Gluonic flux tubes and string modes

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1 Introduction

The stationary states of glue surrounding a static quark-antiquark pair, separated by some distance r, contain important clues to the microscopic origin of quark confinement and the nature of the vacuum in quantum chromodynamics (QCD). Currently, little is known about the properties of such states since they cannot be described using standard perturbative techniques. It is generally believed that at sufficiently large r, the chromoelectric and chromomagnetic fields become confined to a long tube-like region of space connecting the quark and the antiquark. A description of the glue in terms of the collective degrees of freedom associated with the position of the long flux might then be sufficient for reproducing the low-energy spectrum. The oscillating flux can be treated in terms of an effective string theory. The effective QCD string can also be studied without fixed end sources by investigating the stationary states of glue in a box with periodic boundary conditions; such states involve flux tubes (torelons) which wrap around the torus. In either case, the lowest-lying excitations are expected to be the Goldstone modes associated with the spontaneously broken transverse translational symmetry. These modes are a universal feature of any low-energy description of the effective QCD string and have energy separations above the ground state given by multiples of π/r. For the gluonic excitations at small r, no robust expectations from theory presently exist.

The purpose of these notes is to outline the spectrum and nature of the expected string modes for both the toroidal and fixed end cases. Very general properties will be used to deduce the expected pattern of degeneracies and level orderings.

2 Fixed end string levels

The first step in determining the energies of the stationary states of gluons in the presence of a static quark and antiquark, fixed in space some distance r apart, is to classify the levels in terms of the symmetries of the problem. Such a system has cylindrical symmetry about the axis $\hat{\mathbf{r}}$ passing through the quark and the antiquark (the molecular axis). The total angular momentum $\vec{J}_g$ of the gluons is not a conserved quantity, but its projection $\vec{J}_g \cdot \hat{\mathbf{r}}$ onto the molecular axis is and can be used to label the energy levels of the gluons. Here, we adopt the standard notation from the physics of diatomic molecules and denote the magnitude of

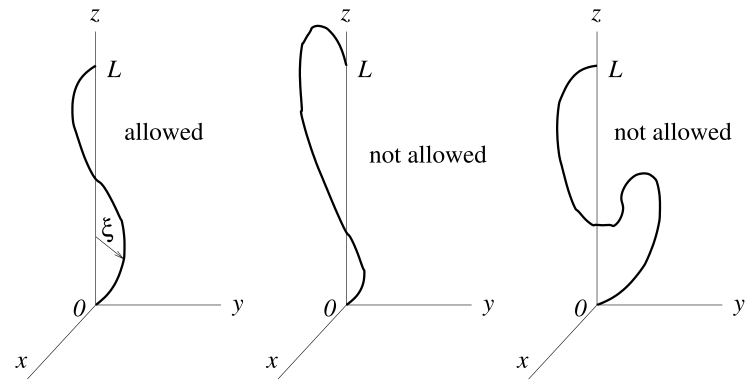

Figure 1: Examples of allowed and disallowed string configurations.

the eigenvalue of $\vec{J}_g \cdot \hat{\mathbf{r}}$ by Λ. States with $\Lambda = 0, 1, 2, 3, 4, \ldots$ are typically denoted by the capital Greek letters $\Sigma, \Pi, \Delta, \Phi, \Gamma, \ldots$, respectively. The energy of the gluons is unaffected by reflections in a plane containing the molecular axis; since such reflections interchange states of opposite handedness, given by the sign of the eigenvalue of $\vec{J}_g \cdot \hat{\mathbf{r}}$, such states must necessarily be degenerate (Λ doubling). However, this doubling does not apply to the Σ states; Σ states which are even (odd) under a reflection in a plane containing the molecular axis are denoted by a superscript $+$ $(-)$. Another symmetry is the combined operation of charge conjugation and spatial inversion about the midpoint between the quark and the antiquark. Here, we denote the eigenvalue of this transformation by η_{CP} which can take values ± 1. States which are even (odd) under this parity–charge-conjugation operation are indicated by the subscripts g (u). Hence, the low-lying gluon levels are labelled Σ_g^+, Σ_g^-, Σ_u^+, Σ_u^-, Π_g, Π_u, Δ_g, Δ_u, and so on.

Next, assume that the fixed ends of the effective QCD string lie along the z-axis. The location of the string can be specified in terms of displacements $\xi_x(z,t)$ and $\xi_y(z,t)$ in the x and y directions, respectively, from the z-axis at time t. The boundary conditions are $\xi_j(0,t) = 0$ and $\xi_j(L,t) = 0$ where $L = r$. Furthermore, we assume that the displacements (and their first derivatives with respect to z and t) are continuous and single-valued for each value of z and t; in other words, string configurations which double-back on themselves or overhang the ends are disallowed (see Fig. 1).

The effective string action, without interactions, is taken to be

$$S = \int dt \int_0^L dz \left[\tfrac{1}{2}\rho \left(\dot{\xi}_x^2 + \dot{\xi}_y^2 \right) - \tfrac{1}{2}\kappa \left(\xi_x'^2 - \xi_y'^2 \right) \right], \tag{1}$$

where ρ is the linear mass density of the string, κ is the string tension, and

$$\dot{\xi}_j = \frac{\partial \xi_j}{\partial t}, \qquad \xi_j' = \frac{\partial \xi_j}{\partial z}. \tag{2}$$

The momentum canonically conjugate to ξ_j is

$$\Pi_j = \frac{\partial L}{\partial \dot{\xi}_j} = \rho \, \dot{\xi}_j, \tag{3}$$

so that the Hamiltonian is

$$H = \int_0^L dz \left\{ \frac{1}{2\rho}\Big(\Pi_x^2 + \Pi_y^2\Big) + \frac{\kappa}{2}\Big(\xi_x'^2 + \xi_y'^2\Big) \right\}, \tag{4}$$

and the equal-time commutation relations are

$$[\ \xi_i(z,t),\ \Pi_j(z',t)\ ] = i\delta_{ij}\delta(z-z'). \tag{5}$$

The system is solved by expressing the displacements in terms of their normal modes. For fixed ends, the normal modes are standing waves $\sin(m\pi z/L)$ having energy $m\omega$ for positive integer m. Using such modes, we can introduce ladder operators:

$$\xi_j(z,t) = \sum_{m=1}^{\infty} \frac{1}{\sqrt{m\omega\rho L}} \sin\left(\frac{m\pi z}{L}\right)\Big(a_{jm}\, e^{-im\omega t} + a_{jm}^\dagger\, e^{im\omega t}\Big), \qquad \omega = \frac{\pi}{L}\sqrt{\frac{\kappa}{\rho}}, \tag{6}$$

for $j = 1, 2 = x, y$. For fixed ends, these are standing waves having energy $m\omega$. Note that the displacement operators are Hermitian, as they should be. The ladder operators satisfy the commutation relations

$$[a_{jm}, a_{j'm'}] = 0, \qquad [a_{jm}, a_{j'm'}^\dagger] = \delta_{jj'}\delta_{mm'}. \tag{7}$$

