Sponsoring Mobile Data: An Economic Analysis of the Impact on Users and Content Providers

Carlee Joe-Wong*, Sangtae Ha†, Mung Chiang*
*Princeton University, Princeton, NJ, USA †University of Colorado, Boulder, CO, USA
Emails: {cjoe, chiangm}@princeton.edu, sangtae.ha@colorado.edu

***Abstract*—In January 2014, AT&T introduced *sponsored data* to the U.S. mobile data market, allowing content providers (CPs) to subsidize users' cost of mobile data. As sponsored data gains traction in industry, it is important to understand its implications. This work considers CPs' choice of how much content to sponsor and the implications for users, CPs, and ISPs (Internet service providers). We first formulate a model of user, CP, and ISP interaction for heterogeneous users and CPs and derive their optimal behaviors. We then show that these behaviors can reverse our intuition as to how user demand and utility change with different user and CP characteristics. While all three parties can benefit from sponsored data, we find that sponsorship disproportionately favors less cost-constrained CPs and more cost-constrained users, exacerbating CP inequalities but making user demand more even. We also show that users' utilities increase more than CPs' with sponsored data. We finally illustrate these results in practice through numerical simulations with data from a commercial pricing trial and introduce a framework for CPs to decide which, in addition to how much, content to sponsor.**

I. Introduction

As demand for mobile data grows, ISPs (Internet service providers) are turning to new types of smart data pricing (SDP) to generate revenue for expanding their network capacity [1]. Though much SDP research studies different ways to charge users for data [2], *sponsored data* instead introduces a new party to data pricing: content providers (CPs). Facebook has long sponsored data for its mobile site in developing markets [3], and AT&T began offering sponsored data in January 2014 [4]. The ISP FreedomPop has introduced free, 100% sponsored data plans [5], and the startups Syntonic Wireless and DataMi[1] offer marketplaces for CPs to sponsor data for different apps.[2]

A. Sponsoring Mobile Data

Under AT&T's sponsored data plan, CPs can split the cost of transferring mobile data traffic over ISP networks with end users. Figure 1 illustrates the data and payment flows between end users, CPs, and ISPs. Sponsored content is represented by the dashed arrow showing payments from CPs to the ISP, while the solid arrow represents user payments.

Content sponsorship has the potential to benefit users, CPs, and ISPs: users are charged lower prices due to CP subsidies, CPs can attract more traffic as users increase their demand, and ISPs can generate more revenue. Yet such plans have raised

[1]http://www.syntonicwireless.com, http://www.datami.com

[2]Throughout the paper, we use "sponsored content" and "sponsored data" interchangeably; CPs sponsor content by sponsoring the data it consumes.

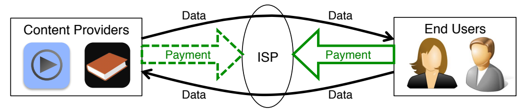
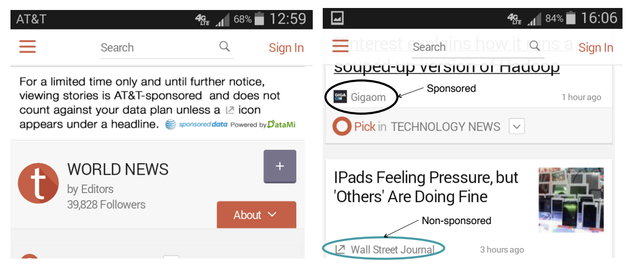


Fig. 1: Data and payment flows between users, CPs, and ISPs.



(a) Sponsored data banner. (b) In-app sponsorship.

Fig. 2: Trove's sponsored data website on LTE.

concerns over the possible advantage they give to larger CPs that can better afford to sponsor data [6]. These concerns echo the controversy over CPs paying for higher Quality-of-Service (QoS) for their users. In this paper, we do not consider QoS: we show that even *without* a QoS component, sponsored data can exacerbate profit and demand unevenness among different CPs, but can also even out demand among different types of users and benefit users more than CPs.

In current sponsored data implementations, CPs decide which content to sponsor and mark it as such in the app, as in Figure 2's example of sponsored news stories on Trove. The CP app routes all sponsored traffic through per-app proxies or VPNs implemented on users' devices, allowing ISPs to identify sponsored traffic and bill the CP. With this architecture, CPs can easily sponsor different content for different users. Though not trivial to implement, per-app VPN support is becoming more widespread in iOS [7] and Android [8].

In our sponsored data model, CPs decide how much data to sponsor for each user, unlike previous works in which ISPs choose the amount sponsored [9], [10]. While ISPs can influence CPs by changing their data prices, sponsored data's effect on the mobile data market will be determined by *how much data CPs decide to sponsor.* It is this decision, and the impact on users and CPs, that we examine in this paper.

B. Modeling Content Provider Behavior

Since CPs decide how much data to sponsor for different users, an accurate model of their behavior must include the full heterogeneity of CP and user data valuations. We therefore explicitly model not only differences in user price sensitivity,

TABLE I: CP benefit from increased demand.

CP Type	Benefit source	Benefit from usage	Example
Revenue	Ad revenue	Linear in usage	Pandora
	Subscriptions	Linear in usage	Vimeo
	Subscriptions	Linear in # of users	Netflix
Promotion	Goodwill	Concave in usage	Promotions
	Usage	Concave in usage	Enterprise

as was done in [11], but also the different benefits that CPs receive from greater user demand, as shown in Table I. While some CPs take in ad revenue, as considered in [12], others rely on subscription revenue, and still other CPs benefit from user goodwill or usage itself. We thus classify CPs as "revenue CPs," which benefit from usage as it contributes to revenue, and "promotion CPs," which directly benefit from usage.

Most revenue CPs rely on either ads or freemium subscriptions to make money [13]. For apps that rely on ads like Pandora or Facebook [14], [15], revenue grows linearly with usage, as the number of ads shown is often proportional to the amount of content consumed (e.g., ads at regular intervals between songs or news stories). Other apps charge users per unit of content consumed, e.g., Vimeo's per-video viewing fees. Still others, like Netflix or freemium apps, offer flat-fee subscriptions that do not depend on usage volume. These CPs are arguably less likely to sponsor data to increase subscriptions, since non-subscribers likely derive little utility from the content itself and would require high subsidies.

Promotion CPs benefit directly from increased usage. For instance, a new photo-sharing app may sponsor data to attract users and usage in the early stages of its release. Another type of CP, enterprises, might subsidize employee usage of company apps, encouraging them to work more while out of the office. In both cases, the CP's benefit from increased demand is concave rather than linear, mirroring users' diminishing marginal utility from more data usage.

C. Implications of Sponsored Data

By fully accounting for user and CP heterogeneity, we find that content sponsorship can:

- *Reverse our intuition as to how user demand changes with different user and CP characteristics.* For instance, while we would expect user utility to decrease as CPs show more ads, with sponsorship revenue CPs may sponsor more data if they show more ads, increasing users' utility.
- *Disproportionately benefit more cost-aware users and less cost-aware CPs*[3]. We thus justify the concern that sponsored data will exacerbate the advantage of larger, less cost-aware CPs [6]. However, sponsored content's benefit to more cost-conscious users implies that it evens out the distributions of demand and utility across users.
- *Benefit users more than CPs.* User utility increases proportionally more than a CP's when the CP chooses the amount of data sponsored so as to maximize its utility.

