Excited States from the Stochastic LapH Method

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goals

- comprehensive survey of spectrum of QCD stationary states in finite volume
- hadron scattering phase shifts, decay widths, matrix elements
- focus: large $32^3$ lattices, $m_\pi \sim 240$ MeV, all 2-hadron operators

extracting excited-state energies

single-hadron and multi-hadron operators

preliminary results in $\rho$-channel: $I = 1$, $S = 0$, $T_{1u}^+$

- used $56 \times 56$ matrix of correlators
- 12 single-hadron operators in first pass, more later
- “$\pi\pi$”, “$\eta\pi$”, “$\phi\pi$”, “$KK$” operators

level identification

preliminary results using $59 \times 59$ matrix of correlators in the bosonic $I = \frac{1}{2}$, $S = 1$, $T_{1u}$

the stochastic LapH method
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Excited states from correlation matrices

- excited-state energies from $N \times N$ Hermitian correlation matrix

\[ C_{ij}(t) = \langle 0 | O_i(t+t_0) \overline{O}_j(t_0) | 0 \rangle \]

- estimate $C_{ij}(t)$ with Monte Carlo method in lattice QCD

- in finite volume, energies are discrete

\[ C_{ij}(t) = \sum_n Z_i^{(n)} Z_j^{(n)\ast} e^{-E_n t}, \quad Z_j^{(n)} = \langle 0 | O_j | n \rangle \]

(no wrap-around)

- for $t$ large such that only lowest $N$ energies contribute, can solve for $E_n$, $Z_j^{(n)}$ using $C(t)$ at two time separations

- $N$ principal correlators $\lambda_\alpha(t, \tau_0)$ are eigenvalues of

\[ C(\tau_0)^{-1/2} C(t) C(\tau_0)^{-1/2} \]

- large time separation:

\[ \lim_{t \to \infty} \lambda_\alpha(t, \tau_0) = e^{-(t-\tau_0)E_\alpha} \]

to extract $N$ lowest-lying stationary state energies in a channel
simplest method: define new correlation matrix \( \tilde{C}(t) \) using a single rotation

\[
\tilde{C}(t) = U^\dagger C(\tau_0)^{-1/2} C(t) C(\tau_0)^{-1/2} U
\]

columns of \( U \) are eigenvectors of \( C(\tau_0)^{-1/2} C(\tau_D) C(\tau_0)^{-1/2} \)

choose \( \tau_0 \) and \( \tau_D \) large enough such that \( \tilde{C}(t) \) remains diagonal for \( t > \tau_D \)

produces results similar to principal correlator method

avoids unpalatable eigenvector “pinning”

effective masses

\[
\tilde{m}_{\alpha}^{\text{eff}}(t) = \frac{1}{\Delta t} \ln \left( \frac{\tilde{C}_{\alpha\alpha}(t)}{\tilde{C}_{\alpha\alpha}(t + \Delta t)} \right)
\]

tend to \( N \) lowest-lying stationary state energies in a channel

exponential fits to \( \tilde{C}_{\alpha\alpha}(t) \) yield energies \( E_\alpha \) and overlaps \( Z_j^{(n)} \)
Quantum numbers in toroidal box

- periodic boundary conditions in cubic box
  - not all directions equivalent $\Rightarrow$
    - using $J^{PC}$ is wrong!!

- label stationary states of QCD in a periodic box using irreps of cubic space group even in continuum limit
  - zero momentum states: little group $O_h$
    - $A_{1a}, A_{2ga}, E_a, T_{1a}, T_{2a}, G_{1a}, G_{2a}, H_a$, $a = g, u$
  - on-axis momenta: little group $C_{4v}$
    - $A_1, A_2, B_1, B_2, E$, $G_1, G_2$
  - planar-diagonal momenta: little group $C_{2v}$
    - $A_1, A_2, B_1, B_2$, $G_1, G_2$
  - cubic-diagonal momenta: little group $C_{3v}$
    - $A_1, A_2, E$, $F_1, F_2, G$
  - include $G$ parity in some meson sectors (superscript $+$ or $-$)
Building blocks for single-hadron operators

