The frontier awaits

- experiments show many excited-state hadrons exist
- significant experimental efforts to map out QCD resonance spectrum → JLab Hall B, Hall D, ELSA, etc.
- great need for *ab initio* calculations → lattice QCD

**Nucleon Mass Spectrum (Exp)**

**Delta Mass Spectrum (Exp)**
The challenge of exploration!

- most excited hadrons are unstable (resonances)
- excited states more difficult to extract in Monte Carlo calculations
  - correlation matrices needed
  - operators with very good overlaps onto states of interest
- must extract all states lying below a state of interest
  - as pion get lighter, more and more multi-hadron states
- best multi-hadron operators made from constituent hadron operators with well-defined relative momenta
  - need for all-to-all quark propagators
- disconnected diagrams
Hadron Spectrum Collaboration (HSC)

- spin-off from the Lattice Hadron Physics Collaboration which was spear-headed by Nathan Isgur and John Negele
- current members:
  - Justin Foley, David Lenkner, Colin Morningstar, Ricky Wong (CMU)
  - John Bulava (DESY, Zeuthen)
  - Eric Engelson, Steve Wallace (U. Maryland)
  - Mike Peardon, Sinead Ryan (Trinity Coll. Dublin)
  - Keisuke Jimmy Juge (U. of Pacific)
  - R. Edwards, B. Joo, D. Richards, C. Thomas (Jefferson Lab.)
  - H.W. Lin (U. Washington), J. Dudek (Old Dominion)
  - N. Mathur (Tata Institute)
Overview of our spectrum project

- obtain stationary state energies of QCD in various boxes
  - 1st milestone: quenched excited states with heavy pion \( \rightarrow \) done
  - 2nd milestone: \( N_f=2 \) excited states with heavy pion \( \rightarrow \) done
  - 3rd milestone: \( N_f=2+1 \) excited states with light pion
    - multi-hadron operators needed \( \rightarrow \) many-to-many quark propagators
    - recent technology breakthrough \( \rightarrow \) new quark smearing with improved variance reduction

- interpretation of finite-volume energies
  - spectrum matching to construct effective hadron theory?
Monte Carlo method

- hadron operators $\phi = \phi[\bar{\psi}, \psi, U] \quad \psi =$quark $\quad U =$gluon field
- temporal correlations from path integrals

$$\langle \phi(t)\phi(0) \rangle = \frac{\int D[\bar{\psi}, \psi, U] \phi(t)\phi(0) e^{-\bar{\psi}M[U]\psi - S[U]}}{\int D[\bar{\psi}, \psi, U] e^{-\bar{\psi}M[U]\psi - S[U]}}$$

- integrate exactly over quark Grassmann fields

$$\langle \phi(t)\phi(0) \rangle = \frac{\int D[U] \det M[U] \left( M^{-1}[U] \cdots \right) e^{-S[U]}}{\int D[U] \det M[U] e^{-S[U]}}$$

- resort to Monte Carlo method to integrate over gluon fields
- generate sequence of field configurations $U_1, U_2, U_3, \ldots, U_N$
  using Markov chain procedure
  - use of parallel computations on supercomputers
  - especially intensive as quark mass (pion mass) gets small
Lattice regularization

- hypercubic space-time lattice regulator needed for Monte Carlo
- quarks reside on sites, gluons reside on links between sites
- lattice excludes short wavelengths from theory (regulator)
- regulator removed using standard renormalization procedures (continuum limit)
- systematic errors
  - discretization
  - finite volume
Excited-state energies from Monte Carlo

- Extracting excited-state energies requires matrix of correlators.
- For a given $N \times N$ correlator matrix $C_{\alpha\beta}(t) = \langle 0 | O_\alpha(t) O_\beta^+(0) | 0 \rangle$, one defines the $N$ principal correlators $\lambda_\alpha(t, t_0)$ as the eigenvalues of
  \[ C(t_0)^{-1/2} C(t) C(t_0)^{-1/2} \]
  where $t_0$ (the time defining the “metric”) is small.
- Can show that $\lim_{t \to \infty} \lambda_\alpha(t, t_0) = e^{-(t-t_0)E_\alpha} (1 + e^{-t\Delta E_\alpha})$.
- $N$ principal effective masses defined by $m_{\alpha}^{\text{eff}}(t) = \ln \left( \frac{\lambda_\alpha(t, t_0)}{\lambda_\alpha(t+1, t_0)} \right)$ now tend (plateau) to the $N$ lowest-lying stationary-state energies.
- Analysis:
  - Fit each principal correlator to single exponential.
  - Optimize on earlier time slice, matrix fit to optimized matrix.
  - Both methods as consistency check.
Operator design issues