After briefly reviewing related work in Section II, we derive the optimal behavior for users, CPs, and ISPs in Section III and analyze the implications in Section IV. In Section V, we consider sponsored data in practice, illustrating our results with data from a commercial pricing trial and proposing a framework for CPs to decide which pieces of content to sponsor. We conclude in Section VI. All proofs are in Appendix A.

[3]We call CPs less "cost-aware" if they experience more utility from user demand relative to the cost of sponsorship.

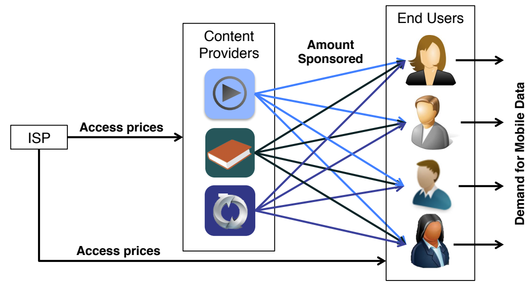


Fig. 3: Three-stage decision hierarchy for users, CPs, and ISPs.

II. Related Work

Much of the prior work on sponsored content either focuses on ISPs' optimal actions in splitting costs between CPs and users or includes QoS prioritization and examines the implications for net neutrality. For instance, CPs might explicitly pay ISPs extra fees for higher QoS [16]. Many such works use game theory to identify ISPs' and CPs' optimal actions and the consequences for network neutrality [17]. When user and CP demand are defined in terms of bandwidth speed (i.e., QoS), [9] considers monopolistic and perfectly competitive ISPs' optimal amount to charge CPs and end users, while [10] proposes a similar model that includes transit and user-facing ISPs. However, these works allow ISPs, instead of CPs, to determine the amount of sponsorship. They therefore do not reflect the full heterogeneity of CP behavior.

Other works study the effects of sponsored data on CPs and end users, e.g., [11] and [18] examine CPs' market share when CPs sponsor data for homogeneous users or pay fixed side payments to ISPs respectively. CP sponsorship of advertisements is explicitly addressed in [12], which accounts for users' probability of viewing different ads and monthly data caps. Implementation challenges are discussed in [19].

III. Sponsored Data Model

We consider three players in the sponsored data ecosystem: users, CPs, and ISPs. They make decisions in three stages, as shown in Figure 3: first, the ISP chooses the prices to charge users and CPs. CPs then decide how much content to sponsor, and finally users choose how much content to consume from each CP, depending on the amount sponsored and ISP price. CPs can sponsor different amounts of data for different types of users. Each party selfishly maximizes its own utility subject to others' decisions. We assume a monopolistic ISP; while many ISPs, e.g., in the US, are oligopolists, due to low churn rates they are often effective monopolies [20].

A. End Users' Decisions

Suppose that N users and M CPs exchange traffic over the ISP's network. Each user i receives a utility $V_{i,j}(x_{i,j}, p_u, \gamma_{i,j})$

from CP j, where $x_{i,j}$ is the monthly volume of content that user i consumes from CP j, p_u is the unit price of data that the ISP charges users, and $\gamma_{i,j}$ is the fraction of content sponsored by CP j for user i.[4] We assume that users incur a linear data cost, as is commonly done [9], [11], [12]. For instance, users can choose one of several data plans that charge different amounts for different monthly data caps [21].

Users are affected by two variables chosen by the CP: the volume of ads per volume of content, s_j, and $\gamma_{i,j}$. We assume that s_j is constant over all of the CP's content; for instance, Pandora plays ads at regular intervals between songs. While in theory s_j can take any positive value, we assume $s_j \in [0,1)$ since most CPs provide more content than ads. We note that $\gamma_{i,j}$ represents the *aggregate* fraction of data sponsored over one month, including sponsorship of ads and content; thus, $\gamma_{i,j} \in [0, 1+s_j]$. The scenario before sponsored data corresponds to taking $\gamma_{i,j} \equiv 0$ for all users i and CPs j. With sponsored data, the user pays for $(1-\gamma_{i,j}+s_j)\, x_{i,j}$ amount of data; thus, her data cost is $p_u\,(1-\gamma_{i,j}+s_j)\, x_{i,j}$.

We suppose that, absent the data cost, a user derives utility $U_{i,j}(x_{i,j}(1+r_{i,j}s_j))$ from consuming $x_{i,j}$ amount of content from CP j, where $U_{i,j}$ is a concave utility function. The factor $r_{i,j}$ represents user i's average ad click-through rate for CP j, i.e., the fraction of ads contributing to user utility. While users rarely derive utility from ads, clicking on an ad indicates that they find it entertaining or useful. We assume that $r_{i,j}$ is independent of sponsorship (the bottleneck to users' clicking on ads is likely their lack of interest, not the data cost). Each user i's utility function for CP j is then

$$V_{i,j} = c_{i,j}\frac{\left(x_{i,j}\,(1+r_{i,j}s_j)\right)^{1-\alpha_{i,j}}}{1-\alpha_{i,j}} - p_u\,(1-\gamma_{i,j}+s_j)\, x_{i,j}, \tag{1}$$

where we take $U_{i,j}(x) = c_{i,j}x^{1-\alpha_{i,j}}/(1-\alpha_{i,j})$, the α-fair utility function with $\alpha_{i,j} \in [0,1)$ and a scaling factor $c_{i,j} > 0$. A user's total utility is the sum of his utilities from each CP: usage of one CP does not affect the utility from others, e.g., browsing Facebook does not affect the utility of watching Hulu. User i's optimal demand for data from each CP j is then

$$x^*_{i,j}\left(p_u\,(1-\gamma_{i,j}+s_j)\right) = \left(\frac{p_u\,(1-\gamma_{i,j}+s_j)}{c_{i,j}\,(1+r_{i,j}s_j)^{-\alpha_{i,j}}}\right)^{\frac{-1}{\alpha_{i,j}}}. \tag{2}$$

B. Content Provider Sponsorship

As discussed in Section I-B, the amount of data that a CP sponsors depends on the CP's benefit from user demand, i.e., whether the CP is a "revenue" or "promotion" CP. These two scenarios can be viewed as special cases of a general CP utility model. As with end users, we suppose that CPs' utility functions include a utility and a cost component. We use $W_{i,j}$ to denote CP j's overall utility function for user i:

$$W_{i,j}\,(\gamma_{i,j}) = \overline{U}_{i,j}(x^*_{i,j}) - p_c\gamma_{i,j}x^*_{i,j}, \tag{3}$$

[4]In practice, a user i can represent a group of users with similar behavior.