In order to show that the above commutation relations are consistent with the commutators of Eq. 5, we need the Fourier series of the periodic Dirac δ-function. Recall the definition of the Fourier series for a periodic function with period T:

$$f(z) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left[a_m \cos\left(\frac{2\pi m z}{T}\right) + b_m \sin\left(\frac{2\pi m z}{T}\right)\right], \tag{8}$$

$$a_m = \frac{2}{T}\int_c^{c+T} dz\ f(z)\cos\left(\frac{2\pi m z}{T}\right), \tag{9}$$

$$b_m = \frac{2}{T}\int_c^{c+T} dz\ f(z)\sin\left(\frac{2\pi m z}{T}\right). \tag{10}$$

Here, the modes are $\sin(m\pi z/L)$ so we need $T = 2L$ and can choose $c = -L$, even though we are only interested in the range $0 \le z \le L$. Each of the modes is *odd* in z, so any linear combinations of the normal modes will be odd, so we can use a Fourier sine series:

$$f(z) = \sum_{m=1}^{\infty} b_m \sin\left(\frac{\pi m z}{L}\right), \tag{11}$$

$$b_m = \frac{2}{L}\int_0^L dz\ f(z)\sin\left(\frac{\pi m z}{L}\right). \tag{12}$$

If $f(z) = \mathrm{sgn}(z)\delta(|z| - z')$, which is odd in z, then $b_m = (2/L)\sin(m\pi z'/L)$ if $0 < z' \le L$. Hence,

$$\delta(z - z') = \frac{2}{L}\sum_{m=1}^{\infty} \sin\left(\frac{\pi m z}{L}\right)\sin\left(\frac{\pi m z'}{L}\right), \quad \text{for } 0 < z \le L. \tag{13}$$

The Hamiltonian is then given by, discarding an irrelevant (but infinitely large) constant,

$$H = \sum_{m=1}^{\infty} m\omega\Big(a_{xm}^\dagger a_{xm} + a_{ym}^\dagger a_{ym}\Big), \qquad \omega = \frac{\pi}{L}\sqrt{\frac{\kappa}{\rho}}. \tag{14}$$

Under rotations $R_z(\phi)$ of angle ϕ about the z-axis, the normal mode operators transform according to

$$
\begin{aligned}
R_z(\phi)\ a_{xm}^\dagger\ R_z^\dagger(\phi) &= \cos\phi\ a_{xm}^\dagger + \sin\phi\ a_{ym}^\dagger, &(15)\\
R_z(\phi)\ a_{ym}^\dagger\ R_z^\dagger(\phi) &= -\sin\phi\ a_{xm}^\dagger + \cos\phi\ a_{ym}^\dagger. &(16)
\end{aligned}
$$

Given the rotational symmetry about the z-axis, it is better to work with left and right circularly polarized modes. Define the circularly polarized modes as

$$a_{m\pm}^\dagger = \frac{1}{\sqrt{2}}\Big(a_{xm}^\dagger \pm i a_{ym}^\dagger\Big). \tag{17}$$

Under a rotation about the z-axis, these operators transform as

$$R_z(\phi)\ a_{m\pm}^\dagger\ R_z^\dagger(\phi) = e^{\mp i\phi}\ a_{m\pm}^\dagger, \tag{18}$$

and the Hamiltonian is

$$H = \sum_{m=1}^{\infty} m\,\omega\Big(a_{m+}^\dagger a_{m+} + a_{m-}^\dagger a_{m-}\Big), \qquad \omega = \frac{\pi}{L}\sqrt{\frac{\kappa}{\rho}}. \tag{19}$$

The right-handed "phonons" are indicated by the $+$ sign, whereas the left-handed modes are indicated by the $-$ sign.

Let $|0\rangle$ denote the ground state of the string, then the string eigenmodes are

$$\prod_{m=1}^{\infty} \frac{(a_{m+}^\dagger)^{n_{m+}}}{\sqrt{n_{m+}!}}\ \frac{(a_{m-}^\dagger)^{n_{m-}}}{\sqrt{n_{m-}!}}\ |0\rangle, \tag{20}$$

where n_{m+} and n_{m-} are the occupation numbers which take values $0, 1, 2, \ldots$. We now wish to determine the symmetry properties of these states. Let $\mathcal{P}_{L/2}$ denote spatial inversion about the point $(0, 0, L/2)$ and C denote charge conjugation. The flux in the effective QCD string has a direction associated with it, so that charge conjugation simply effects a reversal of this direction. This direction is also reversed under $\mathcal{P}_{L/2}$ so that $C\mathcal{P}_{L/2}$ is a symmetry of the system. Also, let σ_x and σ_y denote reflections in the xz and yz planes, respectively. The ground state satisfies

$$
\begin{aligned}
R_z(\phi)\ |0\rangle &= |0\rangle, &(21)\\
C\mathcal{P}_{L/2}\ |0\rangle &= |0\rangle, &(22)\\
\sigma_x\ |0\rangle &= |0\rangle, &(23)\\
\sigma_y\ |0\rangle &= |0\rangle. &(24)
\end{aligned}
$$

To determine the behavior of the operators $a_{m\pm}^\dagger$ under these symmetry operations, one uses Eq. 6 and the following transformation properties of the string displacement:

$$
\begin{aligned}
C\mathcal{P}_{L/2}\ \xi_j(z,t)\ \mathcal{P}_{L/2}^\dagger C^\dagger &= -\xi_j(L-z,t), &(25)\\
\sigma_x\ \xi_x(z,t)\ \sigma_x^\dagger &= \xi_x(z,t), &(26)\\
\sigma_y\ \xi_x(z,t)\ \sigma_y^\dagger &= -\xi_x(z,t), &(27)\\
\sigma_x\ \xi_y(z,t)\ \sigma_x^\dagger &= -\xi_y(z,t), &(28)\\
\sigma_y\ \xi_y(z,t)\ \sigma_y^\dagger &= \xi_y(z,t). &(29)
\end{aligned}
$$

Using Eq. 6 and Eq. 17 and the above transformation properties, one easily determines

$$\begin{aligned} C\mathcal{P}_{L/2}\, a^\dagger_{m\pm}\, \mathcal{P}^\dagger_{L/2} C^\dagger &= (-1)^m a^\dagger_{m\pm}, & (30)\\ \sigma_x\, a^\dagger_{m\pm}\, \sigma^\dagger_x &= a^\dagger_{m\mp}, & (31)\\ \sigma_y\, a^\dagger_{m\pm}\, \sigma^\dagger_y &= -a^\dagger_{m\mp}. & (32) \end{aligned}$$

Hence, if E_0 denotes the energy of the ground state (with the above Hamiltonian, it has been defined to be zero), then the eigenvalues E (energy), Λ, and η_{CP} associated with the string eigenstates are given by

$$\begin{aligned} E &= E_0 + \frac{N\pi}{r}\sqrt{\frac{\kappa}{\rho}}, & (33)\\ N &= \sum_{m=1}^{\infty} m\ (n_{m+} + n_{m-}), & (34)\\ \Lambda &= \left|\sum_{m=1}^{\infty} (n_{m+} - n_{m-})\right|, & (35)\\ \eta_{CP} &= (-1)^N. & (36) \end{aligned}$$

For the Σ states, the evenness or oddness under exchange $(-) \leftrightarrow (+)$ of the circular polarizations yields a superscript + or −, respectively. Using these properties, the orderings and degeneracies of the Goldstone string energy levels and their symmetries are as shown in Table 1. Hence, for $\kappa = \rho$, the $N\pi/r$ behavior and a well-defined pattern of degeneracies and level orderings among the different channels form a very distinctive signature of the onset of the Goldstone modes for the effective QCD string.