- statistical noise increases with temporal separation $t$
- use of very good operators is crucial or noise swamps signal
- recipe for making better operators
  - crucial to construct operators using smeared fields
    - link variable smearing
    - quark field smearing
  - spatially extended operators
  - use large set of operators (variational coefficients)
Three stage approach (PRD72:094506,2005)

- concentrate on **baryons at rest** (zero momentum)
- operators classified according to the irreps of \( O_h \)
  \[ G_{1g}, G_{1u}, G_{2g}, G_{2u}, H_g, H_u \]
- **(1)** basic building blocks: smeared, covariant-displaced quark fields
  \[ (\tilde{D}_j^{(p)} \phi(x))_{A\alpha\beta} \quad p\text{-link displacement (}j = 0, \pm 1, \pm 2, \pm 3) \]
- **(2)** construct **elemental** operators (translationally invariant)
  \[ B^F(x) = \phi_{ABC}^F \varepsilon_{abc} (\tilde{D}_i^{(p)} \psi(x))_{A\alpha\beta} (\tilde{D}_j^{(p)} \psi(x))_{B\beta\gamma} (\tilde{D}_k^{(p)} \psi(x))_{C\gamma\delta} \]
  - flavor structure from isospin
  - color structure from gauge invariance
- **(3)** group-theoretical projections onto irreps of \( O_h \)
  \[ B_i^{\Lambda F}(t) = \frac{d_\Lambda}{g_{O_h}^{D}} \sum_{R \in O_h^{D}} D^{(\Lambda)}_\Lambda (R)^* U_R B_i^{F}(t) U_R^+ \]

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Single-hadron operators

- covariantly-displaced quark fields as building blocks
- group-theoretical projections onto irreps of lattice symmetry group
- displacements of different lengths build up radial structure
- displacements in different directions build up orbital structure

**Reference:** PRD72, 094506 (2005)
Spin identification and other remarks

- Spin identification possible by pattern matching

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- Total numbers of operators is huge $\Rightarrow$ uncharted territory
- Ultimately must face two-hadron scattering states
Quark- and gauge-field smearing

- smeared quark and gluon fields $\rightarrow$ dramatically reduced coupling with short wavelength modes
- **link-variable** smearing (stout links PRD69, 054501 (2004))
  - define $C_\mu(x) = \sum_{\pm(v\mp \mu)} \rho_{\mu \nu} U_\nu(x) U_\mu(x + \hat{v}) U^+_\nu(x + \hat{\mu})$
  - spatially isotropic $\rho_{jk} = \rho$, $\rho_{4k} = \rho_{k4} = 0$
  - exponentiate traceless Hermitian matrix
    $$\Omega_\mu = C_\mu U^+_\mu, \quad Q_\mu = \frac{i}{2} \left( \Omega^+_\mu - \Omega_\mu \right) - \frac{i}{2N} \text{Tr} \left( \Omega^+_\mu - \Omega_\mu \right)$$
  - iterate
    $$U^{(n+1)}_{\mu} = \exp \left( iQ^{(n)}_{\mu} \right) U^{(n)}_{\mu}$$
- initial quark-field smearing (Laplacian using smeared gauge field)
  $$\bar{\psi}(x) = \left( 1 + \frac{\sigma_s}{4n_\sigma} \tilde{\Lambda} \right)^{n_\sigma} \psi(x)$$
Importance of smearing

• Nucleon $G_{1g}$ channel
• Effective masses of 3 selected operators
• Noise reduction from link variable smearing, especially for displaced operators
• Quark-field smearing reduces couplings to high-lying states

$\sigma_s = 4.0$, $n_\sigma = 32$

$\rho_\rho = 2.5$, $n_\rho = 16$

• Less noise in excited states using $\sigma_s = 3.0$

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Operator selection

- operator construction leads to very large number of operators
- rules of thumb for “pruning” operator sets
  - noise is the enemy!
  - prune first using intrinsic noise (diagonal correlators)
  - prune next within operator types (single-site, singly-displaced, etc.) based on condition number of
  - prune across all operators based on condition number
- best to keep a variety of different types of operators, as long as condition numbers maintained
  \[
  \hat{C}_{ij}(t) = \frac{C_{ij}(t)}{\sqrt{C_{ii}(t)C_{jj}(t)}}, \quad t = 1
  \]
- typically use 16 operators to get 8 lowest lying levels
Nucleon $G_{1g}$ effective masses