where $\overline{U}_{i,j}(x) = d_{i,j}x^{1-\beta_{i,j}}/(1-\beta_{i,j})$, with $\beta_{i,j} \in [0,1)$ and $d_{i,j}$ a positive scaling factor, specifies the CP's utility from data usage. Here $x^*_{i,j}$ is the user demand (2). The term $p_c\gamma_{i,j}x^*_{i,j}$ represents the cost of delivering content for each user i; p_c is the unit data price that ISPs charge CPs. Substituting (2) into (3), we now find that (3) equals

$$\begin{aligned} W_{i,j}\,(\gamma_{i,j}) =& \frac{d_{i,j}}{1-\beta_{i,j}}\left(\left(\frac{p_u(1-\gamma_{i,j}+s_j)}{c_{i,j}\,(1+r_{i,j}s_j)^{-\alpha_{i,j}}}\right)^{\frac{-1}{\alpha_{i,j}}}\right)^{1-\beta_{i,j}} \\ & - \frac{p_c\gamma_{i,j}}{1+r_{i,j}s_j}\left(\frac{p_u(1-\gamma_{i,j}+s_j)}{c_{i,j}}\right)^{\frac{-1}{\alpha_{i,j}}}. \end{aligned}$$

We assume that s_j, the fraction of ads per content volume, does not change with sponsorship (e.g., to avoid disrupting users' experience). The CP chooses $\gamma_{i,j}$ to maximize $\sum_{i=1}^N W_{i,j}$, with the constraint $\gamma_{i,j} \in [0, 1+s_j]$. If the CP must sponsor the same amount of content for all users i (e.g., due to lack of information on different users' behavior), $\gamma_{i,j} \equiv \gamma_j$ for all i, turning the optimization into a bounded line search.

We now consider the CP's optimization problem for Section I-B's revenue and promotion CPs:

1) Revenue CPs: We first consider a CP whose utility $\overline{U}_{i,j}$ is its revenue. We assume that the CP's revenue is proportional to user demand, as discussed in Section I-B and Table I.[5] Aside from transport costs paid to the ISP, we do not explicitly consider CP data costs; these may be included by reducing the revenue per unit volume by a constant marginal cost.

We now take $\beta_{i,j} = 0$ and $d_{i,j}$ to be the marginal revenue per unit of content in (3), making (3) the CP's revenue less the cost of sponsoring data. For instance, CPs deriving revenue from a cost-per-click advertising model would take $d_{i,j} = ar_{i,j}s_j$, where a is the revenue per volume of ads clicked on. We can calculate a from the cost per click and average ad volume.[6] Thus, we find the CP utility function

$$\begin{aligned} &(d_{i,j} - p_c\gamma_{i,j})\, x^*_{i,j}\left(p_u\,(1-\gamma_{i,j}+s_j)\right) \\ &= \left(\frac{d_{i,j}-p_c\gamma_{i,j}}{1+r_{i,j}s_j}\right)\left(\frac{p_u(1-\gamma_{i,j}+s_j)}{c_{i,j}}\right)^{\frac{-1}{\alpha_{i,j}}}, \end{aligned} \tag{4}$$

yielding the optimization problem

$$\max_{\gamma_{i,j}} \sum_{i=1}^N \left(\frac{d_{i,j}-p_c\gamma_{i,j}}{1+r_{i,j}s_j}\right)\left(\frac{p_u(1-\gamma_{i,j}+s_j)}{c_{i,j}}\right)^{\frac{-1}{\alpha_{i,j}}} \tag{5}$$

$$\text{s.t. } \gamma_{i,j} \in [0, 1+s_j]. \tag{6}$$

Proposition 1: Suppose that $d_{i,j} < p_c(1+s_j)$ for all users i. Then (5–6) has the optimal solution

$$\gamma^*_{i,j}\,(p_c, p_u) = \max\left\{0, \frac{d_{i,j}}{p_c(1-\alpha_{i,j})} - \frac{\alpha_{i,j}\,(1+s_j)}{1-\alpha_{i,j}}\right\}. \tag{7}$$

For instance, if $d_{i,j} = ar_{i,j}s_j$, users' click-through rates $r_{i,j}$ are generally small ($< 5\%$ [22]) and Prop. 1's condition

[5]CPs with flat subscription fees would choose the minimum $\gamma_{i,j}$ such that each user i's utility is higher than the cost of subscription. As discussed in Section I-B, these CPs will be less likely to sponsor data.

[6]Similarly, $d_{i,j} = as_j$ if the CP receives cost-per-mille ad revenue.

$d_{i,j} < p_c(1+s_j)$ is reasonable. By setting $\gamma^*_{i,j} > 0$ in (7), we can find conditions under which a CP wishes to sponsor data:

Corollary 1: CP j sponsors data for user i ($\gamma^*_{i,j} > 0$) if and only if $d_{i,j} > \alpha_{i,j} p_c (1+s_j)$.

Thus, if a CP's marginal revenue $d_{i,j}$ is sufficiently small compared to its data price p_c, it will not sponsor any data. As user utility becomes more concave ($\alpha_{i,j}$ increases), the CP becomes less likely to sponsor data: its sponsorship has less effect on user demand and is thus less profitable. However, $\gamma^*_{i,j}$ is independent of $c_{i,j}$, indicating that revenue CPs will not discriminate among users with different utility "levels."

2) Promotion CPs: Promotion CPs benefit directly from usage. These CPs can try to attract users by optimizing their users' utilities, e.g., in order to promote their brand or app. Similarly, enterprise CPs will have the same utility function as their employees when sponsoring company apps: the benefit of increased usage for both is its contribution to employee productivity. Though promotion CPs can sponsor ads, we assume that their ad revenue is negligible compared to their utility from usage (e.g., recently released apps may earn very little revenue per ad since advertisers are unfamiliar with their value). Enterprise CPs will likely not show ads.

For promotion CPs, we thus take $\beta_{i,j} = \alpha_{i,j}$ in (3): user and CP utility components have the same shape.[7] By taking the CP and user scaling factors $d_{i,j}$ and $c_{i,j}$ to be different, we can introduce different weights on the utility, e.g., if CPs care less about cost relative to gaining demand than users care about their data cost. We then solve for the optimal $\gamma_{i,j}$:

Proposition 2: Suppose that $\beta_{i,j} = \alpha_{i,j} > 0$. Then $W_{i,j}$ is maximized with respect to $\gamma_{i,j} \in [0, 1+s_j]$ at

$$\gamma^*_{i,j} = \max\left\{0, \frac{(1+s_j)\left(d_{i,j}\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}} p_u - \alpha_{i,j} c_{i,j} p_c\right)}{(1-\alpha_{i,j})c_{i,j}p_c + d_{i,j}\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}} p_u}\right\}. \tag{8}$$

Promotion CPs thus sponsor data only if the price p_u paid by users and the CP utility scaling $d_{i,j}$ are sufficiently high:

Corollary 2: CP j sponsors data for user i if only if $d_{i,j}\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}} p_u > \alpha_{i,j} c_{i,j} p_c$.

Intuitively, if users' data is already inexpensive (small p_u) or the CP cares more about cost than promoting usage (small $d_{i,j}$), the CP has no incentive to sponsor data. In contrast to revenue CPs, however, $\gamma^*_{i,j}$ depends on users' utility scaling factors $c_{i,j}$. Promotion CPs experience diminishing marginal utility with usage volume, so they will be less likely to sponsor data for users with higher $c_{i,j}$, whose demand $x^*_{i,j}$ is already large without sponsorship (cf. (2)).

