Recent highlights with baryons from lattice QCD

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Recent highlights involving baryons in lattice QCD:
- Proton mass decomposition
- Nucleon spin decomposition
- Percent level determination of nucleon axial coupling
- Proton and neutron electromagnetic form factors
- Parton distribution function
- Scattering amplitudes
- Baryon-baryon interactions with HAL QCD method
- $H$-dibaryon warm up

Key progress:
- Achieving much better precision with disconnected diagrams
- Ability to include multi-hadron operators
- More and more studies being done at physical point
Temporal correlations from path integrals

- Stationary-state energies from $N \times N$ Hermitian correlation matrix

$$C_{ij}(t) = \langle 0 | O_i(t + t_0) \bar{O}_j(t_0) | 0 \rangle$$

- Judiciously designed operators $\bar{O}_j$ create states of interest

$$O_j(t) = O_j[\bar{\psi}(t), \psi(t), U(t)]$$

- Correlators from path integrals over quark $\bar{\psi}, \psi$ and gluon $U$ fields

$$C_{ij}(t) = \frac{\int \mathcal{D}(\bar{\psi}, \psi, U) \ O_i(t + t_0) \bar{O}_j(t_0) \ exp \left(-S[\bar{\psi}, \psi, U]\right)}{\int \mathcal{D}(\bar{\psi}, \psi, U) \ exp \left(-S[\bar{\psi}, \psi, U]\right)}$$

- Involves the action in imaginary time

$$S[\bar{\psi}, \psi, U] = \bar{\psi} K[U] \ \psi + S_G[U]$$

- $K[U]$ is fermion Dirac matrix

- $S_G[U]$ is gluon action
Integrating the quark fields

- integrals over Grassmann-valued quark fields done exactly
- meson-to-meson example:
  \[
  \int \mathcal{D}(\bar{\psi}, \psi) \, \psi_a \psi_b \, \bar{\psi}_c \bar{\psi}_d \, \exp (-\bar{\psi}K\psi) = \left(K_{ad}^{-1} K_{bc}^{-1} - K_{ac}^{-1} K_{bd}^{-1}\right) \det K.
  \]
- baryon-to-baryon example:
  \[
  \int \mathcal{D}(\bar{\psi}, \psi) \, \psi_{a_1} \psi_{a_2} \psi_{a_3} \, \bar{\psi}_{b_1} \bar{\psi}_{b_2} \bar{\psi}_{b_3} \, \exp (-\bar{\psi}K\psi) = \left(-K_{a_1b_1}^{-1} K_{a_2b_2}^{-1} K_{a_3b_3}^{-1} + K_{a_1b_1}^{-1} K_{a_2b_3}^{-1} K_{a_3b_2}^{-1} + K_{a_1b_2}^{-1} K_{a_2b_1}^{-1} K_{a_3b_3}^{-1} \right) \det K
  \]
Monte Carlo integration

- Correlators have form

\[ C_{ij}(t) = \frac{\int DU \ det K[U] \ K^{-1}[U] \cdots K^{-1}[U] \ \exp(-S_G[U])}{\int DU \ det K[U] \ \exp(-S_G[U])} \]

- Resort to **Monte Carlo method** to integrate over gluon fields
- Use Markov chain to generate sequence of gauge-field configurations

\[ U_1, U_2, \ldots, U_N \]

- Most computationally demanding parts:
  - Including \( \det K \) in updating
  - Evaluating \( K^{-1} \) in numerator
Monte Carlo method using computers requires formulating integral on space-time lattice (usually hypercubic)

- **quarks** reside on sites, **gluons** reside on links between sites
- integrate over gluon fields on each link

- Metropolis method with global updating proposal
  - RHMC: solve Hamilton equations with Gaussian momenta
- \( \det K \) estimates with integral over pseudo-fermion fields
- systematic errors
  - discretization
  - finite volume
  - unphysical quark masses
Building blocks for single-hadron operators