- 200 quenched configs, $12^3 \times 48$ anisotropic Wilson lattice, $a_s \sim 0.1$ fm, $a_s/a_t \sim 3$, $m_\pi \sim 700$ MeV
- nucleon $G_{1g}$ channel
- green=fixed coefficients, red=principal
Nucleon $H_u$ effective masses

- 200 quenched configs, $12^3 \times 48$ anisotropic Wilson lattice, $a_s \sim 0.1$ fm, $a_s/a_t \sim 3$, $m_\pi \sim 700$ MeV
- nucleon $H_u$ channel
- green=fixed coefficients, red=principal

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Nucleons

- $N_f=2$ on $24^3 \times 64$ anisotropic clover lattice, $a_s \sim 0.11$ fm, $a_s/a_t \sim 3$
- Left: $m_\pi = 578$ MeV  Right: $m_\pi = 416$ MeV  PRD 79, 034505 (2009)

- multi-hadron thresholds above show need for multi-hadron operators to go to lower pion masses!!
Spatial summations

- Baryon at rest is operator of form

\[ B(\vec{p} = 0, t) = \frac{1}{V} \sum_{\vec{x}} \phi_B(\vec{x}, t) \]

- Baryon correlator has a double spatial sum

\[ \langle 0 | \bar{B}(\vec{p} = 0, t) B(\vec{p} = 0, 0) | 0 \rangle = \frac{1}{V^2} \sum_{\vec{x}, \vec{y}} \langle 0 | \bar{\phi}_B(\vec{x}, t) \phi_B(\vec{y}, 0) | 0 \rangle \]

- Computing all elements of propagators exactly not feasible

- Translational invariance can limit summation over source site to a single site for local operators

\[ \langle 0 | \bar{B}(\vec{p} = 0, t) B(\vec{p} = 0, 0) | 0 \rangle = \frac{1}{V} \sum_{\vec{x}} \langle 0 | \bar{\phi}_B(\vec{x}, t) \phi_B(0, 0) | 0 \rangle \]
good baryon-meson operator of total zero momentum has form

\[ B(\vec{p}, t)M(-\vec{p}, t) = \frac{1}{V^2} \sum_{\vec{x}, \vec{y}} \phi_B(\vec{x}, t)\phi_M(\vec{y}, t)e^{i\vec{p} \cdot (\vec{x} - \vec{y})} \]

cannot limit source to single site for multi-hadron operators

disconnected diagrams (scalar mesons) will also need many-to-many quark propagators

quark propagator elements from all spatial sites to all spatial sites are needed!
Laplacian Heaviside quark-field smearing

- new quark-field smearing method  PRD80, 054506 (2009)
- judicious choice of quark-field smearing makes exact computations with all-to-all quark propagators possible (on small volumes)
- to date, quark field smeared using covariant Laplacian

\[ \bar{\psi}(x) = \left(1 + \frac{\sigma_s}{4n_c} \tilde{\Delta} \right)^{n^c} \psi(x) \]

- express in term of eigenvectors/eigenvalues of Laplacian

\[ \bar{\psi}(x) = \left(1 + \frac{\sigma_s}{4n_c} \tilde{\Delta} \right)^{n^c} \sum_k |\varphi_k\rangle \langle \varphi_k | \psi(x) \]

\[ = \sum_k \left(1 + \frac{\sigma_s \lambda_k}{4n_c} \right)^{n^c} |\varphi_k\rangle \langle \varphi_k | \psi(x) \]

- truncate sum and set weights to unity \( \rightarrow \) Laplacian Heaviside
Getting to know the Laplacian

- spectrum of the covariant Laplacian
- left: dependence on lattice size; right: dependence on link smearing
Choosing the smearing cut-off

- Laplacian Heaviside (Laph) quark smearing

\[ \tilde{\psi}(x) = \Theta \left( \sigma_s^2 + \tilde{\Delta} \right) \psi(x) \]

\[ \approx \sum_{k=1}^{N_{\text{max}}} |\phi_k \rangle \langle \phi_k| \psi(x) \]