C. ISP Price Optimization

Like the CPs, the ISP chooses the prices p_c and p_u so as to maximize its profit.[8] We suppose that the ISP has a finite amount of available capacity in its network, e.g., LTE base stations can handle only a finite amount of traffic at any given time. We translate this instantaneous capacity into a maximum monthly demand for data, X, by supposing that the peak demand over time is a function of the total demand for data. We thus introduce the constraint $\sum_{i,j}(1+s_j)\, x^*_{i,j}\left(\pi^*_{i,j}\right) \le X$, where $(1+s_j)\, x^*_{i,j}$ is the total volume of data pushed over the ISP's network by user i for CP j, and $\pi^*_{i,j}(p_c, p_u) = p_u\left(1 - \gamma^*_{i,j}(p_c, p_u) + s_j\right)$ denotes user i's *effective data price* for each CP j. The ISP then wishes to maximize its total profit subject to this capacity constraint, i.e., to solve

$$\max_{p_c, p_u \ge 0} \sum_{i=1}^{N}\sum_{j=1}^{M}\left(\pi^*_{i,j} + p_c\gamma^*_{i,j}\right) x^*_{i,j}\left(\pi^*_{i,j}\right) \tag{9}$$

$$\text{s.t.} \sum_{i=1}^{N}\sum_{j=1}^{M}(1+s_j)\, x^*_{i,j}\left(\pi^*_{i,j}\right) \le X \tag{10}$$

We can solve (9–10) by noting that both $x^*_{i,j}\left(\pi^*_{i,j}\right)$ and $\left(\pi^*_{i,j} + p_c\gamma^*_{i,j}\right) x^*_{i,j}\left(\pi^*_{i,j}\right)$ are decreasing in p_c:

Proposition 3: If each CP optimally chooses $\gamma_{i,j}$ so as to maximize (3), $x^*_{i,j}\left(\pi^*_{i,j}\right)$ and (9) are both decreasing in p_c. Thus, for any given p_u, the optimal p_c is the unique minimum value of p_c for which either (10) is satisfied with equality or $d_{i,j} = p_c(1+s_j)$ for some revenue CP j. The optimal value of p_u can be found by a bounded line search.

IV. Impact on Users and CPs

We now consider the implications of user, CP, and ISP behavior. We show that sponsorship can reverse some trends of user behavior (Section IV-A) and that it favors more cost-aware users and less cost-aware CPs (Section IV-B). Section IV-C shows that users, CPs, and ISPs can all benefit from content sponsorship, but users benefit more than CPs.

We primarily characterize users and CPs by two attributes:

Definition 1 (Price elasticity): User i's *price elasticity* for CP j is defined in the usual economic manner as the % change in usage in response to a 1% change in price.

In our model, each user i's price elasticity for CP j is a constant $\alpha^{-1}_{i,j}$. As $\alpha_{i,j}$ increases, users have lower price elasticity and their demands are less sensitive to price changes.

Definition 2 (Cost awareness): A user i's *cost awareness* for CP j equals $c^{-1}_{i,j}$, the reciprocal of the $U_{i,j}$ scaling factor in users' utility (1).[9] Similarly, a CP's cost awareness is $d^{-1}_{i,j}$.

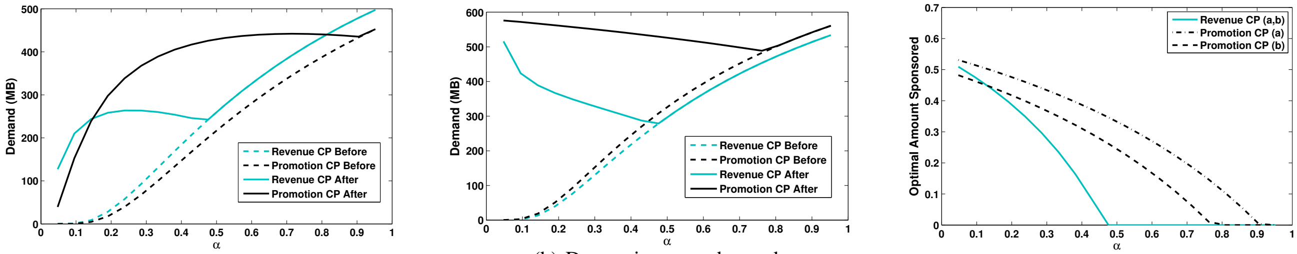
As $c_{i,j}$ or $d_{i,j}$ increases, the user or CP becomes less cost-aware: the utility function's cost term is weighted less compared to the utility from using data. For revenue CPs, "cost awareness" is simply the reciprocal of the marginal revenue per unit of content, e.g., $d_{i,j} = a r_{i,j} s_j$ for advertising revenue.

We use the term "user-CP pair" to refer to a given user's demand for and utility derived from a given CP, as well as

[7] While this appears to be a restrictive assumption, in Appendix B we show numerically that our results qualitatively hold for arbitrary $\beta_{i,j}$.

[8] The ISP may face price constraints, e.g., to avoid user anger over nominally higher prices. We then constrain $p_u \le P_u$, the current user price.

[9] Optimizing $V_{i,j}$ is equivalent to optimizing $V_{i,j}/c_{i,j}$, for which the utility scaling factor equals one and the cost term is scaled by $c^{-1}_{i,j}$.



(a) Non-monotonic user demand. (b) Decreasing user demand. (c) Optimal amounts sponsored.

Fig. 4: Demand can be (a) non-monotonic and (b) decreasing as users become less price-elastic for a revenue ((a) $c_{i,j} = 5.8$, (b) $c_{i,j} = 6.2$) and promotion ((a) $c_{i,j} = 5.3$, $d_{i,j} = 4.8$; (b) $c_{i,j} = 6.5$, $d_{i,j} = 5$) CP with (c) decreasing optimal amounts sponsored $\gamma^*_{i,j}$.

the CP's utility from that user. We also make a reasonable technical assumption on promotion CPs' cost awarenesses: $c_{i,j}p_c > d_{i,j}p_u\left(1 + r_{i,j}s_j\right)^{\alpha_{i,j}} \ln\left(1 + r_{i,j}s_j\right)$. We expect this to hold since users' click-through rates $r_{i,j}$ are generally $< 5\%$ [22], so $\ln\left(1 + r_{i,j}s_j\right) \approx 0$. To focus on the effect of CP rather than ISP actions, in this section we assume that the ISP does not change users' data price p_u with sponsorship, e.g., to avoid user anger over (nominally) higher prices.