3 Toroidal string levels

A string without fixed ends which winds around a box with periodic (toroidal) boundary conditions has different symmetry properties. Here we shall assume that the string loop winds around the torus in the z-direction, and let L be the circumference of the torus in this direction. Let the position of the string in the x and y directions be specified by $q_x(z,t)$ and $q_y(z,t)$, respectively. Once again, assume that the string is stiff enough that $q_x(z,t)$ and $q_y(z,t)$ are single valued (no configurations which double back on themselves). Of course, this assumption could be relaxed by labeling the position along the string by some parameter other than z, but this is an unnecessary complication for our purposes.

The effective string action is taken to be

$$S_T = \int dt \int_0^L dz\ \left[\tfrac{1}{2}\rho\ \left(\dot{q}_x^2 + \dot{q}_y^2\right) - \tfrac{1}{2}\kappa\left(q_x'^2 - q_y'^2\right)\right], \tag{37}$$

where ρ is the linear mass density of the string and κ is the string tension. The momentum canonically conjugate to q_j is

$$\Pi_j = \frac{\partial L}{\partial \dot{q}_j} = \rho\, \dot{q}_j, \tag{38}$$

Table 1: Low-lying string levels for fixed ends. The $N = 1$ level is two-fold degenerate, and the $N = 2, 3, 4$ levels are 5,10,15-fold degenerate, respectively. The $+(-)$ signs refer to right (left) circular polarizations, and positive integers indicate phonon modes. The Σ, Π, Δ, Φ, and Γ levels have $\Lambda = 0, 1, 2, 3$, and 4, respectively, where Λ is the magnitude of the z-projection of angular momentum. Subscripts $g(u)$ indicate evenness (oddness) under $C\mathcal{P}_{L/2}$. The Σ^+ (Σ^-) states are even (odd) under reflections in any plane containing the z-axis.

$N = 0$:	Σ_g^+	$\|0\rangle$	
$N = 1$:	Π_u	$a_{1+}^\dagger\|0\rangle$	$a_{1-}^\dagger\|0\rangle$
$N = 2$:	$\Sigma_g^{+\prime}$	$a_{1+}^\dagger a_{1-}^\dagger\|0\rangle$	
	Π_g	$a_{2+}^\dagger\|0\rangle$	$a_{2-}^\dagger\|0\rangle$
	Δ_g	$(a_{1+}^\dagger)^2\|0\rangle$	$(a_{1-}^\dagger)^2\|0\rangle$
$N = 3$:	Σ_u^+	$(a_{1+}^\dagger a_{2-}^\dagger + a_{1-}^\dagger a_{2+}^\dagger)\|0\rangle$	
	Σ_u^-	$(a_{1+}^\dagger a_{2-}^\dagger - a_{1-}^\dagger a_{2+}^\dagger)\|0\rangle$	
	Π_u'	$a_{3+}^\dagger\|0\rangle$	$a_{3-}^\dagger\|0\rangle$
	Π_u'	$(a_{1+}^\dagger)^2 a_{1-}^\dagger\|0\rangle$	$a_{1+}^\dagger(a_{1-}^\dagger)^2\|0\rangle$
	Δ_u	$a_{1+}^\dagger a_{2+}^\dagger\|0\rangle$	$a_{1-}^\dagger a_{2-}^\dagger\|0\rangle$
	Φ_u	$(a_{1+}^\dagger)^3\|0\rangle$	$(a_{1-}^\dagger)^3\|0\rangle$
$N = 4$:	$\Sigma_g^{+\prime\prime}$	$a_{2+}^\dagger a_{2-}^\dagger\|0\rangle$	
	$\Sigma_g^{+\prime\prime}$	$(a_{1+}^\dagger)^2(a_{1-}^\dagger)^2\|0\rangle$	
	$\Sigma_g^{+\prime\prime}$	$(a_{1+}^\dagger a_{3-}^\dagger + a_{1-}^\dagger a_{3+}^\dagger)\|0\rangle$	
	Σ_g^-	$(a_{1+}^\dagger a_{3-}^\dagger - a_{1-}^\dagger a_{3+}^\dagger)\|0\rangle$	
	Π_g'	$a_{4+}^\dagger\|0\rangle$	$a_{4-}^\dagger\|0\rangle$
	Π_g'	$(a_{1+}^\dagger)^2 a_{2-}^\dagger\|0\rangle$	$(a_{1-}^\dagger)^2 a_{2+}^\dagger\|0\rangle$
	Π_g'	$a_{1+}^\dagger a_{1-}^\dagger a_{2+}^\dagger\|0\rangle$	$a_{1+}^\dagger a_{1-}^\dagger a_{2-}^\dagger\|0\rangle$
	Δ_g'	$a_{1+}^\dagger a_{3+}^\dagger\|0\rangle$	$a_{1-}^\dagger a_{3-}^\dagger\|0\rangle$
	Δ_g'	$(a_{2+}^\dagger)^2\|0\rangle$	$(a_{2-}^\dagger)^2\|0\rangle$
	Δ_g'	$(a_{1+}^\dagger)^3 a_{1-}^\dagger\|0\rangle$	$a_{1+}^\dagger(a_{1-}^\dagger)^3\|0\rangle$
	Φ_g	$(a_{1+}^\dagger)^2 a_{2+}^\dagger\|0\rangle$	$(a_{1-}^\dagger)^2 a_{2-}^\dagger\|0\rangle$
	Γ_g	$(a_{1+}^\dagger)^4\|0\rangle$	$(a_{1-}^\dagger)^4\|0\rangle$

so that the Hamiltonian is

$$H = \int_0^L dz \left\{ \frac{1}{2\rho}\left(\Pi_x^2 + \Pi_y^2\right) + \frac{\kappa}{2}\left(q_x'^2 + q_y'^2\right) \right\}, \tag{39}$$

and the equal-time commutation relations are

$$[\ q_i(z,t),\ \Pi_j(z',t)\ ] = i\delta_{ij}\delta(z-z'). \tag{40}$$

Now define the "center of mass" and the total transverse momentum by

$$Q_j(t) = \frac{1}{L}\int_0^L dz\ q_j(z,t), \tag{41}$$

$$P_j(t) = \int_0^L dz\ \Pi_j(z,t), \tag{42}$$

which satisfy the equal-time commutation relations

$$[\ Q_j(t),\ P_k(t)\ ] = i\delta_{jk}. \tag{43}$$

The Hamiltonian can be diagonalized by expressing the string location and momentum in terms of the normal modes, introducing ladder operators:

$$q_j(z,t) = Q_j + \frac{t}{\rho L}P_j + \sum_{m\neq 0} \frac{1}{\sqrt{2\rho L\Omega_m}}\left(a_{jm}\, e^{-i\Omega_m t + ik_m z} + a_{jm}^\dagger\, e^{i\Omega_m t - ik_m z}\right), \tag{44}$$

for $j = 1,2 = x,y$ and where $Q_j = Q_j(0)$, $P_j = P_j(0) = P_j(t)$, and

$$k_m = \frac{2\pi}{L}\ m, \qquad \Omega_m = \frac{2\pi}{L}\sqrt{\frac{\kappa}{\rho}}\ |m|. \tag{45}$$