- choose smearing cut-off based on minimizing excited-state contamination, keep noise small
  - behavior of nucleon $t=1$ effective masses
Tests of Laplacian Heaviside smearing

- comparison of $\rho$-meson effective masses using same number of gauge-field configurations

- typically need about 32 modes on $16^3$ lattice
- about 128 modes on $24^3$ lattice
Nucleon operator pruning

- $N_f=2+1$ on $16^3 \times 128$ lattice, $m_\pi = 380$ MeV (100 configs, 32 eigvecs)

![Graph showing Nucleon Mass Spectrum (Exp)]
Delta operator pruning

- $N_f=2+1$ on $16^3 \times 128$ lattice, $m_\pi = 380$ MeV (481 configs, 32 eigvecs)

Delta Mass Spectrum (Exp)
Sigma operator pruning

- $N_f=2+1$ on $16^3 \times 128$ lattice, $m_{\pi} = 380$ MeV (100 configs, 32 eigvecs)
Isovector G-parity odd mesons

- $N_f=2+1$ on $16^3 \times 128$ lattice, $m_\pi = 380$ MeV (100 configs, 32 eigvecs)

---

**Simulated Isovector Meson Spectrum, $V=16^3$**

**Experimental Isovector Meson Spectrum**

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- $a$ mesons
- $\pi$ mesons

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Kaons

- $N_f=2+1$ on $16^3 \times 128$ lattice, $m_\pi = 380$ MeV (100 configs, 32 eigvecs)
Stochastic estimation of quark propagators

- new Laph quark smearing method allows exact computation of all-to-all quark propagators
- *but* number of Laplacian eigenvectors needed becomes prohibitively large on large lattices
  - 128 modes needed on $24^3$ lattice
- computational method is rather cumbersome, too
- need to resort to stochastic estimation
Stochastic estimation

- quark propagator is just inverse of Dirac matrix $M$
- noise vectors $\eta$ satisfying $E(\eta_i) = 0$ and $E(\eta_i \eta_j^*) = \delta_{ij}$ are useful for stochastic estimates of inverse of a matrix $M$
- $Z_4$ noise is used $\{1, i, -1, -i\}$
- define $X(\eta) = M^{-1} \eta$ then

$$E(X_i \eta_j^*) = E\left(\sum_k M^{-1}_{ik} \eta_k \eta_j^*\right) = \sum_k M^{-1}_{ik} E(\eta_k \eta_j^*) = \sum_k M^{-1}_{ik} \delta_{kj} = M^{-1}_{ij}$$

- if can solve $M X^{(r)} = \eta^{(r)}$ for each of $N_R$ noise vectors $\eta^{(r)}$ then we have a Monte Carlo estimate of all elements of $M^{-1}$:

$$M^{-1}_{ij} \approx \frac{1}{N_R} \sum_{r=1}^{N_R} X_i^{(r)} \eta_j^{(r)*}$$

- variances in above estimates usually unacceptably large
- introduce variance reduction using source *dilution*
Source dilution for single matrix inverse

- dilution introduces a complete set of projections:
  \[ P^{(a)} P^{(b)} = \delta^{ab} P^{(a)}, \quad \sum_a P^{(a)} = 1, \quad P^{(a)\dagger} = P^{(a)} \]
- observe that
  \[
  M^{-1}_{ij} = M^{-1}_{ik} \delta_{kj} = \sum_a M^{-1}_{ik} P^{(a)} = \sum_a M^{-1}_{ik} P^{(a)} \delta_{kj} P^{(a)}
  \]
  \[
  = \sum_a M^{-1}_{ik} P^{(a)} E(\eta_k \eta_j^*) P^{(a)}_{jj} = \sum_a M^{-1}_{ik} E(P^{(a)}_{kk'} \eta_k \eta_j^* P^{(a)}_{jj})
  \]
- define
  \[
  \eta^{[a]}_k = P^{(a)}_{kk'} \eta_{k'}, \quad \eta^{[a]*}_j = \eta_j^* P^{(a)}_{jj}, \quad X^{[a]}_k = M^{-1}_{kj} \eta^{[a]}_j
  \]
  so that
  \[
  M^{-1}_{ij} = \sum_a E(X^{[a]}_i \eta^{[a]*}_j)
  \]
- Monte Carlo estimate is now
  \[
  M^{-1}_{ij} \approx \frac{1}{N_R} \sum_{r=1}^{N_R} \sum_a X^{(r)[a]}_i \eta^{(r)[a]*}_j
  \]
- \[\sum_a \eta^{[a]}_i \eta^{[a]*}_j \] has same expected value as \[\eta_i \eta_j^*\], but reduced variance (statistical zeros \(\rightarrow\) exact)
Earlier schemes