- Building blocks: covariantly-displaced LapH-smeared quark fields
- Stout links $\tilde{U}_j(x)$
- Laplacian-Heaviside (LapH) smeared quark fields
  \[ \tilde{\psi}_{a\alpha}(x) = S_{ab}(x, y) \psi_{b\alpha}(y), \quad S = \Theta \left( \sigma_s^2 + \tilde{\Delta} \right) \]
- 3d gauge-covariant Laplacian $\tilde{\Delta}$ in terms of $\tilde{U}$
- Displaced quark fields:
  \[ q^A_{a\alpha j} = D^{(j)} \tilde{\psi}^{(A)}_{a\alpha}, \quad \bar{q}^A_{a\alpha j} = \bar{\tilde{\psi}}^{(A)}_{a\alpha} \gamma_4 D^{(j)\dagger} \]
- Displacement $D^{(j)}$ is product of smeared links:
  \[ D^{(j)}(x, x') = \tilde{U}_{j_1}(x) \tilde{U}_{j_2}(x+d_2) \tilde{U}_{j_3}(x+d_3) \ldots \tilde{U}_{j_p}(x+d_p) \delta_{x', x+d_p+1} \]
- To good approximation, LapH smearing operator is
  \[ S = V_s V_s^\dagger \]
  - Columns of matrix $V_s$ are eigenvectors of $\tilde{\Delta}$
Extended operators for single hadrons

- quark displacements build up orbital, radial structure

Meson configurations

\[
\Phi_{AB}^{\alpha\beta}(p, t) = \sum_x e^{ip \cdot (x + \frac{1}{2} (d\alpha + d\beta))} \delta_{ab} \overline{q}_b^B(x, t) q_a^A(x, t)
\]

\[
\Phi_{ABC}^{\alpha\beta\gamma}(p, t) = \sum_x e^{ip \cdot x} \varepsilon_{abc} \overline{q}_c^C(x, t) \overline{q}_b^B(x, t) \overline{q}_a^A(x, t)
\]

- group-theory projections onto irreps of lattice symmetry group

\[
\overline{M}_l(t) = c^{(l)*}_{\alpha\beta} \Phi_{AB}^{\alpha\beta}(t) \quad \overline{B}_l(t) = c^{(l)*}_{\alpha\beta\gamma} \Phi_{ABC}^{\alpha\beta\gamma}(t)
\]

- definite momentum \( p \), irreps of little group of \( p \)
Stable hadron mass success

- low-lying mass spectrum successfully determined
- level of precision: isospin breaking now relevant

[Image of a graph showing hadron masses with various labels such as \( H, H^*, H_s, H_s^*, B_c, B_c^* \) and others, along the x-axis and y-axis labeled as (MeV).]


- challenge: scattering amplitudes and resonances
Matrix elements from lattice QCD

- standard method for matrix element calculations requires 3-point functions

- excited-state contamination removed by taking $t_{\text{sep}}$, $t_{\text{ins}}$, and $t_{\text{sep}} - t_{\text{ins}}$ large

- in practice, difficult to achieve due to signal-to-noise

- current requires renormalization for comparison to $\overline{\text{MS}}$

- nonperturbative/perturbative methods
Proton mass decomposition

- recent determination of mass decomposition of proton
  [Y. Yang, J. Liang, Y. Bi, Y. Chen, T. Draper, K. F. Liu, Z. Liu, PRL 121, 212001 (2018)]

- rest mass $M$ of proton given by [Ji PRL 74, 1071 (1995)]

\[ M = -\langle T_{44} \rangle = \langle H_m \rangle + \langle H_E \rangle(\mu) + \langle H_g \rangle(\mu) + \frac{1}{4} \langle H_a \rangle, \]

- $\langle T_{\mu\nu} \rangle$ expectation value of energy momentum tensor in hadron quark condensate $H_m = \sum_{u,d,s} \ldots \int d^3x m \overline{\psi}\psi$

- quark energy $H_E = \sum_{u,d,s} \ldots \int d^3x \overline{\psi}(\vec{D} \cdot \vec{\gamma})\psi$