- building blocks: covariantly-displaced LapH-smeared quark fields
- stout links $\tilde{U}_j(x)$
- Laplacian-Heaviside (LapH) smeared quark fields

$$\tilde{\psi}_{a\alpha}(x) = S_{ab}(x, y) \psi_{b\alpha}(y), \quad S = \Theta \left( \sigma^2_s + \tilde{\Delta} \right)$$

- 3d gauge-covariant Laplacian $\tilde{\Delta}$ in terms of $\tilde{U}$
- displaced quark fields:

$$q^A_{a\alpha j} = D^{(j)}(x) \tilde{\psi}^{(A)}_{a\alpha}, \quad \bar{q}^A_{a\alpha j} = \bar{\tilde{\psi}}^{(A)}_{a\alpha} \gamma_4 D^{(j)\dagger}$$

- displacement $D^{(j)}$ is product of smeared links:

$$D^{(j)}(x, x') = \tilde{U}_{j_1}(x) \tilde{U}_{j_2}(x+d_2) \tilde{U}_{j_3}(x+d_3) \ldots \tilde{U}_{j_p}(x+d_p) \delta_{x', x+d_p+1}$$

- to good approximation, LapH smearing operator is

$$S = V_s V_s^\dagger$$

- columns of matrix $V_s$ are eigenvectors of $\tilde{\Delta}$
Extended operators for single hadrons

- quark displacements build up orbital, radial structure

Meson configurations

\[ \Phi_{AB}^{\alpha\beta}(t) = \sum_x e^{ip \cdot (x + \frac{1}{2}(d_\alpha + d_\beta))} \delta_{ab} q_B^{B}(x, t) q^{A}_{a\alpha}(x, t) \]

\[ \Phi_{ABC}^{\alpha\beta\gamma}(p, t) = \sum_x e^{ip \cdot x} \varepsilon_{abc} \overline{q}^C_{c\gamma}(x, t) \overline{q}^B_{b\beta}(x, t) \overline{q}^A_{a\alpha}(x, t) \]

- group-theory projections onto irreps of lattice symmetry group

\[ M_l(t) = c^{(l)*}_{\alpha\beta} \Phi_{AB}^{\alpha\beta}(t) \quad B_l(t) = c^{(l)*}_{\alpha\beta\gamma} \Phi_{ABC}^{\alpha\beta\gamma}(t) \]

- definite momentum \( p \), irreps of little group of \( p \)
Testing single-hadron operators

- meson effective masses on \((24^3|390)\) ensemble

\[
LSD0 \ A^+_J [\bar{u}\gamma_5(1235)] \\
p_{\text{lat}} = (0,0,1)
\]

\[
SS0 \ A^+_J [\bar{u}\gamma_5(1235)] \\
p_{\text{lat}} = (0,0,1)
\]

\[
LSD5 \ B^+ \ [\bar{u}\gamma_5(1260)] \\
p_{\text{lat}} = (0,1,1)
\]

\[
TSD1 \ B^+ \ [p] \\
p_{\text{lat}} = (0,1,1)
\]

\[
SD0 \ A^+_J [x] \\
p_{\text{lat}} = (1,1,1)
\]

\[
SS0 \ A^+_J [x] \\
p_{\text{lat}} = (1,1,1)
\]

\[
TSD4 \ E^+ \ [\bar{u}\gamma_5(1650)] \\
p_{\text{lat}} = (0,0,1)
\]

\[
LSD0 \ A^+_J [n] \\
p_{\text{lat}} = (0,1,1)
\]

\[
SD7 \ A^+_J [\omega] \\
p_{\text{lat}} = (1,1,1)
\]
Testing single-hadron operators (con’t)

- (left and center) pion energies on \((32^3|240)\) ensemble
- (right) nucleon and \(\Delta\) baryons

![Graphs showing pion energies and nucleon and \(\Delta\) baryons](image-url)
Two-hadron operators

- our approach: superposition of products of single-hadron operators of definite momenta

\[
c^{I_{3a}I_{3b}}_{p_a\lambda_a; p_b\lambda_b} B^{I_{3a}S_a}_{p_a\Lambda_a\lambda_i a} B^{I_{3b}S_b}_{p_b\Lambda_b\lambda_i b}
\]