A. Variation in Demand and Utility

We first show some numerical examples of interesting user behaviors before deriving conditions under which they are observed. In all simulations in the paper, unless otherwise noted we use the following parameters: ISP prices are $p_u = p_c =$ \$10/GB, which approximates current ISP data prices (e.g., AT&T's data plans vary from \$7.50/GB to \$25/GB [21]). Revenue CPs are assumed to make money from advertising with $a =$ \$1800 per GB of ads, based on a \$2 revenue per ad click [23] and an 880 KB average ad size, e.g., a short video. We assume that CPs carry an additional 15% of ads per content volume ($s_j = 0.15$), e.g., a 30-second ad for a 200-second video, and that users' ad click-through rate is 2% ($r_{i,j} = 0.02$) [22], [23]. Revenue CPs then have $d_{i,j} = ar_{i,j}s_j = 5.4$. In all figures, we use "before" and "after" to respectively denote the scenarios before (i.e., without) and after (with) sponsorship.

Without sponsorship, user demand $x^*_{i,j} = (p_u(1 + s_j)/c_{i,j})^{\frac{-1}{\alpha_{i,j}}}/(1 + r_{i,j}s_j)$; thus, if $p_u(1 + s_j) > c_{i,j}$, users' demands increase as they become less price-elastic (i.e., as $\alpha_{i,j}$ increases). However, as Figure 4a shows, with sponsorship demand can both increase ($\alpha_{i,j} \leq 0.238$ for revenue CPs, ≤ 0.714 for promotion CPs) and decrease ($\alpha_{i,j} \in [0.238, 0.476]$ for revenue CPs, $\alpha_{i,j} \in [0.714, 0.905]$ for promotion CPs) as $\alpha_{i,j}$ increases even when $p_u(1 + s_j)c_{i,j}$. We can derive sufficient conditions under which demand decreases as price elasticity decreases for all $\alpha_{i,j}$, as in Figure 4b:

Proposition 4: User demand increases as the CP sponsors more data ($\gamma_{i,j}$ increases). Moreover, a CP with $\gamma^*_{i,j} > 0$ will experience smaller demand $x^*_{i,j}$ as users become less price-elastic ($\alpha_{i,j}$ increases) if $p_u\left(p_c(1 + s_j) - d_{i,j}\right) < p_c c_{i,j}$.

Intuitively, as users become less price-elastic ($\alpha_{i,j}$ increases), they do not increase their demands as much in response to CPs' sponsoring data to lower prices. CPs thus do not benefit as much from sponsorship and sponsor less data.

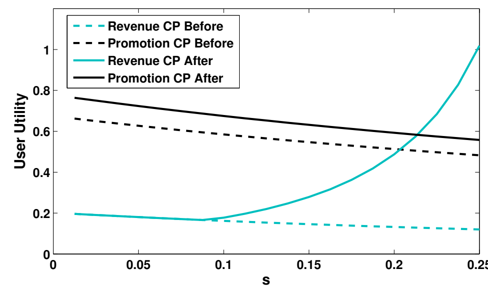


Fig. 5: User utilities increase with the fraction of ads shown s_j for a user of revenue CPs ($\alpha_{i,j} = 0.3$, $c_{i,j} = 4$) and decrease for a user of promotion CPs ($\alpha_{i,j} = 0.4$, $c_{i,j} = 4$, $d_{i,j} = 2$).

In Figure 4c, we see that $\gamma^*_{i,j}$ indeed decreases to zero as $\alpha_{i,j}$ increases. As $\gamma^*_{i,j}$ decreases, user demand $x^*_{i,j}$ also eventually decreases in Figures 4a and 4b. Proposition 4 shows that if a user's cost awareness is low enough ($c_{i,j}$ is high), the decrease in $\gamma^*_{i,j}$ will cause user demand to decrease at any $\alpha_{i,j}$ with $\gamma^*_{i,j} > 0$, as in Figure 4b.

We next consider the amount of ads sponsored, s_j. Without sponsorship, user utility decreases as s_j increases, since users must pay for the ads' data and do not experience much utility from ads. However, Figure 5 shows that with sponsorship, user utility can increase with s_j if $\gamma^*_{i,j} > 0$ ($s_j \geq 0.09$), though promotion CPs always observe a lower user utility as s_j increases. We now see that this increase is due to revenue CPs sponsoring more data as s_j increases:

Proposition 5: Consider a revenue CP j earning its revenue from ads. User i's utility $V_{i,j}(x^*_{i,j}\left(\pi^*_{i,j}\right), p_u, \gamma^*_{i,j})$ from CP j increases with ads shown (s_j) if $ar_{i,j} > p_c$ and $\gamma^*_{i,j} > 0$.

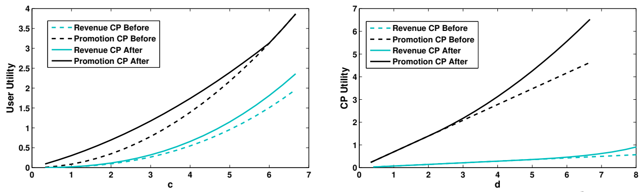
Revenue CPs' marginal revenue $ar_{i,j}s_j$ increases with s_j. Thus, if the expected revenue per volume of ads, $ar_{i,j}$, is sufficiently high, CPs sponsor more data as s_j increases. The resulting decrease in users' effective prices $\pi^*_{i,j}$ is enough to offset their lower utilities from ads. The a, $r_{i,j}$, and p_c values used in Figure 5 satisfy Prop. 5's condition, indicating that users will gain more utility from CPs that show more ads if those CPs also sponsor data. For instance, if the New York Times shows more ads than the Washington Post, it can attract more usage by sponsoring more data.

B. Distributions of Demand and Utility

We next examine the global implications of changes in demand and utility due to content sponsorship. In particular,

TABLE II: Jain's fairness index for distributions of (demands, user utilities, CP utilities) before and after sponsorship over all user-CP pairs.

Variable	Before (Revenue CP)	After (Revenue CP)	Before (Promotion CP)	After (Promotion CP)
$\alpha_{i,j}$ (Figure 4a)	(0.674, 0.214, 0.674)	(0.912, 0.219, 0.719)	(0.65, 0.21, 0.256)	(0.922, 0.228, 0.267)
$\alpha_{i,j}$ (Figure 4b)	(0.694, 0.218, 0.694)	(0.961, 0.224, 0.757)	(0.709, 0.221, 0.276)	(0.998, 0.237, 0.289)
$c_{i,j}$ (Figure 6a)	(0.503, 0.503, 0.503)	(0.503, 0.503, 0.503)	(0.57, 0.57, 0.768)	(0.856, 0.676, 0.868)
$d_{i,j}$ (Figure 6b)	(1, 1, 0.769)	(0.643, 0.859, 0.673)	(1, 1, 0.769)	(0.769, 0.927, 0.705)



(a) User utilities, varied $c_{i,j}$.

(b) CP utilities, varied $d_{i,j}$.

Fig. 6: (a) User utility with a revenue ($\alpha_{i,j} = 0.4$) and promotion ($\alpha_{i,j} = 0.5$, $d_{i,j} = 3$) CP; (b) CP utility for a user of revenue ($\alpha_{i,j} = 0.4$, $c_{i,j} = 4$) and promotion ($\alpha_{i,j} = 0.5$, $c_{i,j} = 4$) CPs.

we characterize the distributions of demand and utility:

Definition 3 (Fairness): Let the *fairness* of a set of user-CP variables $\{y_{i,j}\}$ (e.g., demand $x_{i,j}$) be given by a function $F(\{y_{i,j}\})$, where F is a homogeneous Schur-concave fairness measure satisfying the axioms in [24].