- glue field energy $H_g = \int d^3x \frac{1}{2}(B^2 - E^2)$

- anomaly term $H_a = \sum_{u,d,s} \ldots \int d^3x \gamma_m m \overline{\psi}\psi - \int d^3x \frac{\beta(g)}{g}(E^2 + B^2)$

- $\langle H_m \rangle, \langle H_a \rangle, \langle H_E + H_g \rangle$ scale and scheme independent

- obtain from renormalized quark and gluon momentum fractions $\langle H_g \rangle = \frac{3}{4} M \langle x \rangle_g$ and $\langle H_E \rangle = \frac{3}{4} M \langle x \rangle_q - \frac{3}{4} \langle H_m \rangle$

- anomaly term from $\langle H_a \rangle = M - \langle H_m \rangle$
determined mass $M$ from two-point correlator
used previous determination of $\langle H_m \rangle$ (2016)
momentum fractions from
\[
\langle x \rangle_{q,g} \equiv -\frac{\langle N | \frac{4}{3} T_{44}^{q,g} | N \rangle}{M \langle N | N \rangle},
\]
\[
T_{44}^q = \int d^3 x \overline{\psi}(x) \frac{1}{2} (\gamma_4 \overleftrightarrow{D}_4 - \frac{1}{4} \sum_{i=0,1,2,3} \gamma_i \overleftrightarrow{D}_i) \psi(x),
\]
\[
T_{44}^g = \int d^3 x \frac{1}{2} (E(x)^2 - B(x)^2).
\]
renormalization
\[
\langle x \rangle_{u,d,s}^R = Z_{QQ}^{MS}(\mu) \langle x \rangle_{u,d,s} + \delta Z_{QQ}^{MS}(\mu) \sum_{q=u,d,s} \langle x \rangle_q + Z_{GQ}^{MS}(\mu) \langle x \rangle_g
\]
\[
\langle x \rangle_g^R = \sum_{q=u,d,s} Z_{GQ}^{MS}(\mu) \langle x \rangle_q + Z_{GG}^{MS} \langle x \rangle_g,
\]
obtained results on 4 ensembles ($N_f = 2 + 1$ DWF action, overlap valence)

disconnected insertions: cluster-decomposition error reduction, all time slices looped over

extrapolate with global fit including finite volume, spacing corrections, chiral behavior

- quark energy $32(4)(4)\%$
- glue energy $36(5)(4)\%$
- quark condensate $9(2)(1)\%$
- trace anomaly $23(1)(1)\%$
- with $N_f = 2 + 1$
Nucleon spin decomposition

- spin decomposition of nucleon
  
  \[ J_N = \sum_{q=u,d,s,c\ldots} \left( \frac{1}{2} \Delta \Sigma_q + L_q \right) + J_g \]

- from Ji sum rule [Ji, PRL78, 610 (1997)]
  
  \[ \langle N(p, s')|\bar{q}\gamma_{\mu}\gamma_5 q|N(p, s)\rangle = \bar{u}_N(p, s') \left[ g_A^{\mu} \gamma_5 \right] u_N(p, s), \]

\[ \langle N(p', s')|\bar{q}\gamma_{\{\mu}^{\{\mu D_{\nu}\}^{\nu}} q|N(p, s)\rangle = \bar{u}_N(p', s') \Lambda_{\mu\nu}^q(Q^2) u_N(p, s), \]

\[ \Lambda_{\mu\nu}^q(Q^2) = A_{20}^q(g)(Q^2) \gamma_{\{\mu P^{\nu}\{\mu}^{\nu}} + B_{20}^q(g)(Q^2) \frac{\sigma_{\{\mu\alpha} q_{\alpha P^{\nu}\{\mu}}}{2m} \]

\[ + C_{20}^q(g)(Q^2) \frac{1}{m} Q_{\{\mu} Q^{\nu\{\mu}} \]
Nucleon spin decomposition (con’t)

- quark(gluon) total angular momentum and quark momentum fraction and spin from

\[
J_{q(g)} = \frac{1}{2} [A_{20}^{q(g)}(0) + B_{20}^{q(g)}(0)]
\]

\[
\langle x \rangle_q = A_{20}^q(0), \quad \Delta \Sigma_q = g_A^q
\]