- fixed total momentum \( p = p_a + p_b \), fixed \( \Lambda_a, i_a, \Lambda_b, i_b \)
- group-theory projections onto little group of \( p \) and isospin irreps
- restrict attention to certain classes of momentum directions
  - on axis \( \pm \hat{x}, \pm \hat{y}, \pm \hat{z} \)
  - planar diagonal \( \pm \hat{x} \pm \hat{y}, \pm \hat{x} \pm \hat{z}, \pm \hat{y} \pm \hat{z} \)
  - cubic diagonal \( \pm \hat{x} \pm \hat{y} \pm \hat{z} \)
- crucial to know and fix all phases of single-hadron operators for all momenta
  - each class, choose reference direction \( p_{\text{ref}} \)
  - each \( p \), select one reference rotation \( R^p_{\text{ref}} \) that transforms \( p_{\text{ref}} \) into \( p \)
- efficient creating large numbers of two-hadron operators
- generalizes to three, four, \ldots hadron operators
Testing our two-meson operators

- (left) $K\pi$ operator in $T_{1u} I = \frac{1}{2}$ channels
- (center and right) comparison with localized $\pi\pi$ operators

$$(\pi\pi)^{A_{1g}^+}(t) = \sum_x \pi^+(x, t) \pi^+(x, t),$$

$$(\pi\pi)^{T_{1u}^+}(t) = \sum_{x,k=1,2,3} \left\{ \pi^+(x, t) \Delta_k \pi^0(x, t) - \pi^0(x, t) \Delta_k \pi^+(x, t) \right\}$$

- less contamination from higher states in our $\pi\pi$ operators
Ensembles and run parameters

- plan to use three Monte Carlo ensembles
  - \((32^3|240)\): 412 configs \(32^3 \times 256\), \(m_\pi \approx 240\) MeV, \(m_\pi L \sim 4.4\)
  - \((24^3|240)\): 584 configs \(24^3 \times 128\), \(m_\pi \approx 240\) MeV, \(m_\pi L \sim 3.3\)
  - \((24^3|390)\): 551 configs \(24^3 \times 128\), \(m_\pi \approx 390\) MeV, \(m_\pi L \sim 5.7\)

- anisotropic improved gluon action, clover quarks (stout links)
- QCD coupling \(\beta = 1.5\) such that \(a_s \sim 0.12\) fm, \(a_t \sim 0.035\) fm
- strange quark mass \(m_s = -0.0743\) nearly physical (using kaon)
- work in \(m_u = m_d\) limit so \(SU(2)\) isospin exact
- generated using RHMC, configs separated by 20 trajectories

- stout-link smearing in operators \(\xi = 0.10\) and \(n_\xi = 10\)
- LapH smearing cutoff \(\sigma_s^2 = 0.33\) such that
  - \(N_v = 112\) for \(24^3\) lattices
  - \(N_v = 264\) for \(32^3\) lattices

- source times:
  - 4 widely-separated \(t_0\) values on \(24^3\)
  - 8 \(t_0\) values used on \(32^3\) lattice
correlator software **last_laph** completed summer 2013
  - testing of all flavor channels for single and two-mesons completed
first focus on the resonance-rich $\rho$-channel: $I = 1$, $S = 0$, $T^+_1u$
experiment: $\rho(770)$, $\rho(1450)$, $\rho(1570)$, $\rho_3(1690)$, $\rho(1700)$
first results: $56 \times 56$ matrix of correlators (24$^3$|390) ensemble
  - 12 single-hadron (quark-antiquark) operators
  - 17 “$\pi\pi$” operators
  - 14 “$\eta\pi$” operators, 3 “$\phi\pi$” operators
  - 10 “$K\bar{K}$” operators
inclusion of all possible 2-meson operators
3-meson operators currently neglected
still finalizing analysis code
$I = 1, \ S = 0, \ T_{1u}^+$ channel

- effective masses $\tilde{m}^{\text{eff}}(t)$ for levels 0 to 15
- dashed lines show energies from single exponential fits
effective masses \( \tilde{m}^{\text{eff}}(t) \) for levels 16 to 31
- dashed lines show energies from single exponential fits

\( I = 1, \ S = 0, \ T_{1u}^+ \) energy extraction, continued
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Level identification