These operators satisfy the commutation relations

$$[a_{jm}, a_{j'm'}] = 0, \qquad [a_{jm}, a_{j'm'}^\dagger] = \delta_{jj'}\delta_{mm'}, \tag{46}$$

$$[a_{jm}, P_k] = 0, \qquad [a_{jm}, Q_k] = 0, \qquad [Q_j, P_k] = i\delta_{jk}. \tag{47}$$

In order to show that the above commutation relations are consistent with the commutators of Eq. 40, set $c = 0$ and $T = L$ in Eqs. 8-10 to show that

$$\delta(z-z') = \frac{1}{L} + \frac{2}{L}\sum_{m=1}^{\infty} \cos\left(\frac{2\pi m}{L}(z-z')\right). \tag{48}$$

Note that $q_j(z,t)$ are Hermitian operators and satisfy the boundary conditions $q_j(0,t) = q_j(L,t)$ and $q_j'(0,t) = q_j'(L,t)$. Satisfying both of these equations results in the 2ω energy quantization, instead of $\omega = (\pi/L)\sqrt{\kappa/\rho}$ as with fixed ends. With periodic boundary conditions, the normal modes are traveling plane waves having energy Ω_m. Also note that Eq. 44 is consistent with Eq. 41 given that

$$Q_j(t) = Q_j + \frac{t}{\rho L}P_j, \tag{49}$$

since $\int_0^L dz\ \exp(2\pi i m z/L) = 0$ for non-zero integer m.

In terms of the ladder operators, the Hamiltonian is given by, discarding an irrelevant constant,

$$H = \frac{1}{2\rho L}\left(P_x^2 + P_y^2\right) + \sum_{m\neq 0} \Omega_m\left(a_{xm}^\dagger a_{xm} + a_{ym}^\dagger a_{ym}\right), \qquad \Omega_m = \frac{2\pi}{L}\sqrt{\frac{\kappa}{\rho}}\ |m|. \tag{50}$$

As in the case of the string with fixed ends, it is convenient to transform to right and left-circularly polarized waves,

$$a_{mR}^\dagger = \frac{1}{\sqrt{2}}\left(a_{xm}^\dagger + i a_{ym}^\dagger\right), \tag{51}$$

$$a_{mL}^\dagger = \frac{1}{\sqrt{2}}\left(a_{xm}^\dagger - i a_{ym}^\dagger\right), \tag{52}$$

so that the final form of the Hamiltonian is

$$H = \frac{1}{2\rho L}\left(P_x^2 + P_y^2\right) + \sum_{m\neq 0} \Omega_m\left(a_{mR}^\dagger a_{mR} + a_{mL}^\dagger a_{mL}\right), \qquad \Omega_m = \frac{2\pi}{L}\sqrt{\frac{\kappa}{\rho}}\ |m|. \tag{53}$$

The ground state satisfies

$$P_x|0\rangle = P_y|0\rangle = 0. \tag{54}$$

Since we are not interested in the simple transverse-translational modes, we work in the zero transverse momentum sector and consider only the eigenstates

$$\prod_{m\neq 0} \frac{(a_{mR}^\dagger)^{n_{mR}}}{\sqrt{n_{mR}!}}\ \frac{(a_{mL}^\dagger)^{n_{mL}}}{\sqrt{n_{mL}!}}\ |0\rangle. \tag{55}$$

Under a rotation about the z-axis, the ladder operators transform as

$$R_z(\phi)\ a_{mR}^\dagger\ R_z^\dagger(\phi) \ = \ e^{-i\phi}\ a_{mR}^\dagger, \tag{56}$$

$$R_z(\phi)\ a_{mL}^\dagger\ R_z^\dagger(\phi) \ = \ e^{i\phi}\ a_{mL}^\dagger. \tag{57}$$

Next, consider how these operators transform under translations along the longitudinal z-direction. Let $T_z(b)$ denote the translation in the z-direction by length b. We require that

$$T_z(b)\ q_j(z,t)\ T_z^\dagger(b) = q_j(z+b,t). \tag{58}$$

From Eq. 44, we see that this means

$$T_z(b)\ a_{mR}^\dagger\ T_z^\dagger(b) \ = \ e^{-ibk_m}\ a_{mR}^\dagger, \tag{59}$$

$$T_z(b)\ a_{mL}^\dagger\ T_z^\dagger(b) \ = \ e^{-ibk_m}\ a_{mL}^\dagger. \tag{60}$$

Let P_z denote the generator of such longitudinal translations $T_z(b) = \exp(-ibP_z)$, then

$$P_z\ a_{mR}^\dagger\ P_z^\dagger \ = \ k_m\ a_{mR}^\dagger, \tag{61}$$

$$P_z\ a_{mL}^\dagger\ P_z^\dagger \ = \ k_m\ a_{mL}^\dagger, \tag{62}$$

which shows that each phonon mode has longitudinal momentum k_m. Let $\mathcal{P}_{L/2}$ denote spatial inversion about the point $(0, 0, L/2)$ and C denote charge conjugation. The flux in the effective QCD string has a direction associated with it, so that charge conjugation simply effects a reversal of this direction. This direction is also reversed under $\mathcal{P}_{L/2}$ so that $C\mathcal{P}_{L/2}$ is a possible symmetry of the system. Also, let σ_x and σ_y denote reflections in the xz and yz planes, respectively. The ground state satisfies

$$R_z(\phi)\ |0\rangle = |0\rangle, \tag{63}$$
$$P_z\ |0\rangle = 0, \tag{64}$$
$$C\mathcal{P}_{L/2}\ |0\rangle = |0\rangle, \tag{65}$$
$$\sigma_x\ |0\rangle = |0\rangle, \tag{66}$$
$$\sigma_y\ |0\rangle = |0\rangle. \tag{67}$$

To determine the behavior of the operators $a^\dagger_{mR}$ and $a^\dagger_{mR}$ under these symmetry operations, one uses Eq. 44 and the following transformation properties of the string coordinates:

$$C\mathcal{P}_{L/2}\ q_j(z,t)\ \mathcal{P}^\dagger_{L/2}C^\dagger = -q_j(L-z,t), \tag{68}$$
$$\sigma_x\ q_x(z,t)\ \sigma_x^\dagger = q_x(z,t), \tag{69}$$
$$\sigma_y\ q_x(z,t)\ \sigma_y^\dagger = -q_x(z,t), \tag{70}$$
$$\sigma_x\ q_y(z,t)\ \sigma_x^\dagger = -q_y(z,t), \tag{71}$$
$$\sigma_y\ q_y(z,t)\ \sigma_y^\dagger = q_y(z,t). \tag{72}$$