Table II shows the fairness of demands and utilities before and after sponsorship for the numerical examples given in this section. We use Jain's index as the fairness function; an index nearer to 1 indicates greater fairness. We also compare the fairness before and after sponsorship analytically using the following metric:

Definition 4 (Relative benefit): User i's *relative benefit* from CP j's sponsorship is the ratio of his or her utility $V_{i,j}\left(x^*_{i,j}(\pi^*_{i,j}), p_u, \gamma^*_{i,j}\right)$ with optimal sponsorship to the utility $V_{i,j}\left(x^*_{i,j}\left(p_u(1+s_j)\right), p_u, 0\right)$ before sponsorship. Similarly, a CP j's relative benefit from user i is the ratio of its utility $W_{i,j}$ with optimal sponsorship to that before sponsorship.

The *relative demand* of a user-CP pair is analogously defined as the ratio of demand with optimal sponsorship $x^*_{i,j}\left(\pi^*_{i,j}\right)$ to that without sponsorship. Table III shows the relative benefit and demand for both types of CPs and users.

To analyze the fairness of different distributions, we will make use of the following lemma:

Lemma 1: Consider two vectors $\vec{x}$ and $\vec{y}$ of length K with elements in increasing order (i.e., $\vec{x}_k > \vec{x}_l$ for $k > l$), such that $\vec{y}_k/\vec{x}_k \leq \vec{y}_l/\vec{x}_l$ $\forall$ indices $k \leq l$. Then $F(\vec{y}) \leq F(\vec{x})$.

We first consider the effect of a user's price elasticity on her relative demand and benefit. We find that the distribution of demand can become more or less fair:

Proposition 6 (Fairness across price elasticity): Consider a set S of users varying only in their price elasticities $\alpha^{-1}_{i,j}$ and a CP j with the same $d_{i,j}$ for all users. Within S, relative demand decreases as users become less price-elastic.

Suppose that $p_u(1+s_j) < c_{i,j}$. Then user demands across different price elasticities are distributed less fairly $\left(F\left(\{x_{i,j}\}_{(i,j)\in S}\right)\right.$ decreases) with sponsorship.

Conversely, if CP j is a revenue CP and $p_u(1+s_j) > c_{i,j}\exp\left(d_{i,j}/(p_c(1+s_j) - d_{i,j})\right)$, then $F\left(\left\{x^*_{i,j}\right\}_{(i,j)\in S}\right)$ increases with sponsorship. If CP j is a promotion CP, then if $p_u(1+s_j) > c_{i,j}$ and all users in a set $S_1 \subseteq S$ satisfy $\alpha_{i,j}p_c(1+s_j) > \exp(\alpha_{i,j})d_{i,j}(1+r_{i,j}s_j)^{\alpha_{i,j}}$, $F\left(\left\{x^*_{i,j}\right\}_{(i,j)\in S_1}\right)$ increases with sponsorship.

The results extend to multiple CPs with the same s_j and $d_{i,j}$.

As users become less price-elastic ($\alpha_{i,j}$ increases), their relative demands decrease since they increase their demands less in response to pricing changes. Thus, using Lemma 1, the fairness of user demands $x^*_{i,j}$ with and without sponsorship depends on whether $x^*_{i,j}$ increases or decreases as $\alpha_{i,j}$ increases (from Figure 4a, $x^*_{i,j}$ can be non-monotonic). In Figures 4a and 4b's simulations, we find that the fairness of user demands and utilities increases with sponsorship (Table II). While this is not always the case for users with different price elasticities (Prop. 6), *sponsorship does favor more cost-aware users, allowing them to disproportionately increase their utility and demand*:

Proposition 7 (Fairness across user cost awareness): Consider a set of users who vary only in their cost awarenesses $c^{-1}_{i,j}$ and a CP j with the same $d_{i,j}$ for all users. If CP j is a promotion CP, the distributions of user demands and utilities across different cost awarenesses become more fair ($F\left(\left\{x^*_{i,j}\right\}\right)$ and $F\left(\{V_{i,j}\}\right)$ increase) with sponsorship. Users' relative demands and benefits increase as $c^{-1}_{i,j}$ increases.

If CP j is a revenue CP, relative demand and benefit is independent of cost awareness, so fairness does not change.

User demand increases as users become less cost-aware, with and without sponsorship. Thus, we might expect less cost-aware users to benefit more from sponsorship; indeed, with revenue CPs, relative benefit is independent of cost awareness. However, promotion CPs experience diminishing marginal utility from greater user demand, and we see from (8) that they sponsor less content for less cost-aware (high $c_{i,j}$) users. Promotion CPs thus increase the effective prices $\pi^*_{i,j}$ as $c_{i,j}$ increases, dampening user demand enough to ensure that relative demand decreases with $c_{i,j}$ (increases with $c^{-1}_{i,j}$).

Figure 6a illustrates Prop. 7's result; user utility not only increases as $c_{i,j}$ increases, but is never less than the utility before sponsorship. Table II shows that the demand and user utility distributions become more fair for the promotion CP; fairness does not change for the revenue CP.

If we instead consider homogeneous users and vary CP cost awareness, we find the opposite effect: the distributions of demand and utility always become less fair.

Proposition 8 (Fairness across CP cost awareness): Consider a set of homogeneous users and a set of either

TABLE III: Relative demands and benefits for users and CPs with optimal sponsorship.

	Revenue CP	Promotion CP
Relative demand	$\left(\frac{1}{1-\alpha_{i,j}} - \frac{d_{i,j}}{p_c(1-\alpha_{i,j})(1+s_j)}\right)^{\frac{-1}{\alpha_{i,j}}}$	$\left(1-\alpha_{i,j}+\frac{d_{i,j}p_u(1+r_{i,j}s_j)^{\alpha_{i,j}}}{c_{i,j}p_c}\right)^{\frac{1}{\alpha_{i,j}}}$
Relative user benefit	$\left(\frac{1}{1-\alpha_{i,j}} - \frac{d_{i,j}}{p_c(1-\alpha_{i,j})(1+s_j)}\right)^{1-\frac{1}{\alpha_{i,j}}}$	$\left(1-\alpha_{i,j}+\frac{d_{i,j}p_u(1+r_{i,j}s_j)^{\alpha_{i,j}}}{c_{i,j}p_c}\right)^{\frac{1}{\alpha_{i,j}}-1}$
Relative CP benefit	$\frac{\alpha_{i,j}}{1-\alpha_{i,j}}\left(\frac{p_c(1+s_j)}{d_{i,j}}-1\right)\left(\frac{1}{1-\alpha_{i,j}} - \frac{d_{i,j}}{p_c(1-\alpha_{i,j})(1+s_j)}\right)^{\frac{-1}{\alpha_{i,j}}}$	$\left(1-(1-\alpha_{i,j})(1+r_{i,j}s_j)^{\alpha_{i,j}-1}+\frac{c_{i,j}p_c\alpha_{i,j}(1-\alpha_{i,j})}{d_{i,j}p_u(1+r_{i,j}s_j)}\right) \times \left(1-\alpha_{i,j}+\frac{d_{i,j}p_u(1+r_{i,j}s_j)^{\alpha_{i,j}}}{c_{i,j}p_c}\right)^{\frac{1}{\alpha_{i,j}}-1}$

revenue or promotion CPs varying only in their cost awareness $d_{i,j}^{-1}$. The demand and CP utility distributions across different CP cost awarenesses become less fair ($F\left(\left\{x_{i,j}^*\right\}\right)$ and $F\left(\left\{W_{i,j}\right\}\right)$ decrease) with sponsorship.