- Level identification inferred from $Z$ overlaps with probe operators.
- Analogous to experiment: infer resonances from scattering cross sections.
- Keep in mind:
  - Probe operators $\bar{O}_j$ act on vacuum, create a “probe state” $|\Phi_j\rangle$.
  - $Z$’s are overlaps of probe state with each eigenstate:
    $$|\Phi_j\rangle \equiv \bar{O}_i|0\rangle,$$
    $$Z_j^{(n)} = \langle \Phi_j | n \rangle$$
  - Have limited control of “probe states” produced by probe operators:
    - Ideal to be $\rho$, single $\pi\pi$, and so on.
    - Use of small $-a$ expansions to characterize probe operators.
    - Use of smeared quark, gluon fields.
    - Field renormalizations.
  - Mixing is prevalent.
  - Identify by dominant probe state(s) whenever possible.
Level identification

- overlaps for various operators

\[ \pi A_2^* \pi \ A_2 \pi \ SS1 \ OA \]

\[ \pi(140) \pi(140) \]

\[ K A_2 \ SS1 \ K A_2 \ SS1 \ OA \]

\[ K(497) K^*(497) \]

\[ \pi A_2^* \ SS0 \pi A_2^* \ SS0 \ PD \]

\[ \pi(140) \pi(140) \]

\[ \eta \ E \ SS1 \ \eta \ LS1 \ OA \]

\[ \omega(782) \pi(140) \]

\[ K A_2 ^* SS0 \ K A_2 ^* SS0 \ PD \]

\[ K(497) K^*(497) \]

\[ \phi \ E \ SS1 \ \phi \ LS1 \ OA \]

\[ \phi(1020) \pi(140) \]

\[ \pi A_2^* \ SS0 \pi A_2^* \ SS0 \ CD \]

\[ \pi(140) \pi(140) \]

\[ \eta T_{1u} \ SS0 \pi A_{1u} \ SS0 \]

\[ \omega(782) a_1(980) \]

\[ \pi A_2^* \ SS1 \pi A_2^* \ TSD0 \ OA \]

\[ \pi(140) \pi(1300) \]
illustrate problem with real scalar field $\varphi(x)$

consider three operators $\Phi_j$ for $j = 1, 2, 3$ defined by

$$\Phi_j(x) = \frac{1}{2a} \left( \varphi(x + \hat{j}) - \varphi(x - \hat{j}) \right).$$

this is forward-backward finite difference approx to derivative

carry $T_1$ irrep of the octahedral point group $O$

classical small-$a$ expansion:

$$\Phi_j(x) \approx \partial_j \varphi(x) + \frac{1}{6} a^2 \partial_j^3 \varphi(x) + O(a^4).$$

first term $\partial_j \varphi(x)$ is spin $J = 1$

second term contains both $J = 1$ and $J = 3$

radiative corrections modify relative weights (calculate in lattice perturbation theory, but difficult)
link variables in terms of continuum gluon field

\[ U_\mu(x) = \mathcal{P} \exp \left\{ ig \int_x^{x+\hat{\mu}} d\eta \cdot A(\eta) \right\}, \]

classical small-\(a\) expansion of displaced quark field:

\[ U_j(x)U_k(x+\hat{j})\psi_\alpha(x+\hat{j}+\hat{k}) = \exp(aD_j) \exp(aD_k) \psi_\alpha(x). \]

where \(D_j = \partial_j + igA_j\) is covariant derivative

must take smearing of fields into account

radiative corrections of expansion coefficients (hopefully small due to smearing)

work in progress for our probe operators
isovector meson continuum probe operators

\[ M_{\mu j_1 j_2 \ldots} = \chi^d \Gamma_\mu D_{j_1} D_{j_2} \cdots \psi^u, \quad \chi = \overline{\psi} \gamma_4 \]

where \( \Gamma_0 = 1 \) and \( \Gamma_k = \gamma_k \) (analogous table inserting \( \gamma_4, \gamma_5, \gamma_4\gamma_5 \))