Furthermore, we know that

$$C\mathcal{P}_{L/2}\ Q_j\ \mathcal{P}^\dagger_{L/2}C^\dagger = -Q_j, \tag{73}$$
$$C\mathcal{P}_{L/2}\ P_j\ \mathcal{P}^\dagger_{L/2}C^\dagger = -P_j, \tag{74}$$
$$C\mathcal{P}_{L/2}\ P_z\ \mathcal{P}^\dagger_{L/2}C^\dagger = -P_z. \tag{75}$$

Using Eq. 44 and Eq. 17 with the above transformation properties, one easily determines

$$C\mathcal{P}_{L/2}\ a^\dagger_{mR}\ \mathcal{P}^\dagger_{L/2}C^\dagger = -a^\dagger_{-mR}, \tag{76}$$
$$C\mathcal{P}_{L/2}\ a^\dagger_{mL}\ \mathcal{P}^\dagger_{L/2}C^\dagger = -a^\dagger_{-mL}, \tag{77}$$
$$\sigma_x\ a^\dagger_{mR}\ \sigma_x^\dagger = a^\dagger_{mL}, \tag{78}$$
$$\sigma_x\ a^\dagger_{mL}\ \sigma_x^\dagger = a^\dagger_{mR}, \tag{79}$$
$$\sigma_y\ a^\dagger_{mR}\ \sigma_y^\dagger = -a^\dagger_{mL}, \tag{80}$$
$$\sigma_y\ a^\dagger_{mL}\ \sigma_y^\dagger = -a^\dagger_{mR}. \tag{81}$$

The symmetries of the system are, thus, as follows. For states with zero total longitudinal momentum, the symmetries are exactly the same as for the fixed end case. We denote these levels using $\Sigma_g^+(0)$, $\Sigma_g^-(0)$, $\Sigma_u^+(0)$, $\Sigma_u^-(0)$, $\Pi_g(0)$, $\Pi_u(0)$, $\Delta_g(0)$, $\Delta_u(0)$, and so on, where the zero in parentheses indicates that these levels correspond to states having zero longitudinal momentum. For non-zero longitudinal momentum, $C\mathcal{P}_{L/2}$ is no longer a symmetry since it reverses the longitudinal momentum. Hence, these levels may be labeled $\Sigma^+(p)$, $\Sigma^-(p)$, $\Pi(p)$, $\Delta(p)$, and so on. Here, $p = \pm 1, \pm 2, \pm 3, \ldots$ and corresponds to longitudinal momentum $2\pi p/L$. Note that the energy is independent of the sign (direction) of the longitudinal momentum. Also, the σ_x and σ_y symmetries produce Λ-doubling again, except for the $\Lambda = 0$ states, which still require the $\pm$ superscript.

Hence, if E_0 denotes the energy of the ground state (with the above Hamiltonian, it has been defined to be zero), then the eigenvalues E (energy), longitudinal momentum k_z, Λ, and η_{CP} (for the $k_z = 0$ states) associated with the string eigenmodes are given by

$$E = E_0 + \frac{2N\pi}{L}\sqrt{\frac{\kappa}{\rho}}, \quad (82)$$

$$k_z = \frac{2M\pi}{L}, \quad (83)$$

$$N = \sum_{m\neq 0} |m| \; (n_{mR} + n_{mL}), \quad (84)$$

$$M = \sum_{m\neq 0} m \; (n_{mR} + n_{mL}), \quad (85)$$

$$\Lambda = \left| \sum_{m\neq 0} (n_{mR} - n_{mL}) \right|. \quad (86)$$

For zero-momentum states, we make even and odd $C\mathcal{P}_{L/2}$ states using symmetric and antisymmetric superpositions, respectively, under $a^\dagger_{mR} \to -a^\dagger_{-mR}$ and $a^\dagger_{mL} \to -a^\dagger_{-mL}$. For Σ states, we make Σ^+ and Σ^- states using symmetric and antisymmetric superpositions, respectively, under interchange of right and left-handed modes.

Using these properties, the orderings and degeneracies of the Goldstone string energy levels and their symmetries are as shown in Tables 2–6. Hence, for $\kappa = \rho$, the $2N\pi/L$ behavior and a well-defined pattern of degeneracies and level orderings among the different channels form a very distinctive signature of the onset of the Goldstone modes for the effective QCD string.

Table 2: Low-lying torelon string levels. Note that R and L refer to right and left circular polarizations, respectively, and the signed integers refer to the phonon mode. A positive integer indicates a mode with longitudinal momentum in the positive z-direction, whereas a negative integer indicates a mode with oppositely directed longitudinal momentum. The Σ, Π, Δ, Φ, and Γ levels have $\Lambda = 0, 1, 2, 3$, and 4, respectively, where Λ is the magnitude of the z-projection of angular momentum. The total longitudinal momentum of each level, in terms of the fundamental quantum $2\pi/L$, is indicated in parentheses. For the states having zero longitudinal momentum, the levels which are even and odd under $C\mathcal{P}_{L/2}$ are indicated by subscripts g and u, respectively. The Σ^+ and Σ^- states are even and odd, respectively, under reflections in any plane containing the z-axis. Note that the $N = 1$ level is 4-fold degenerate and the $N = 2$ level is 14-fold degenerate.

$N = 0$:	$\Sigma_g^+(0)$	$\|0\rangle$	
$N = 1$:	$\Pi(1)$	$a^\dagger_{+1R}\|0\rangle$	$a^\dagger_{+1L}\|0\rangle$
	$\Pi(-1)$	$a^\dagger_{-1R}\|0\rangle$	$a^\dagger_{-1L}\|0\rangle$
$N = 2$:	$\Sigma_g^+(0)'$	$(a^\dagger_{+1R}a^\dagger_{-1L} + a^\dagger_{-1R}a^\dagger_{+1L})\|0\rangle$	
	$\Sigma_u^-(0)$	$(a^\dagger_{+1R}a^\dagger_{-1L} - a^\dagger_{-1R}a^\dagger_{+1L})\|0\rangle$	
	$\Delta_g(0)$	$a^\dagger_{+1R}a^\dagger_{-1R}\|0\rangle$	$a^\dagger_{+1L}a^\dagger_{-1L}\|0\rangle$
	$\Sigma^+(2)$	$a^\dagger_{+1R}a^\dagger_{+1L}\|0\rangle$	
	$\Sigma^+(-2)$	$a^\dagger_{-1R}a^\dagger_{-1L}\|0\rangle$	
	$\Pi(2)$	$a^\dagger_{+2R}\|0\rangle$	$a^\dagger_{+2L}\|0\rangle$
	$\Pi(-2)$	$a^\dagger_{-2R}\|0\rangle$	$a^\dagger_{-2L}\|0\rangle$
	$\Delta(2)$	$(a^\dagger_{+1R})^2\|0\rangle$	$(a^\dagger_{+1L})^2\|0\rangle$
	$\Delta(-2)$	$(a^\dagger_{-1R})^2\|0\rangle$	$(a^\dagger_{-1L})^2\|0\rangle$

Table 3: The $N = 3$ torelon string levels. See Table 2 for a description of the notation used. The $N = 3$ level is 40-fold degenerate.