- Introduce $Z_N$ noise in color, spin, space, time
  \[ \eta_{\alpha\alpha}(\tilde{x},t) \]

- Time dilution (particularly effective)
  \[ P_{a\alpha;b\beta}^{(B)}(\tilde{x},t;\tilde{y},t') = \delta_{ab} \delta_{\alpha\beta} \delta(\tilde{x},\tilde{y}) \delta_{Bt} \delta_{B't'}, \quad B = 0,1,\ldots,N_t-1 \]

- Spin dilution
  \[ P_{a\alpha;b\beta}^{(B)}(\tilde{x},t;\tilde{y},t') = \delta_{ab} \delta_{B\alpha} \delta_{B\beta} \delta(\tilde{x},\tilde{y}) \delta_{t't'}, \quad B = 0,1,2,3 \]

- Color dilution
  \[ P_{a\alpha;b\beta}^{(B)}(\tilde{x},t;\tilde{y},t') = \delta_{Ba} \delta_{Bb} \delta_{\alpha\beta} \delta(\tilde{x},\tilde{y}) \delta_{t't'}, \quad B = 0,1,2 \]

- Spatial dilutions?
  - even-odd
**Dilution tests (old method)**

- 100 quenched configs, $12^3 \times 48$ anisotropic Wilson lattice

![Graph showing relative error $\sigma$ vs. $1/N_{invr}^{1/2}$ for $C(t=5)$ for single-site nucleon](image)
New stochastic Laph method

- Introduce $Z_N$ noise in Laph subspace
  \[ \rho_{\alpha k}(t) \quad t = \text{time}, \alpha = \text{spin}, k = \text{eigenvector number} \]

- Time dilution (particularly effective)
  \[ P_{\alpha k;\beta l}^{(B)}(t; t') = \delta_{kl} \delta_{\alpha\beta} \delta_{Bl} \delta_{B't'}, \quad B = 0, 1, \ldots, N_t - 1 \]

- Spin dilution
  \[ P_{\alpha k;\beta l}^{(B)}(t; t') = \delta_{kl} \delta_{B\alpha} \delta_{B\beta} \delta_{It'}, \quad B = 0, 1, 2, 3 \]

- Laplacian eigenvector dilution
  - define \[ P_{\alpha k;\beta l}^{(B)}(t; t') = \delta_{Bk} \delta_{Bl} \delta_{\alpha\beta} \delta_{It'}, \quad B = 0, 1, 2, N_{\text{eig}} - 1 \]
  - group projectors together
    - by blocking
    - as interlaced
Old stochastic versus new stochastic

- new method (open symbols) has dramatically decreased variance
- test using a triply-displaced-T nucleon operator
Old stochastic versus new stochastic (zoom in)

- zoom in of triply-displaced-T nucleon plot on last slide
Old stochastic versus new stochastic

- comparison using single-site $\pi$ operator
Old stochastic versus new stochastic

- zoom in of $\pi$ plot on previous slide
Mild volume dependence

- $16^3$ lattice versus $20^3$ lattice, both old and new stochastic methods
- test using triply-displaced-$T$ nucleon operator
Mild volume dependence

- zoom in of plot on previous slide
Source-sink factorization

- baryon correlator has form
  \[ C_{\overline{I}\overline{I}} = c_{ijk}^{(l)} c_{\overline{ijk}}^{(\overline{l})} Q^A_{i\overline{i}} Q^B_{j\overline{j}} Q^C_{k\overline{k}} \]

- stochastic estimates with dilution
  \[ C_{\overline{I}\overline{I}} \approx \frac{1}{N_R} \sum_r \sum_{d_A d_B d_C} c_{ijk}^{(l)} c_{\overline{ijk}}^{(\overline{l})} (\phi_i^{(Ar)}[d_A] \eta_{\overline{i}}^{(Ar)}[d_A]^*) \]
  \[ \times \left( \phi_j^{(Br)}[d_B] \eta_{\overline{j}}^{(Br)}[d_B]^* \right) \left( \phi_k^{(Cr)}[d_C] \eta_{\overline{k}}^{(Cr)}[d_C]^* \right) \]