<table>
<thead>
<tr>
<th>( J^P_G )</th>
<th>( O^G_h ) irrep</th>
<th>Basis operator</th>
</tr>
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<tr>
<td>0++</td>
<td>( A_{1g}^+ )</td>
<td>( M_0 )</td>
</tr>
<tr>
<td>1−+</td>
<td>( T_{1u}^+ )</td>
<td>( M_1 )</td>
</tr>
<tr>
<td>1−−</td>
<td>( T_{1u}^- )</td>
<td>( M_{01} )</td>
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<tr>
<td>0+-</td>
<td>( A_{1g}^- )</td>
<td>( M_{11} + M_{22} + M_{33} )</td>
</tr>
<tr>
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<td>( T_{1g}^- )</td>
<td>( M_{23} - M_{32} )</td>
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<tr>
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<td>( T_{2g}^- )</td>
<td>( M_{23} + M_{32} )</td>
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<td>( A_{1g}^+ )</td>
<td>( M_{011} + M_{022} + M_{033} )</td>
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<td>( M_{023} - M_{032} )</td>
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isovector meson continuum probe operators

\[ M_{\mu j_1 j_2 \ldots} = \chi^d \Gamma_{\mu} D_{j_1} D_{j_2} \cdots \psi^u, \quad \chi = \bar{\psi} \gamma_4 \]

where \( \Gamma_0 = 1 \) and \( \Gamma_k = \gamma_k \) (analogous table inserting \( \gamma_4, \gamma_5, \gamma_4 \gamma_5 \))

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<th>( J^P )</th>
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<td>0--</td>
<td>( A_{1u}^- )</td>
<td>( M_{123} + M_{231} + M_{312} - M_{321} - M_{213} - M_{132} )</td>
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<tr>
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<td>( T_{1u}^+ )</td>
<td>( M_{111} + M_{122} + M_{133} )</td>
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<tr>
<td>1--</td>
<td>( T_{1u}^+ )</td>
<td>( 2M_{111} + M_{221} + M_{331} + M_{212} + M_{313} )</td>
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<tr>
<td>1--</td>
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<td>( M_{221} + M_{331} - M_{212} - M_{313} )</td>
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<tr>
<td>2--</td>
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<td></td>
<td>( T_{2u}^- )</td>
<td>( M_{221} - M_{331} + M_{313} - M_{212} )</td>
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<tr>
<td>2--</td>
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<td>( M_{123} + M_{213} - 2M_{321} - 2M_{312} + M_{231} + M_{132} )</td>
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<td>( M_{331} - M_{212} + M_{313} - M_{122} + M_{133} - M_{221} )</td>
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Identifying resonances

- resonances: finite-volume “precursor states”
- probes: *optimized* single-hadron operators
  - analyze matrix of just single-hadron operators $O_{i}^{[SH]}$ ($12 \times 12$)
  - perform single-rotation as before to build probe operators
    \[ O_{m}^{[SH]} = \sum_{i} v_{i}^{(m)*} O_{i}^{[SH]} \]
- obtain $Z'$ factors of these probe operators
  \[ Z_{m}^{(n)} = \langle 0 | O_{m}^{[SH]} | n \rangle \]
### List of tentative level identifications

<table>
<thead>
<tr>
<th>Level</th>
<th>Dominant Probe</th>
<th>Level</th>
<th>Dominant Probe</th>
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<td>$\omega(782)\pi(140)$ (PD)</td>
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<td>$\pi(140)\pi(140)$ (OA)</td>
<td>17</td>
<td>$\phi(1020)\pi(140)$ (PD)</td>
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<td>$\eta(547)b_1(1235)$ (AR)</td>
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<td>12</td>
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<td>46</td>
<td>$\rho(\ ?)$</td>
</tr>
<tr>
<td>13</td>
<td>$\rho(1570)$</td>
<td>48</td>
<td>$\rho(\ ?)$</td>
</tr>
<tr>
<td>14</td>
<td>$K^*(892)K_c(497)$ (OA)</td>
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</table>
Summary and comparison with experiment

- left: energies of $\bar{q}q$-dominant states as ratios over $m_N$ for $(24^3|390)$ ensemble (resonance precursor states)
- right: experiment
Issues

- address presence of 3 and 4 meson states
- in other channels, address scalar particles in spectrum
  - scalar probe states need vacuum subtractions
  - hopefully can neglect due to OZI suppression
- infinite-volume resonance parameters from finite-volume energies
  - Luscher method too cumbersome, restrictive in applicability
  - need for new hadron effective field theory techniques
Bosonic $I = \frac{1}{2}, \ S = 1, \ T_{1u}$ channel