$N = 3$:	$\Sigma^+(1)$	$(a^\dagger_{-1L}a^\dagger_{+2R} + a^\dagger_{-1R}a^\dagger_{+2L})\vert 0\rangle$	
	$\Sigma^+(-1)$	$(a^\dagger_{+1R}a^\dagger_{-2L} + a^\dagger_{+1L}a^\dagger_{-2R})\vert 0\rangle$	
	$\Sigma^-(1)$	$(a^\dagger_{-1L}a^\dagger_{+2R} - a^\dagger_{-1R}a^\dagger_{+2L})\vert 0\rangle$	
	$\Sigma^-(-1)$	$(a^\dagger_{+1R}a^\dagger_{-2L} - a^\dagger_{+1L}a^\dagger_{-2R})\vert 0\rangle$	
	$\Pi(1)'$	$(a^\dagger_{+1R})^2 a^\dagger_{-1L}\vert 0\rangle$	$a^\dagger_{-1R}(a^\dagger_{+1L})^2\vert 0\rangle$
	$\Pi(1)'$	$a^\dagger_{+1R}a^\dagger_{-1R}a^\dagger_{+1L}\vert 0\rangle$	$a^\dagger_{+1R}a^\dagger_{+1L}a^\dagger_{-1L}\vert 0\rangle$
	$\Pi(-1)'$	$(a^\dagger_{-1R})^2 a^\dagger_{+1L}\vert 0\rangle$	$a^\dagger_{+1R}(a^\dagger_{-1L})^2\vert 0\rangle$
	$\Pi(-1)'$	$a^\dagger_{+1R}a^\dagger_{-1R}a^\dagger_{-1L}\vert 0\rangle$	$a^\dagger_{-1R}a^\dagger_{+1L}a^\dagger_{-1L}\vert 0\rangle$
	$\Delta(1)$	$a^\dagger_{-1R}a^\dagger_{+2R}\vert 0\rangle$	$a^\dagger_{-1L}a^\dagger_{+2L}\vert 0\rangle$
	$\Delta(-1)$	$a^\dagger_{+1R}a^\dagger_{-2R}\vert 0\rangle$	$a^\dagger_{+1L}a^\dagger_{-2L}\vert 0\rangle$
	$\Phi(1)$	$(a^\dagger_{+1R})^2 a^\dagger_{-1R}\vert 0\rangle$	$(a^\dagger_{+1L})^2 a^\dagger_{-1L}\vert 0\rangle$
	$\Phi(-1)$	$a^\dagger_{+1R}(a^\dagger_{-1R})^2\vert 0\rangle$	$a^\dagger_{+1L}(a^\dagger_{-1L})^2\vert 0\rangle$
	$\Sigma^+(3)$	$(a^\dagger_{+1R}a^\dagger_{+2L} + a^\dagger_{+1L}a^\dagger_{+2R})\vert 0\rangle$	
	$\Sigma^+(-3)$	$(a^\dagger_{-1L}a^\dagger_{-2R} + a^\dagger_{-1R}a^\dagger_{-2L})\vert 0\rangle$	
	$\Sigma^-(3)$	$(a^\dagger_{+1R}a^\dagger_{+2L} - a^\dagger_{+1L}a^\dagger_{+2R})\vert 0\rangle$	
	$\Sigma^-(-3)$	$(a^\dagger_{-1L}a^\dagger_{-2R} - a^\dagger_{-1R}a^\dagger_{-2L})\vert 0\rangle$	
	$\Pi(3)$	$a^\dagger_{+3R}\vert 0\rangle$	$a^\dagger_{+3L}\vert 0\rangle$
	$\Pi(3)$	$(a^\dagger_{+1R})^2 a^\dagger_{+1L}\vert 0\rangle$	$a^\dagger_{+1R}(a^\dagger_{+1L})^2\vert 0\rangle$
	$\Pi(-3)$	$(a^\dagger_{-1R})^2 a^\dagger_{-1L}\vert 0\rangle$	$a^\dagger_{-1R}(a^\dagger_{-1L})^2\vert 0\rangle$
	$\Pi(-3)$	$a^\dagger_{-3R}\vert 0\rangle$	$a^\dagger_{-3L}\vert 0\rangle$
	$\Delta(3)$	$a^\dagger_{+1R}a^\dagger_{+2R}\vert 0\rangle$	$a^\dagger_{+1L}a^\dagger_{+2L}\vert 0\rangle$
	$\Delta(-3)$	$a^\dagger_{-1R}a^\dagger_{-2R}\vert 0\rangle$	$a^\dagger_{-1L}a^\dagger_{-2L}\vert 0\rangle$
	$\Phi(3)$	$(a^\dagger_{+1R})^3\vert 0\rangle$	$(a^\dagger_{+1L})^3\vert 0\rangle$
	$\Phi(-3)$	$(a^\dagger_{-1R})^3\vert 0\rangle$	$(a^\dagger_{-1L})^3\vert 0\rangle$

Table 4: The $N = 4$ torelon string levels which have zero longitudinal momentum. See Table 2 for a description of the notation used. The $N = 4$ level is 105-fold degenerate.