Sponsored content favors less cost-aware CPs, as some have feared [6]. Less cost-aware CPs receive higher utilities even without sponsorship, due to their greater valuation of user demand. As we would expect, they can also sponsor more data, disproportionately increasing their utility. Figure 6b illustrates Prop. 8 by showing CP utilities as $d_{i,j}$ varies. Sponsorship increases the unevenness of these utilities, making their distribution more unfair (Table II).

C. User and CP Benefits

Having compared user and CP utility among different users and CPs, in this section we compare users' and CPs' utilities to each other. We first show that if demand increases with content sponsorship as in Prop. 4, user utility also increases:

Lemma 2: User i's utility $V_{i,j}$ from CP j increases with sponsorship if and only if the CP sponsors content ($\gamma_{i,j} > 0$).

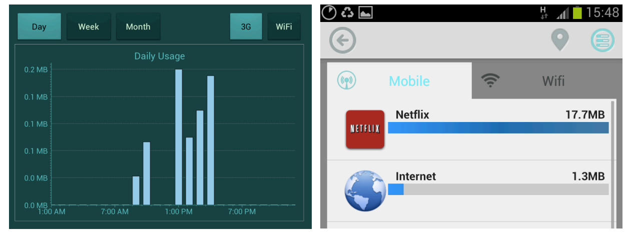
Note that Lemma 2 does not require the amount sponsored to be chosen optimally; it is enough for CPs to sponsor any amount of data. We use this result to show that users, CPs, and ISPs can all benefit from content sponsorship:

Proposition 9 (Overall utility increase): Suppose the ISP's chosen price p_u^* is not greater than users' data price p_u before sponsorship. Then users and CPs do not decrease their utilities and the ISP does not decrease its profit with sponsorship.

If users' data price p_u does not increase, their effective prices $\pi_{i,j}^*$ will decrease with sponsorship; user demands then increase, which benefits users, CPs, and ISPs. CPs and users, however, do not benefit equally. We consider the ratio of CP to user utility when $\gamma_{i,j}^* > 0$ (i.e., some data is sponsored) and show that it decreases with sponsorship, meaning that *CPs experience lower increases in utility compared to users.*

Proposition 10 (User and CP utilities): Table IV gives the ratio of CP to user utility for each user-CP pair before and after sponsorship. This ratio is always lower after sponsorship.

Examining Table IV, we see that surprisingly, CPs benefit more relative to users (i.e., the CP-user utility ratio increases) as their data price p_c increases: CPs then sponsor less data, bringing the utility ratio closer to that before sponsorship.



(a) Total usage over time. (b) Per-app monthly usage.

Fig. 7: Partial screenshots of the trial app.

Proposition 10's result is surprising, as the $\gamma_{i,j}^*$ are chosen to maximize CP, not user, utility: thus, we would expect CPs to benefit more than users. However, while both users and CPs benefit from higher user demand with sponsorship, CPs must drive this demand by paying subsidies to ISPs. These subsidies lower CPs' utility gains relative to users'.

V. Sponsored Data in Practice

We illustrate Section IV's results with data from a small-scale commercial pricing trial. We then turn to a practical question of sponsored data and provide a framework for CPs to decide which pieces of content to sponsor, e.g., specific videos in an app.

A. Numerical Evaluation

1) Trial Data: We estimate user utility parameters $(\alpha_{i,j}, c_{i,j})$ from a small-scale pricing trial with a U.S. ISP. We recruited 22 users and divided them into groups with data plans ranging from \$14/GB to \$20/GB. We then recorded users' per-app cellular data usage during June 2013. Figure 7 shows screenshots of the trial app: users could see their hourly usage as well as their monthly usage of different apps. Since our trial only ran for one month, we did not vary the monthly data plan prices for individual users, but different users experienced different data prices.

We consider sixty user-CP pairs in our simulation, consisting of six CPs and ten users. We group the apps from our trial data into three revenue CPs (for social networking, browsing, and video) and three promotion CPs (social networking, downloads, and email). We then estimate the $(\alpha_{i,j}, c_{i,j})$ parameters for each app by fitting the demand curve $x_{i,j}^*(p) = (p/c_{i,j})^{-1/\alpha_{i,j}}$ to all users' monthly usage of the app, where p is the user's \$/GB data plan price. We generate parameters for ten different users of each CP by assuming they are normally distributed around the parameters estimated for apps of this CP type. Figure 8 shows the resulting $\alpha_{i,j}$ and

TABLE IV: CP-to-user utility ratios before and after sponsorship.

	Revenue CP	Promotion CP
Before sponsorship	$\frac{(1-\alpha_{i,j})d_{i,j}}{(r_{i,j}s_j+\alpha_{i,j})p_u(1+s_j)}$	$\frac{d_{i,j}(1+r_{i,j}s_j)}{(\alpha_{i,j}+r_{i,j}s_j)c_{i,j}}$
With optimal sponsorship	$\frac{p_c\alpha_{i,j}(1-\alpha_{i,j})}{p_u(r_{i,j}s_j+\alpha_{i,j})}$	$\frac{d_{i,j}p_u\left(1+r_{i,j}s_j-(1-\alpha_{i,j})(1+r_{i,j}s_j)^{\alpha_{i,j}}\right)+\alpha_{i,j}(1-\alpha_{i,j})c_{i,j}p_c}{p_u c_{i,j}(\alpha_{i,j}+r_{i,j}s_j)}$

$c_{i,j}$ values; while the $\alpha_{i,j}$ values cluster towards the very high and very low, a more comprehensive user base would likely exhibit greater behavioral variation.

2) Simulation Results: We assume that ISPs charge their optimal data prices $p_u^* = \$17$/GB and $p_c^* = \$6.19$/GB, which are close to ISPs' currently offered prices [21]; the maximum traffic is $X = 30$ GB. Revenue CPs take $d_{i,j} = ar_{i,j}s_j = 5.4$ as calculated in Section IV-A.

Figure 9a shows the distribution of the optimal amounts sponsored $\gamma_{i,j}^*$ for each user-CP pair. While $\gamma_{i,j}^* = 0$ for 28% of the user-CP pairs, a few CPs sponsor >100% of users' content (i.e., they also sponsor some ads, which add 15% to the content volume). For instance, some enterprises may sponsor all traffic on a corporate email app. Despite the $\alpha_{i,j}$ estimates clustering around very low and very high values, except for the concentration at 0 the $\gamma_{i,j}$ values are approximately evenly distributed between 0 and 1.15, likely due to different cost awarenesses $c_{i,j}^{-1}$ and $d_{i,j}^{-1}$.