- gluon momentum fraction from \(\mathcal{O}_{\mu\nu}^g = 2 \text{Tr}[G_{\mu\sigma}G_{\nu\sigma}]\) with

\[
\mathcal{O}_g^g \equiv \mathcal{O}_{44}^g - \frac{1}{3} \mathcal{O}_{jj}^g
\]

\[
\langle N(p, s')|\mathcal{O}_g^g|N(p, s)\rangle = \left( -4E_N^2 - \frac{2}{3}P^2 \right) \langle x \rangle_g,
\]

- one ensemble at physical point \(48^3 \times 96\) twisted mass clover-improved \(a = 0.0939(3)\) fm from nucleon mass

- \(u, d\) disconnected diagrams by exact deflation + one-end-trick

- \(s\) disconnected diagrams by truncated solver method

- renormalization factors determined nonperturbatively
nucleon spin (left) and momentum (right) decompositions
striped segments → valence; solid → sea quark and gluon

\( J_N \)

\( \langle X \rangle \)
Nucleon axial coupling

- recent percent level determination of $g_A$

$$\epsilon_\pi = \frac{m_\pi}{4\pi F_\pi}$$

$g_A$ model average

- $g_A^{LQCD}(\epsilon_\pi, a = 0)$
- $g_A^{PDG} = 1.2723(23)$

- use of Feynman-Hellman method

$$g_A = 1.2711(103)^s(39)^x(15)^a(19)^V(04)^I(55)^M$$

- errors: statistical, chiral, spacing, volume, isospin, model selection
Nucleon axial coupling (con’t)

- comparison to other determinations

![Graph showing comparison of axial coupling values](image-url)
Proton/neutron electromagnetic form factors

- recent study of proton and neutron electromagnetic form factors
- one ensemble $N_f = 2 + 1 + 1$ twisted mass with $m_\pi = 130$ MeV
- two ensembles $N_f = 2$ twisted mass with $m_\pi = 130$ MeV and two volumes $Lm_\pi \sim 3$ and $Lm_\pi \sim 4$
- unprecedented precision of disconnected diagram contributions
  - hierarchical probing
  - low mode deflation
  - large numbers of smeared point sources to reduce gauge noise
- disconnected diagrams have nonnegligible effects
- thorough investigation of excited-state contamination
- further study of finite-volume effects at low $Q^2$ needed
Proton/neutron electromagnetic form factors (con’t)

- comparison of $N_f = 2 + 1 + 1$ results to experiment
Proton/neutron electromagnetic form factors (con’t)

- comparing $N_f = 2 + 1 + 1$ and $N_f = 2$ (hollow symbols ignore disconnected)
Light-cone parton distribution function

- first determination of unpolarized helicity parton distribution function at the physical point with nonperturbative renormalization and large momenta treated [C. Alexandrou, K. Cichy, M. Constantinou, K. Jansen, A. Scapellato, F. Steffens, PRL 121, 112001 (2018)]

- extracting PDFs from their moments impractical
- used method proposed by Ji [X. Ji, PRL110, 262002 (2013)] with subsequent refinements
  - compute spatial correlations between boosted nucleon states
  - Fourier transforms produce quasi-PDFs
  - take infinite-momentum limit via a refined matching procedure
  - target mass corrections
  - renormalization scheme for Wilson line operators

- one $48^3 \times 96$ twisted mass $N_f = 2$ ensemble $a = 0.0938(3)(2)$ fm and $m_{\pi}L = 2.98(1)$ at physical point
unpolarized PDFs for three momenta compared to some phenomenological curves
polarized PDFs for three momenta compared to some phenomenological curves

\[ \Delta u - \Delta d \]
Excited states from correlation matrices

- in finite volume, energies are discrete (neglect wrap-around)
  \[ C_{ij}(t) = \sum_n Z_i^{(n)} Z_j^{(n)*} e^{-E_n t}, \quad Z_j^{(n)} = \langle 0 | O_j | n \rangle \]