$N=4$:	$\Sigma_g^+(0)''$	$(a^\dagger_{-2R}a^\dagger_{+2L} + a^\dagger_{+2R}a^\dagger_{-2L})\|0\rangle$	
	$\Sigma_g^+(0)''$	$(a^\dagger_{+1R}a^\dagger_{-1R}a^\dagger_{+1L}a^\dagger_{-1L})\|0\rangle$	
	$\Sigma_g^+(0)''$	$((a^\dagger_{+1R})^2(a^\dagger_{-1L})^2 + (a^\dagger_{-1R})^2(a^\dagger_{+1L})^2)\|0\rangle$	
	$\Sigma_u^-(0)'$	$(a^\dagger_{-2R}a^\dagger_{+2L} - a^\dagger_{+2R}a^\dagger_{-2L})\|0\rangle$	
	$\Sigma_u^-(0)'$	$((a^\dagger_{+1R})^2(a^\dagger_{-1L})^2 - (a^\dagger_{-1R})^2(a^\dagger_{+1L})^2)\|0\rangle$	
	$\Pi_g(0)$	$((a^\dagger_{+1R})^2a^\dagger_{-2L} - (a^\dagger_{-1R})^2a^\dagger_{+2L})\|0\rangle$	
	$\Pi_g(0)$	$((a^\dagger_{-1L})^2a^\dagger_{+2R} - (a^\dagger_{+1L})^2a^\dagger_{-2R})\|0\rangle$	
	$\Pi_g(0)$	$(a^\dagger_{+1R}a^\dagger_{+1L}a^\dagger_{-2L} - a^\dagger_{-1R}a^\dagger_{-1L}a^\dagger_{+2L})\|0\rangle$	
	$\Pi_g(0)$	$(a^\dagger_{+1R}a^\dagger_{+1L}a^\dagger_{-2R} - a^\dagger_{-1R}a^\dagger_{-1L}a^\dagger_{+2R})\|0\rangle$	
	$\Pi_u(0)$	$((a^\dagger_{+1R})^2a^\dagger_{-2L} + (a^\dagger_{-1R})^2a^\dagger_{+2L})\|0\rangle$	
	$\Pi_u(0)$	$((a^\dagger_{-1L})^2a^\dagger_{+2R} + (a^\dagger_{+1L})^2a^\dagger_{-2R})\|0\rangle$	
	$\Pi_u(0)$	$(a^\dagger_{+1R}a^\dagger_{+1L}a^\dagger_{-2L} + a^\dagger_{-1R}a^\dagger_{-1L}a^\dagger_{+2L})\|0\rangle$	
	$\Pi_u(0)$	$(a^\dagger_{+1R}a^\dagger_{+1L}a^\dagger_{-2R} + a^\dagger_{-1R}a^\dagger_{-1L}a^\dagger_{+2R})\|0\rangle$	
	$\Delta_g(0)'$	$(a^\dagger_{-1R}(a^\dagger_{+1L})^2a^\dagger_{-1L} + a^\dagger_{+1R}a^\dagger_{+1L}(a^\dagger_{-1L})^2)\|0\rangle$	
	$\Delta_g(0)'$	$a^\dagger_{+2R}a^\dagger_{-2R}\|0\rangle$	$a^\dagger_{+2L}a^\dagger_{-2L}\|0\rangle$
	$\Delta_g(0)'$	$((a^\dagger_{+1R})^2a^\dagger_{-1R}a^\dagger_{-1L} + a^\dagger_{+1R}(a^\dagger_{-1R})^2a^\dagger_{+1L})\|0\rangle$	
	$\Delta_u(0)$	$((a^\dagger_{+1R})^2a^\dagger_{-1R}a^\dagger_{-1L} - a^\dagger_{+1R}(a^\dagger_{-1R})^2a^\dagger_{+1L})\|0\rangle$	
	$\Delta_u(0)$	$(a^\dagger_{-1R}(a^\dagger_{+1L})^2a^\dagger_{-1L} - a^\dagger_{+1R}a^\dagger_{+1L}(a^\dagger_{-1L})^2)\|0\rangle$	
	$\Phi_g(0)$	$((a^\dagger_{-1L})^2a^\dagger_{+2L} - (a^\dagger_{+1L})^2a^\dagger_{-2L})\|0\rangle$	
	$\Phi_g(0)$	$((a^\dagger_{+1R})^2a^\dagger_{-2R} - (a^\dagger_{-1R})^2a^\dagger_{+2R})\|0\rangle$	
	$\Phi_u(0)$	$((a^\dagger_{-1L})^2a^\dagger_{+2L} + (a^\dagger_{+1L})^2a^\dagger_{-2L})\|0\rangle$	
	$\Phi_u(0)$	$((a^\dagger_{+1R})^2a^\dagger_{-2R} + (a^\dagger_{-1R})^2a^\dagger_{+2R})\|0\rangle$	
	$\Gamma_g(0)$	$(a^\dagger_{+1R})^2(a^\dagger_{-1R})^2\|0\rangle$	$(a^\dagger_{+1L})^2(a^\dagger_{-1L})^2\|0\rangle$

Table 5: The $N = 4$ torelon string levels which have two quanta of longitudinal momentum. See Table 2 for a description of the notation used. The $N = 4$ level is 105-fold degenerate.

$N = 4$:	$\Sigma^+(2)'$	$(a^\dagger_{-1L}a^\dagger_{+3R} + a^\dagger_{-1R}a^\dagger_{+3L})\vert 0\rangle$	
	$\Sigma^+(2)'$	$((a^\dagger_{+1R})^2a^\dagger_{+1L}a^\dagger_{-1L} + a^\dagger_{+1R}a^\dagger_{-1R}(a^\dagger_{+1L})^2)\vert 0\rangle$	
	$\Sigma^+(-2)'$	$(a^\dagger_{+1R}a^\dagger_{-3L} + a^\dagger_{+1L}a^\dagger_{-3R})\vert 0\rangle$	
	$\Sigma^+(-2)'$	$((a^\dagger_{-1R})^2a^\dagger_{+1L}a^\dagger_{-1L} + a^\dagger_{+1R}a^\dagger_{-1R}(a^\dagger_{-1L})^2)\vert 0\rangle$	
	$\Sigma^-(2)$	$(a^\dagger_{-1L}a^\dagger_{+3R} - a^\dagger_{-1R}a^\dagger_{+3L})\vert 0\rangle$	
	$\Sigma^-(2)$	$((a^\dagger_{+1R})^2a^\dagger_{+1L}a^\dagger_{-1L} - a^\dagger_{+1R}a^\dagger_{-1R}(a^\dagger_{+1L})^2)\vert 0\rangle$	
	$\Sigma^-(-2)$	$(a^\dagger_{+1R}a^\dagger_{-3L} - a^\dagger_{+1L}a^\dagger_{-3R})\vert 0\rangle$	
	$\Sigma^-(-2)$	$((a^\dagger_{-1R})^2a^\dagger_{+1L}a^\dagger_{-1L} - a^\dagger_{+1R}a^\dagger_{-1R}(a^\dagger_{-1L})^2)\vert 0\rangle$	
	$\Pi(2)'$	$a^\dagger_{+1R}a^\dagger_{-1R}a^\dagger_{+2L}\vert 0\rangle$	$a^\dagger_{+1L}a^\dagger_{-1L}a^\dagger_{+2R}\vert 0\rangle$
	$\Pi(2)'$	$a^\dagger_{+1R}a^\dagger_{-1L}a^\dagger_{+2R}\vert 0\rangle$	$a^\dagger_{-1R}a^\dagger_{+1L}a^\dagger_{+2L}\vert 0\rangle$
	$\Pi(2)'$	$a^\dagger_{-1R}a^\dagger_{+1L}a^\dagger_{+2R}\vert 0\rangle$	$a^\dagger_{+1R}a^\dagger_{-1L}a^\dagger_{+2L}\vert 0\rangle$
	$\Pi(-2)'$	$a^\dagger_{+1R}a^\dagger_{-1R}a^\dagger_{-2L}\vert 0\rangle$	$a^\dagger_{+1L}a^\dagger_{-1L}a^\dagger_{-2R}\vert 0\rangle$
	$\Pi(-2)'$	$a^\dagger_{+1R}a^\dagger_{-1L}a^\dagger_{-2R}\vert 0\rangle$	$a^\dagger_{-1R}a^\dagger_{+1L}a^\dagger_{-2L}\vert 0\rangle$
	$\Pi(-2)'$	$a^\dagger_{-1R}a^\dagger_{+1L}a^\dagger_{-2R}\vert 0\rangle$	$a^\dagger_{+1R}a^\dagger_{-1L}a^\dagger_{-2L}\vert 0\rangle$
	$\Delta(2)'$	$(a^\dagger_{+1R})^2a^\dagger_{-1R}a^\dagger_{+1L}\vert 0\rangle$	$a^\dagger_{+1R}(a^\dagger_{+1L})^2a^\dagger_{-1L}\vert 0\rangle$
	$\Delta(2)'$	$a^\dagger_{-1R}a^\dagger_{+3R}\vert 0\rangle$	$a^\dagger_{-1L}a^\dagger_{+3L}\vert 0\rangle$
	$\Delta(2)'$	$(a^\dagger_{+1R})^3a^\dagger_{-1L}\vert 0\rangle$	$a^\dagger_{-1R}(a^\dagger_{+1L})^3\vert 0\rangle$
	$\Delta(-2)'$	$a^\dagger_{+1R}(a^\dagger_{-1R})^2a^\dagger_{-1L}\vert 0\rangle$	$a^\dagger_{-1R}a^\dagger_{+1L}(a^\dagger_{-1L})^2\vert 0\rangle$
	$\Delta(-2)'$	$a^\dagger_{+1R}a^\dagger_{-3R}\vert 0\rangle$	$a^\dagger_{+1L}a^\dagger_{-3L}\vert 0\rangle$
	$\Delta(-2)'$	$(a^\dagger_{-1R})^3a^\dagger_{+1L}\vert 0\rangle$	$a^\dagger_{+1R}(a^\dagger_{-1L})^3\vert 0\rangle$
	$\Phi(2)$	$a^\dagger_{+1R}a^\dagger_{-1R}a^\dagger_{+2R}\vert 0\rangle$	$a^\dagger_{+1L}a^\dagger_{-1L}a^\dagger_{+2L}\vert 0\rangle$
	$\Phi(-2)$	$a^\dagger_{+1R}a^\dagger_{-1R}a^\dagger_{-2R}\vert 0\rangle$	$a^\dagger_{+1L}a^\dagger_{-1L}a^\dagger_{-2L}\vert 0\rangle$
	$\Gamma(2)$	$(a^\dagger_{+1R})^3a^\dagger_{-1R}\vert 0\rangle$	$(a^\dagger_{+1L})^3a^\dagger_{-1L}\vert 0\rangle$
	$\Gamma(-2)$	$a^\dagger_{+1R}(a^\dagger_{-1R})^3\vert 0\rangle$	$a^\dagger_{+1L}(a^\dagger_{-1L})^3\vert 0\rangle$