- also have results for the kaon channel: $I = \frac{1}{2}, \ S = 1, \ T_{1u}$
- experiment: $K^*(892), \ K^*(1410), \ K^*(1680), \ K_3^*(1780)$
- first results: $59 \times 59$ matrix of correlators $(24^3|390)$ ensemble
  - 10 single-hadron (quark-antiquark) operators
  - 25 “$K\pi$” operators
  - 12 “$K\eta$” operators, 12 “$K\phi$” operators
Bosonic $I = \frac{1}{2}$, $S = 1$, $T_{1u}$ channel

- Effective masses $\tilde{m}^{\text{eff}}(t)$ for levels 0 to 14
- Dashed lines show energies from single exponential fits

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Excited States
Bosonic $I = \frac{1}{2}$, $S = 1$, $T_{1u}$ channel

- effective masses $\tilde{m}^{\text{eff}}(t)$ for levels 15 to 29
- dashed lines show energies from single exponential fits
Bosonic $I = \frac{1}{2}, \ S = 1, \ T_{1u}$ channel

- effective masses $\tilde{m}^\text{eff}(t)$ for levels 30 to 44
- dashed lines show energies from single exponential fits
Bosonic $I = \frac{1}{2}$, $S = 1$, $T_{1u}$ channel

- preliminary estimates of $Z$ overlaps for various operators:
The scalar isoscalar sector

- $5 \times 5$ correlator matrix mixing glueball $G$, two $\pi\pi$, an $\eta\eta$, and a $\bar{q}q$ operator for $(24^3|390)$ ensemble.
temporal correlations involving our two-hadron operators need
- slice-to-slice quark lines (from all spatial sites on a time slice to all spatial sites on another time slice)
- sink-to-sink quark lines

isoscalar mesons also require sink-to-sink quark lines

solution: the stochastic LapH method!
do not need exact inverse of Dirac matrix $K[U]$
use noise vectors $\eta$ satisfying $E(\eta_i) = 0$ and $E(\eta_i \eta_j^*) = \delta_{ij}$
$Z_4$ noise is used $\{1, i, -1, -i\}$
solve $K[U]X^{(r)} = \eta^{(r)}$ for each of $N_R$ noise vectors $\eta^{(r)}$, then obtain a Monte Carlo estimate of all elements of $K^{-1}$

$$K_{ij}^{-1} \approx \frac{1}{N_R} \sum_{r=1}^{N_R} X^{(r)}_i \eta^{(r)*}_j$$

variance reduction using noise dilution
dilution introduces projectors

$$P^{(a)} P^{(b)} = \delta^{ab} P^{(a)}, \quad \sum_a P^{(a)} = 1, \quad P^{(a)\dagger} = P^{(a)}$$

define

$$\eta^{[a]} = P^{(a)} \eta, \quad X^{[a]} = K^{-1} \eta^{[a]}$$

to obtain Monte Carlo estimate with drastically reduced variance

$$K_{ij}^{-1} \approx \frac{1}{N_R} \sum_{r=1}^{N_R} \sum_a X^{(r)[a]}_i \eta^{(r)[a]}_j$$
Stochastic LapH method

- introduce $Z_N$ noise in the LapH subspace
  \[ \rho_{\alpha k}(t), \quad t = \text{time}, \ \alpha = \text{spin}, \ k = \text{eigenvector number} \]

- four dilution schemes:

\[
\begin{align*}
  P_{ij}^{(a)} &= \delta_{ij} & a &= 0 & \text{(none)} \\
  P_{ij}^{(a)} &= \delta_{ij}\delta_{ai} & a &= 0, 1, \ldots, N-1 & \text{(full)} \\
  P_{ij}^{(a)} &= \delta_{ij}\delta_{a/Ki/N} & a &= 0, 1, \ldots, K-1 & \text{(interlace-$K$)} \\
  P_{ij}^{(a)} &= \delta_{ij}\delta_{a,i \mod k} & a &= 0, 1, \ldots, K-1 & \text{(block-$K$)}
\end{align*}
\]

- apply dilutions to
  - time indices (full for fixed src, interlace-16 for relative src)
  - spin indices (full)
  - LapH eigenvector indices (interlace-8 mesons, interlace-4 baryons)
Quark line estimates in stochastic LapH