- not practical to do fits using above form
- define new correlation matrix \( \tilde{C}(t) \) using a single rotation
  \[ \tilde{C}(t) = U^\dagger C(\tau_0)^{-1/2} C(t) C(\tau_0)^{-1/2} U \]
  columns of \( U \) are eigenvectors of \( C(\tau_0)^{-1/2} C(\tau_D) C(\tau_0)^{-1/2} \)
- choose \( \tau_0 \) and \( \tau_D \) large enough so \( \tilde{C}(t) \) diagonal for \( t > \tau_D \)
- effective energies
  \[ \tilde{m}_\alpha^{\text{eff}}(t) = \frac{1}{\Delta t} \ln \left( \frac{\tilde{C}_{\alpha\alpha}(t)}{\tilde{C}_{\alpha\alpha}(t + \Delta t)} \right) \]
  tend to \( N \) lowest-lying stationary state energies in a channel
- 2-exponential fits to \( \tilde{C}_{\alpha\alpha}(t) \) yield energies \( E_\alpha \) and overlaps \( Z_j^{(n)} \)
Two-hadron operators

- our approach: superposition of products of single-hadron operators of definite momenta

\[ c^{I_3 a I_3 b}_{p_a \lambda_a; p_b \lambda_b} B^{I_a I_3 a S_a}_{p_a \Lambda_a \lambda_a i_a} B^{I_b I_3 b S_b}_{p_b \Lambda_b \lambda_b i_b} \]

- fixed total momentum \( p = p_a + p_b \), fixed \( \Lambda, i_a, \Lambda_b, i_b \)
- group-theory projections onto little group of \( p \) and isospin irreps
- crucial to know and fix all phases of single-hadron operators for all momenta
  - each class, choose reference direction \( p_{\text{ref}} \)
  - each \( p \), select one reference rotation \( R^p_{\text{ref}} \) that transforms \( p_{\text{ref}} \) into \( p \)
- efficient creating large numbers of two-hadron operators
- generalizes to three, four, \ldots hadron operators
Quark propagation

- quark propagator is inverse $K^{-1}$ of Dirac matrix
  - rows/columns involve lattice site, spin, color
  - very large $N_{\text{tot}} \times N_{\text{tot}}$ matrix for each flavor
  - $N_{\text{tot}} = N_{\text{site}} N_{\text{spin}} N_{\text{color}}$
  - for $64^3 \times 128$ lattice, $N_{\text{tot}} \sim 403$ million
- not feasible to compute (or store) all elements of $K^{-1}$
- solve linear systems $Kx = y$ for source vectors $y$
- translation invariance can drastically reduce number of source vectors $y$ needed
- multi-hadron operators and isoscalar mesons require large number of source vectors $y$
Quark line diagrams

- temporal correlations involving our two-hadron operators need
  - slice-to-slice quark lines (from all spatial sites on a time slice to all spatial sites on another time slice)
  - sink-to-sink quark lines

- isoscalar mesons also require sink-to-sink quark lines

- solution: the stochastic LapH method! [CM et al., PRD83, 114505 (2011)]
Quantum numbers in toroidal box

- periodic boundary conditions in cubic box
  - not all directions equivalent $\Rightarrow$ using $J^{PC}$ is wrong!!

- label stationary states of QCD in a periodic box using irreps of cubic space group even in continuum limit
  - zero momentum states: little group $O_h$
    $$A_{1a}, A_{2a}, E_a, T_{1a}, T_{2a}, \ G_{1a}, G_{2a}, H_a, \quad a = g, u$$
  - on-axis momenta: little group $C_{4v}$
    $$A_1, A_2, B_1, B_2, E, \quad G_1, G_2$$
  - planar-diagonal momenta: little group $C_{2v}$
    $$A_1, A_2, B_1, B_2, \quad G_1, G_2$$
  - cubic-diagonal momenta: little group $C_{3v}$
    $$A_1, A_2, E, \quad F_1, F_2, G$$