Table 6: The $N = 4$ torelon string levels which have four quanta of longitudinal momentum. See Table 2 for a description of the notation used. The $N = 4$ level is 105-fold degenerate.

$N = 4$:	$\Sigma^+(4)$	$a^\dagger_{+2R}a^\dagger_{+2L}\|0\rangle$	
	$\Sigma^+(4)$	$(a^\dagger_{+1R}a^\dagger_{+3L} + a^\dagger_{+1L}a^\dagger_{+3R})\|0\rangle$	
	$\Sigma^+(4)$	$(a^\dagger_{+1R})^2(a^\dagger_{+1L})^2\|0\rangle$	
	$\Sigma^+(-4)$	$(a^\dagger_{-1R})^2(a^\dagger_{-1L})^2\|0\rangle$	
	$\Sigma^+(-4)$	$(a^\dagger_{-1L}a^\dagger_{-3R} + a^\dagger_{-1R}a^\dagger_{-3L})\|0\rangle$	
	$\Sigma^+(-4)$	$a^\dagger_{-2R}a^\dagger_{-2L}\|0\rangle$	
	$\Sigma^-(4)$	$(a^\dagger_{+1R}a^\dagger_{+3L} - a^\dagger_{+1L}a^\dagger_{+3R})\|0\rangle$	
	$\Sigma^-(-4)$	$(a^\dagger_{-1L}a^\dagger_{-3R} - a^\dagger_{-1R}a^\dagger_{-3L})\|0\rangle$	
	$\Pi(4)$	$(a^\dagger_{+1R})^2a^\dagger_{+2L}\|0\rangle$	$(a^\dagger_{+1L})^2a^\dagger_{+2R}\|0\rangle$
	$\Pi(4)$	$a^\dagger_{+4R}\|0\rangle$	$a^\dagger_{+4L}\|0\rangle$
	$\Pi(4)$	$a^\dagger_{+1R}a^\dagger_{+1L}a^\dagger_{+2R}\|0\rangle$	$a^\dagger_{+1R}a^\dagger_{+1L}a^\dagger_{+2L}\|0\rangle$
	$\Pi(-4)$	$a^\dagger_{-4R}\|0\rangle$	$a^\dagger_{-4L}\|0\rangle$
	$\Pi(-4)$	$(a^\dagger_{-1R})^2a^\dagger_{-2L}\|0\rangle$	$(a^\dagger_{-1L})^2a^\dagger_{-2R}\|0\rangle$
	$\Pi(-4)$	$a^\dagger_{-1R}a^\dagger_{-1L}a^\dagger_{-2R}\|0\rangle$	$a^\dagger_{-1R}a^\dagger_{-1L}a^\dagger_{-2L}\|0\rangle$
	$\Delta(4)$	$(a^\dagger_{+2R})^2\|0\rangle$	$(a^\dagger_{+2L})^2\|0\rangle$
	$\Delta(4)$	$(a^\dagger_{+1R})^3a^\dagger_{+1L}\|0\rangle$	$a^\dagger_{+1R}(a^\dagger_{+1L})^3\|0\rangle$
	$\Delta(4)$	$a^\dagger_{+1R}a^\dagger_{+3R}\|0\rangle$	$a^\dagger_{+1L}a^\dagger_{+3L}\|0\rangle$
	$\Delta(-4)$	$(a^\dagger_{-2R})^2\|0\rangle$	$(a^\dagger_{-2L})^2\|0\rangle$
	$\Delta(-4)$	$a^\dagger_{-1R}a^\dagger_{-3R}\|0\rangle$	$a^\dagger_{-1L}a^\dagger_{-3L}\|0\rangle$
	$\Delta(-4)$	$(a^\dagger_{-1R})^3a^\dagger_{-1L}\|0\rangle$	$a^\dagger_{-1R}(a^\dagger_{-1L})^3\|0\rangle$
	$\Phi(4)$	$(a^\dagger_{+1R})^2a^\dagger_{+2R}\|0\rangle$	$(a^\dagger_{+1L})^2a^\dagger_{+2L}\|0\rangle$
	$\Phi(-4)$	$(a^\dagger_{-1R})^2a^\dagger_{-2R}\|0\rangle$	$(a^\dagger_{-1L})^2a^\dagger_{-2L}\|0\rangle$
	$\Gamma(4)$	$(a^\dagger_{+1R})^4\|0\rangle$	$(a^\dagger_{+1L})^4\|0\rangle$
	$\Gamma(-4)$	$(a^\dagger_{-1R})^4\|0\rangle$	$(a^\dagger_{-1L})^4\|0\rangle$