- each of our quark lines is the product of matrices
  \[ Q = D^{(j)} S K^{-1} \gamma_4 S D^{(k)\dagger} \]
- displaced-smeared-diluted quark source and quark sink vectors:
  \[ q^{[b]}(\rho) = D^{(j)} V_s P^{(b)} \rho \]
  \[ \varphi^{[b]}(\rho) = D^{(j)} S K^{-1} \gamma_4 V_s P^{(b)} \rho \]
- estimate in stochastic LapH by \((A, B\) flavor, \(u, v\) compound: space, time, color, spin, displacement type)
  \[ Q_{uv}^{(AB)} \approx \frac{1}{N_R} \delta_{AB} \sum_{r=1}^{N_R} \sum_{b} \varphi^{[b]}(\rho^r) q^{[b]}(\rho^r)^* \]
- occasionally use \(\gamma_5\)-Hermiticity to switch source and sink
  \[ Q_{uv}^{(AB)} \approx \frac{1}{N_R} \delta_{AB} \sum_{r=1}^{N_R} \sum_{b} \overline{q}^{[b]}(\rho^r) \overline{\varphi}^{[b]}(\rho^r)^* \]
- defining \(\overline{q}(\rho) = -\gamma_5 \gamma_4 q(\rho)\) and \(\overline{\varphi}(\rho) = \gamma_5 \gamma_4 \varphi(\rho)\)
baryon correlator has form
\[ C_{\bar{l}l} = c_{ijk} c_{\bar{ij}k}^{(l)} Q_i^A Q_j^B Q_k^C \]

stochastic estimate with dilution
\[
C_{\bar{l}l} \approx \frac{1}{N_R} \sum_r \sum_{d_A d_B d_C} c_{ijk} c_{\bar{ij}k}^{(l)} \left( \varphi_i^{(Ar)}[d_A] \varphi_j^{(Br)}[d_B] \varphi_k^{(Cr)}[d_C] \right)
\]
\times \left( \varphi_i^{(\bar{Ar})}[d_A] \varphi_j^{(\bar{Br})}[d_B] \varphi_k^{(\bar{Cr})}[d_C] \right)

define baryon source and sink
\[
B_l^{(r)[d_A d_B d_C]} (\varphi^A, \varphi^B, \varphi^C) = c_{ijk}^{(l)} \varphi_i^{(Ar)}[d_A] \varphi_j^{(Br)}[d_B] \varphi_k^{(Cr)}[d_C]
\]
\[
B_{\bar{l}}^{(r)[d_A d_B d_C]} (\varphi^A, \varphi^B, \varphi^C) = c_{ijk}^{(l)} \varphi_i^{(\bar{Ar})}[d_A] \varphi_j^{(\bar{Br})}[d_B] \varphi_k^{(\bar{Cr})}[d_C]
\]

correlator is dot product of source vector with sink vector
\[
C_{\bar{l}l} \approx \frac{1}{N_R} \sum_r \sum_{d_A d_B d_C} B_l^{(r)[d_A d_B d_C]} (\varphi^A, \varphi^B, \varphi^C) B_{\bar{l}}^{(r)[d_A d_B d_C]} (\varphi^A, \varphi^B, \varphi^C)^* \]
Correlators and quark line diagrams

- baryon correlator

\[ C_{\bar{l}l} \approx \frac{1}{N_R} \sum_r \sum_{dAdbdc} B_l^{(r)[dAdbdc]} (\varphi^A, \varphi^B, \varphi^C) B_{\bar{l}}^{(r)[dAdbdc]} (\varrho^A, \varrho^B, \varrho^C)^* \]

- express diagrammatically

- meson correlator

C. Morningstar

Excited States
More complicated correlators

- two-meson to two-meson correlators (non isoscalar mesons)


Conclusion and future work

- goal: comprehensive survey of energy spectrum of QCD stationary states in a finite volume
- stochastic LapH method works very well
  - allows evaluation of all needed quark-line diagrams
  - source-sink factorization facilitates large number of operators
  - `last_laph` software completed for evaluating correlators
- showed first results in $\rho$-channel: $I = 1, S = 0, T_{1u}^+$ using $56 \times 56$ matrix of correlators
- preliminary results using $59 \times 59$ matrix of correlators in the bosonic $I = \frac{1}{2}, S = 1, T_{1u}$
- large number of channels to study over the next year!
- first peek: results on $(32^3|240)$ ensemble look even better so far!!
- study various scattering phase shifts also planned
- infinite-volume resonance parameters from finite-volume energies $\longrightarrow$ need new effective field theory techniques