- include $G$ parity in some meson sectors (superscript $+$ or $-$)
### Spin content of cubic box irreps

- numbers of occurrences of $\Lambda$ irreps in $J$ subduced

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Common hadrons

- irreps of commonly-known hadrons at rest

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<td>$\Delta, \Omega$</td>
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Local multi-hadron operators

- comparison of $\pi(k)\pi(-k)$ and localized $\sum_x \pi(x)\pi(x)$ operators

- much more contamination from higher states with local multi-hadron operators
The challenge of excited states

- stationary state energies $I = 1$, $S = 0$, $T_{1u}^+$ channel on $(32^3 \times 256)$ anisotropic lattice $m_\pi \sim 240$ MeV

![Graph showing levels and mixing types]
Level identification

- Level identification inferred from $|Z|^2$ overlaps with probe operators
- Overlaps for various operators

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Staircase of energy levels

- stationary state energies $I = 0$, $S = -1$, $G_{1g}^+$ channel on $(32^3 \times 256)$ anisotropic lattice
- challenge: dashed horizontal lines show 3 and 4 particle thresholds

$I = 0$, $S = -1$, $G_{1g}$ Spectrum
Comparison with experiment

- right: $G_{1g}$ energies of $\bar{q}q$-dominant states as ratios over $m_N$ for $(32^3|240)$ ensemble (resonance precursor states)
- left: experiment

$G_{1g}$ Spectrum Comparison

![Graph showing comparison between experiment and lattice results for $G_{1g}$ spectrum.]

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Staircase of energy levels

- stationary state energies $I = 0$, $S = -1$, $G_{1u}^+$ channel on $(32^3 \times 256)$ anisotropic lattice

$I = 0$, $S = -1$, $G_{1u}$ Spectrum
Comparison with experiment

- right: $G_{1u}$ energies of $\bar{q}q$-dominant states as ratios over $m_N$ for $(32^3|240)$ ensemble (resonance precursor states)
- left: experiment

$G_{1u}$ Spectrum Comparison

Experiment  |  Lattice
--- | ---
$\Lambda_{1/2}(1405)$ |  $\Lambda_{1/2}(1670)$
$\Lambda_{1/2}(1800)$ |  $\Lambda_{3/2}(2100)$

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Staircase of energy levels

- stationary state energies $I = 0, S = -1, H_g^+$ channel on $(32^3 \times 256)$ anisotropic lattice

$I = 0, S = -1, H_g$ Spectrum

![Graph showing the energy levels for $I = 0, S = -1, H_g$ on an anisotropic lattice](image.png)
Comparison with experiment

- right: $H_g$ energies of $\bar{q}q$-dominant states as ratios over $m_N$ for $(32^3|240)$ ensemble (resonance precursor states)
- left: experiment

![H_g Spectrum Comparison](image)

$H_g$ Spectrum Comparison

Experiment

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<td>$\Lambda_{s2}^{(2110)}$</td>
<td>2</td>
</tr>
</tbody>
</table>

Lattice

<table>
<thead>
<tr>
<th>State</th>
<th>$E/m_N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda_{s2}^{(2350)}$</td>
<td>3</td>
</tr>
</tbody>
</table>
Staircase of energy levels

- stationary state energies $I = 0, S = -1, H_u^+$ channel on $(32^3 \times 256)$ anisotropic lattice

$I = 0, S = -1, H_u$ Spectrum
Comparison with experiment

- right: $H_u$ energies of $\bar{q}q$-dominant states as ratios over $m_N$ for $(32^3|240)$ ensemble (resonance precursor states)
- left: experiment

$H_u$ Spectrum Comparison

$E/m_N$

Experiment | Lattice

$\Lambda_{32}(1520)$ | $\Lambda_{32}(1690)$ | $\Lambda_{52}(1830)$ | $\Lambda_{72}(2100)$
Scattering amplitudes from lattice QCD

- finite-volume energies $E$ related to infinite-volume $S$ matrix
  [M. Lüscher, NPB354, 531 (1991)]