- define
  \[ \Gamma^{(r)}_{I}[d_A d_B d_C] = c_{ijk}^{(l)} \phi_i^{(Ar)}[d_A] \phi_j^{(Br)}[d_B] \phi_k^{(Cr)}[d_C] \]
  \[ \Omega^{(r)}_{I}[d_A d_B d_C] = c_{ijk}^{(l)} \eta_i^{(Ar)}[d_A] \eta_j^{(Br)}[d_B] \eta_k^{(Cr)}[d_C] \]

- correlator becomes dot product of source vector with sink vector
  \[ C_{\overline{I}\overline{I}} \approx \frac{1}{N_R} \sum_r \sum_{d_A d_B d_C} \Gamma^{(r)}_{I}[d_A d_B d_C] \Omega_{\overline{I}}^{(r)[d_A d_B d_C]^*} \]

- store ABC permutations to handle Wick orderings
Moving $\pi$ and $a$ mesons

- first step towards including multi-hadron operators:
  - moving single hadrons
  - results below have one unit of on-axis momentum
  - projections onto space group irreps (see J. Foley talk)
Configuration generation

- significant time on USQCD (DOE) and NSF computing resources
- anisotropic clover fermion action (with stout links) and anisotropic improved gauge action
  - tunings of couplings, aspect ratio, lattice spacing done
- anisotropic Wilson configurations generated during clover tuning
- current goal:
  - three lattice spacings: \( a = 0.125 \text{ fm}, 0.10 \text{ fm}, 0.08 \text{ fm} \)
  - three volumes: \( V = (3.2 \text{ fm})^4, (4.0 \text{ fm})^4, (5.0 \text{ fm})^4 \)
  - 2+1 flavors, \( m_\pi \sim 350 \text{ MeV}, 220 \text{ MeV}, 180 \text{ MeV} \)
- USQCD Chroma software suite
Resonances in a box: an example

- Consider simple 1D quantum mechanics example
- Hamiltonian

\[ H = \frac{1}{2} p^2 + V(x) \quad V(x) = (x^4 - 3) e^{-x^2/2} \]
1D example spectrum

- Spectrum has two bound states, two resonances for $E<4$
Scattering phase shifts

- define even- and odd-parity phase shifts $\delta_{\pm}$
- phase between transmitted and incident wave
Spectrum in box (periodic b.c.)

- Spectrum is discrete in box (momentum quantized)
- Narrow resonance is avoided level crossing, broad resonance?

Dotted curves are $V=0$ spectrum
Unstable particles (resonances)

- our computations done in a periodic box
  - momenta quantized
  - discrete energy spectrum of stationary states → single hadron, 2 hadron, …
- how to extract resonance info from box info?
- approach 1: crude scan
  - if goal is exploration only → “ferret” out resonances
  - spectrum in a few volumes
  - placement, pattern of multi-particle states known
  - resonances → level distortion near energy with little volume dependence
Unstable particles (resonances)

- **approach 2**: phase-shift method
  - if goal is high precision → work much harder!
  - relate finite-box energy of multi-particle *model* to infinite-volume phase shifts
  - evaluate energy spectrum in several volumes to compute phase shifts using formula from previous step
  - deduce resonance parameters from phase shifts
  - early references
    - B. DeWitt, PR 103, 1565 (1956) (sphere)

- **approach 3**: histogram method
  - recent work for pion-nucleon system:

- **new approach**: construct effective theory of hadrons?
Summary

- goal: to wring out hadron spectrum from QCD Lagrangian using Monte Carlo methods on a space-time lattice
  - baryons, mesons (and glueballs, hybrids, tetraquarks, …)
- discussed extraction of excited states in Monte Carlo calculations
  - correlation matrices needed
  - operators with very good overlaps onto states of interest
- must extract all states lying below a state of interest
  - as pion get lighter, more and more multi-hadron states
- multi-hadron operators \(\rightarrow\) relative momenta
  - need for slice-to-slice quark propagators
- new stochastic Laph method \(\rightarrow\) end game in sight?
- interpretation of finite-box energies