $R = \frac{\Delta N}{\Delta t}$, then (2) becomes:

$$R_{sc} = R_{inc} n_{tar} \sigma.$$  \hspace{1cm} (3)

The concept of $\sigma$ will become generalized, but it always represents the effective area of the target for interacting with the particle.

Since nuclear dimensions are of the order $10^{-14}$ m, a convenient unit for area is the barn, which equals $10^{-28}$ m$^2$.

**Generalizations of the cross section**

**Two spheres**

Before, in Figure 2, we considered the projectile to be small compared to the target. If we instead consider two spheres of similar sizes; a projectile of radius $R_1$ and a target of radius $R_2$, then all of the details of the scattering process from before are the same, with the exception that the cross section is now $\sigma = \pi (R_1 + R_2)^2$ (since there will be a hit if $b \leq R_1 + R_2$). This demonstrates how $\sigma$ is not, in general, the cross section of the target, but instead the effective area of the projectile scattering the target.

**Other kinds**

Suppose that for hard sphere scattering, a portion of the sphere caused the projectile to stick, or become captured, and another portion caused it to scatter. In this case, we can define the cross sections $\sigma_{scatter}$ and $\sigma_{capture}$ by number scattered or captured:

$$N_{scattered} = N_{inc} n_{tar} \sigma_{scatter}$$

$$N_{captured} = N_{inc} n_{tar} \sigma_{capture}$$  \hspace{1cm} (4)

We form other sorts of cross sections for other processes, such as an ionization cross section, a fission cross section, and a cross section only for inelastic collisions.

The **total cross section** is just the sum of all the partial cross sections, for example:

$$\sigma_{tot} = \sigma_{sc} + \sigma_{cap} + \sigma_{ion}.$$  \hspace{1cm} (5)

“Scattering” is really a term reserved for process where a projectile deflects and leaves behind the same target. This excludes things ionization, fission, and capture.
Separate categorization: **Elastic** collisions are those which leave the internal motions of the target unchanged. **Inelastic** collisions are those which change the internal motions of the target. An electron losing some kinetic energy by exciting the energy levels of an atom is an example of an inelastic collision.

Lastly, different cross sections can depend on energy. For example, below the ionization energy of the target, $\sigma_{\text{ion}} = 0$.

**The differential scattering cross section**

If we take the direction of the incident beam to be the $z$-axis, then the direction of a scattered projectile is specified by $\theta$ and $\phi$. We can consider the number of particles scattered into a solid angle $d\Omega$ as shown in Figure 4. The (differential) number of particles scattered into $d\Omega$ is then given by,

$$dN_{\text{sc}} = N_{\text{inc}} n_{\text{tar}} d\sigma,$$

where $d\sigma$ is the effective cross-sectional area of the target for a projectile scattering into $d\Omega$. We can factor out $d\Omega$ to define the **differential scattering cross section** $\frac{d\sigma}{d\Omega}$ as follows:

$$d\sigma = \frac{d\sigma}{d\Omega} d\Omega$$

We can rewrite (6) as,

$$dN_{\text{sc}} = N_{\text{inc}} n_{\text{tar}} \frac{d\sigma}{d\Omega}(\theta, \phi) d\Omega,$$

emphasizing the dependence of the differential cross section on the direction of observation.

We can determine the total number of particles scattered $N_{\text{sc}}$ by integrating (6) over all $\theta$ and $\phi$. This implies that the **total scattering cross section**, $\sigma$, is given by,

$$\sigma = \int \frac{d\sigma}{d\Omega}(\theta, \phi) d\Omega$$

**Calculating the differential cross section**

Assume axial symmetry (which would follow for a situation with spherical symmetry), as in Figure 5. In principle, we can calculate a particle’s trajectory to determine $\theta(b)$ or $b(\theta)$. Considering particles incident on the annulus shown, the cross sectional area is

$$d\sigma = 2\pi b db,$$
Figure 5: Projectiles incident between \( b \) and \( b + db \) are scattered between angles \( \theta \) and \( \theta + d\theta \).

Figure 6: An illustration of hard sphere scattering. Note the assumed “law of reflection”, which can be shown.

and the same particles emerge in a solid angle

\[
d\Omega = 2\pi \sin \theta \, d\theta. \tag{11}\]

Therefore, the differential cross section is:

\[
\frac{d\sigma}{d\Omega} = \frac{b}{\sin \theta} \frac{db}{d\theta}. \tag{12}\]

So to calculate the differential cross section for a particle, we need to find \( \theta(b) \) or \( b(\theta) \) by calculating its trajectory, and then differentiate to plug into (12). Try to calculate \( \frac{d\sigma}{d\Omega} \) for the hard sphere scattering shown in Figure 6.

**Rutherford scattering**

Consider alpha particles (helium nuclei) scattering off the nuclei of gold atoms in a thin (so there are no repeat collisions) foil, as shown in Figure 7. Ignoring the attractive force from the electrons in the foil, the

Figure 7: Rutherford scattering. The orbit is a hyperbola, symmetric about \( \mathbf{u} \). The position of the particle can be measured by the angle \( \psi \). As \( t \to \infty \), \( \psi \to \psi_0 \).
cause of scattering is the Coulomb force, which we will write as

\[ F = \frac{\gamma}{r^2}. \tag{13} \]

Since \( \psi \rightarrow -\psi_0 \) as \( t \rightarrow -\infty \), \( \theta = \pi - 2\psi_0 \). To determine \( b \), we evaluate the change in momentum, \( \Delta p = p' - p \), in two ways. Assuming the gold atom is fixed in position, \( p \) and \( p' \) have equal magnitudes. From Figure 8, it is easy to determine that

\[ |\Delta p| = 2p \sin(\theta/2) \tag{14} \]

For the second evaluation of \( \Delta p \), we use Newton’s second law, \( \Delta p = \int F \, dt \). Examining the figures, we can see that \( \Delta p \) and \( u \) point in the same direction, so we can write

\[ |\Delta p| = \int_{-\infty}^{\infty} F_u \, dt \tag{15} \]

Using \( F_u = \frac{\gamma}{r^2} \cos \psi \), \( dt = d\psi/\dot{\psi} \), and \( m r^2 \dot{\psi} = l = bp \), we can write:

\[ |\Delta p| = \int_{-\psi_0}^{\psi_0} \frac{\gamma \cos \psi}{r^2} \frac{d\psi}{bp/mr^2} = \frac{\gamma m}{bp} 2 \sin \phi_0 = \frac{2\gamma m}{bp} \cos(\theta/2). \tag{16} \]

Relating \[14\] to \[16\], we can solve for \( b \):

\[ b = \frac{\gamma m \cos(\theta/2)}{p^2 \sin(\theta/2)} = \frac{\gamma}{m v^2} \cot(\theta/2) \tag{17} \]

Finally, we can differentiate and solve for the differential scattering cross section, giving the celebrated Rutherford scattering formula:

\[ \frac{d\sigma}{d\Omega} = \left( \frac{kqQ}{4E \sin^2(\theta/2)} \right)^2, \tag{18} \]

where \( E \) is the kinetic energy. Surprisingly, though derived using wrong physics, it agrees exactly with the formula derived using quantum mechanics.

Cross sections in various frames