- introduce $K$-matrix (Wigner 1946)
  \[ S = (1 + iK)(1 - iK)^{-1} = (1 - iK)^{-1}(1 + iK) \]

- $JLSa$ basis: total ang mom $J$, orbital $L$, spin $S$, species channel $a$

- introduce
  \[ K_{L'S'a'; LSa}(E) = q_{cm,a'}^{-L'-\frac{1}{2}} \tilde{K}_{L'S'a'; LSa}(E_{cm}) q_{cm,a}^{-L-\frac{1}{2}} \]

- below 3-particle thresholds, quantization condition is
  \[ \det(1 - B^{(P)}\tilde{K}) = \det(1 - \tilde{K}B^{(P)}) = 0 \]

- or
  \[ \det(\tilde{K}^{-1} - B^{(P)}) = 0 \]

- Hermitian “box matrix” $B^{(P)}$ encodes effects of cubic finite-volume
Scattering amplitudes from lattice QCD (con’t)

- quantization condition relates single energy $E$ to entire $K$-matrix
- cannot solve for $K$-matrix (except single channel, single wave)
- approximate $K$-matrix with functions depending on handful of fit parameters
- obtain estimates of fit parameters using many energies
- quantization condition involves infinite-dimensional determinant
  - make practical by (a) transforming to a block-diagonal basis and (b) truncating in orbital angular momentum
- meson-meson scattering becoming mature
- only a few meson-baryon scattering attempts
- baryon-baryon scattering currently gestating
Decay of \( \Delta \)

- recent study of \( \Delta(1232) \rightarrow N\pi \) amplitude
  
  [C.W. Andersen, J. Bulava, B. Hörz, CM, PRD 97, 014506 (2018)]

- included \( L = 1 \) wave only (for now)
- large \( 48^3 \times 128 \) isotropic lattice, \( m_\pi \approx 280 \) MeV, \( a \sim 0.076 \) fm
- Breit-Wigner fit gives \( m_\Delta/m_\pi = 4.738(47) \) and \( g_{\Delta N\pi} = 19.0(4.7) \) in agreement with experiment \( \sim 16.9 \)
Another recent △ study

- Preliminary results $L = 2.8$ fm, $a = 0.116$ fm, $m_\pi = 260$ MeV

- no slice-to-slice propagators
- three total momenta
- ground and excited states
- single partial wave

$m_{\Delta 3/2,1} = 1414(36)$ MeV
$g_{\Delta - nN} = 26(7)$
Our $\Delta$ study in progress

- Preliminary results $L = 4.2 \text{ fm}$, $a = 0.065 \text{ fm}$, $m_\pi = 200 \text{ MeV}$

[C. Andersen, B. Hörz, J. Bulava, CM, in prep.]

- five total momenta
- ground and excited states
- preliminary statistics: expect 6 times smaller errors
- light pion mass $\rightarrow$ small elastic region
Our $\Delta$ study in progress (con’t)

- Fits include irreps which mix $S$ and $P$ waves
- Relies on automated determination of $B$-matrix elements
  [CM et al., NPB924, 477 (2017)]
- Finite-volume spectrum:
Λ(1405) → Σπ study in progress

- Preliminary results $L = 3.2$ fm, $a = 0.065$ fm, $m_\pi = 280$ MeV
  [B.Hörz, C.Andersen, J.Bulava, M.Hansen, D.Möhler, CM, H.Wittig, in prep.]

- $G_{1u}(0)$ below inelastic threshold only

- fit form
  \[
  \frac{q}{\mu} \cot \delta_0 = \frac{1}{a_0 \mu} + \frac{\mu r}{2} \left( \frac{q}{\mu} \right)^2
  \]

- best fit:
  \[
  \frac{m_R}{\mu} = 6.143(77), \quad \frac{1}{a_0 \mu} = -2.41(57), \quad \frac{\mu r}{2} = -2.9(1.1),
  \]
  \[
  m_R = 1399(24) \text{ MeV}
  \]
recent determination of time-like pion form factor
[C. Andersen, J. Bulava, B. Hörz, CM, NPB939, 145 (2019)]

extracted using
\[ |F_\pi(E_{cm})|^2 = g_\Lambda(\gamma) \left( q_{cm} \frac{\partial \delta_1}{\partial q_{cm}} + u \frac{\partial \phi_1^{(d,\Lambda)}}{\partial u} \right) \frac{3\pi E_{cm}^2}{2q_{cm}^5 L^3} |\langle 0 | V^{(d,\Lambda)} | d\Lambda n \rangle|^2 \]

where
\[ \gamma = \frac{E}{E_{cm}}, \quad u = \frac{Lq_{cm}}{2\pi}, \quad g_\Lambda(\gamma) = \begin{cases} \gamma^{-1}, & \Lambda = A_1^+ \\ \gamma, & \text{otherwise} \end{cases} \]

and \( \delta_1 \) is the physical phase shift, and
\[ B_{11}^{(d,\Lambda)} = (q_{cm}/m_\pi)^3 \cot \phi_1^{(d,\Lambda)} \]
gives the pseudophase \( \phi_1^{(d,\Lambda)} \)

we compute the matrix element
\[ V^{(d,\Lambda)} = \sum_\mu b_\mu^{(d,\Lambda)} V_{R,\mu}, \quad \sum_\mu b_\mu^{(d,\Lambda)}* b_\mu^{(d,\Lambda)} = 1, \]
\[ V_{R,\mu} = Z_V (1 + ab V m_1 + ab V \text{Tr} M_q) V_{I,\mu}, \quad V_{I,\mu} = V_\mu + ac V \tilde{\sigma}_\nu T_{\mu\nu}, \]
\[ V^a_\mu = \frac{1}{2} \bar{\psi} \gamma_\mu \tau^a \psi, \quad \tilde{\sigma}_\nu T^a_{\mu\nu} = \frac{1}{2} i \tilde{\sigma}_\nu \bar{\psi} \sigma_{\mu\nu} \tau^a \psi \]
Time-like pion form factor results

- Results for CLS N200 ensemble $48^3 \times 128$ with $a = 0.064$ fm and $m_\pi = 280$ MeV (curve is fit with thrice-subtracted dispersion)
Time-like pion form factor results

- results for CLS J303 ensemble $64^3 \times 192$ with $a = 0.050$ fm and $m_\pi = 260$ MeV (curve is fit with thrice-subtracted dispersion)

- similar method is now being used for $\Delta$ transition form factor needed by Deep Underground Neutrino Experiment
Baryon-baryon interactions in HAL QCD method

- HAL QCD collaboration has extensively studied $NN$ interactions
- their method extracts observables from non-local kernels associated with tempo-spatial correlation functions
- controversy: disagreements with direct method
- recent study shows discrepancy is from misidentification of energies in direct method
- used the $\Xi\Xi(1S_0)$ temporal correlation functions

- accelerate progress in baryon-baryon scattering with this resolution
Recent $H$-dibaryon study

- obtained results at the $SU(3)$ flavor symmetric point
- used baryon-baryon operators since previous study showed hexaquark operators would not saturate signal
- found several finite-volume energies below $\Lambda\Lambda$ threshold
- scattering amplitude analysis needed to determine if bound/resonance
- warm up exercise (small lattices, pion much too heavy)
- future work on larger lattices and lighter pions will involve stochastic LapH method
Effective masses for spin-0 and spin-1 operators of different flavor irreps using 3 ensembles

Horizontal black lines show two-octet baryon threshold
Conclusion

- recent highlights involving baryons in lattice QCD
  - proton mass decomposition
  - nucleon spin decomposition
  - percent level determination of nucleon axial coupling
  - proton and neutron electromagnetic form factors
  - parton distribution function
  - scattering amplitudes
  - baryon-baryon interactions with HAL QCD method
  - $H$-dibaryon warm up

- key progress
  - achieving much better precision with disconnected diagrams
  - ability to include multi-hadron operators
  - more and more studies being done at physical point

- excited-baryon resonances in near future