Figure 9b shows the ratios of CP to user utility for each user-CP pair before and after sponsorship; as in Prop. 10, the ratios decrease with sponsorship, indicating that users benefit more than CPs. The decrease is most apparent for larger CP-to-user ratios, likely because these correspond to users with very low utility. Low user and CP utilities increase the most with sponsorship, as shown in Figure 9c, though both CP and user utilities generally increase. In fact, the fairness of the CP and user utility distributions over all user-CP pairs increases with sponsorship. Jain's index for user utilities increases from 0.158 to 0.172, as is consistent with Prop. 7 for users facing one CP, and Jain's index for CP utilities increases from 0.126 to 0.134. While sponsorship makes CP utilities more unfair with homogeneous users (Prop. 8), in reality CPs face different user demands, allowing the distribution of CP utility to become (slightly) more fair in our simulation.

Not only do user and CP utilities become more fair with sponsorship, but user demand also increases (Figure 10a). However, different CPs experience different changes in the distribution of these demands. Figure 10b shows that some CPs experience more even and some more uneven demand with sponsorship. Similarly, each individual user faces all six CPs. While the CPs have the same cost awarenesses $d_{i,j}^{-1}$ and fractions of ads sponsored s_j for each user, users' price elasticities $\alpha_{i,j}^{-1}$ and cost awarenesses $c_{i,j}^{-1}$ do vary for each CP. Thus, the distribution of demands for different CPs changes from user to user, as shown in Figure 10c.

B. Which Content to Sponsor?

While Props. 1 and 2 give CPs the optimal amount of content to sponsor, they do not help decide *which* content to sponsor. Since most users choose the content they view (e.g., which videos to watch), sponsorship can influence demand for different content.[10] If the CP sponsors more popular content, more users will benefit, but CPs may wish to promote less popular content by sponsoring it.

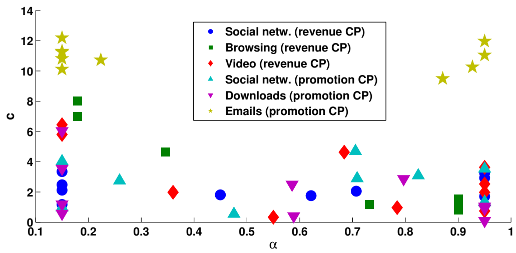


Fig. 8: Estimated user utility parameters for each user-CP pair in simulations with trial data.

We suppose that a CP j has K_j types of content (e.g., videos), each with a probability $v_{i,j}^k\left(p_{i,j}^k\right) \in [0,1]$ of being viewed by type i users, where $p_{i,j}^k$ is the fraction of content type k that is sponsored. The CP chooses the $p_{i,j}^k$ so as to maximize an objective function $G\left(\left\{v_{i,j}^k\left(p_{i,j}^k\right)\right\}\right)$. For instance, to spread demand across different types of content the CP can equalize the $v_{i,j}^k\left(p_{i,j}^k\right)$ (i.e., make G a fairness function [24]). To target more popular content, it can maximize the sum of all viewing probabilities: $G = \sum_k v_{i,j}^k\left(p_{i,j}^k\right)$.

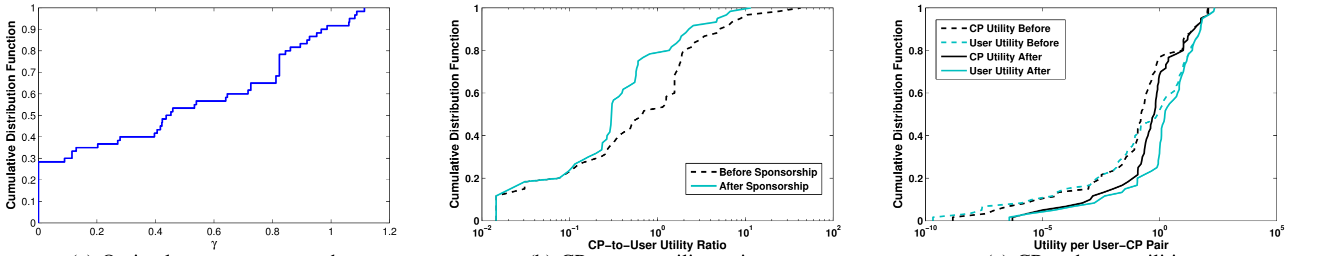
The CP maximizes G subject to two constraints. First, the total fraction of content sponsored should equal $\gamma_{i,j}^*$, as given by (7) or (8): $\sum_k p_{i,j}^k z_k = \gamma_{i,j}^* \sum_k z_k$, where z_k is the volume of type k content. Second, users' overall demand should be $x_{i,j}^*(\pi_{i,j}^*)$, i.e., $\sum_k v_{i,j}^k\left(p_{i,j}^k\right) z_k = x_{i,j}^*(\pi_{i,j}^*)$. If the CP cannot sponsor fractions of each content type (e.g., a content type is a single video), we constrain $p_{i,j}^k \in \{0,1\}$. We then have a knapsack problem with multiple constraints [25].

VI. Conclusion

In this work, we derive the optimal sponsored data behaviors for users, CPs, and ISPs and then consider their implications for heterogeneous CPs and users. In particular, we find that sponsorship can reverse some of our intuition on how user demand changes with price sensitivity and the amount of ads shown by CPs. These changes affect the distributions of CP and user utilities: while sponsored data exacerbates CP inequality, it can even out user demand and utility, and it benefits users proportionally more than CPs. We illustrate our results with data from an ISP pricing trial.

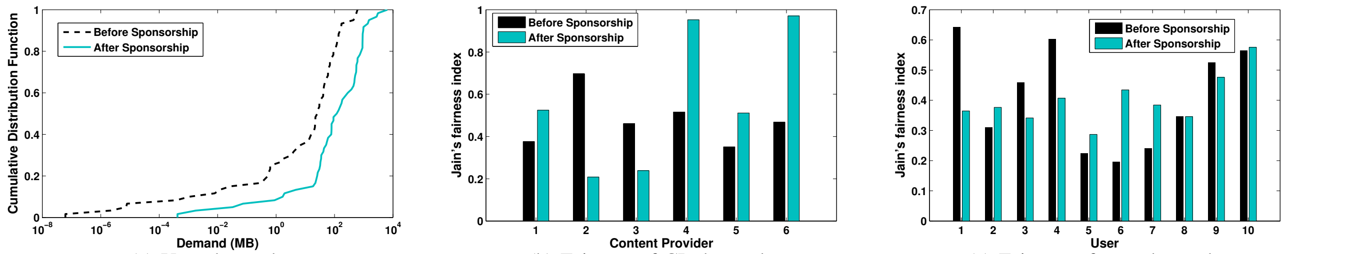
Though sponsored data offers a seemingly simple new choice to CPs, its effect will be felt throughout the mobile data market: as we show in this work, CPs' newfound market power over mobile data can significantly change user demand

[10] Since we assume that CPs fix the ratio of ads to content, sponsoring ads will affect overall demand but not which content users choose to view.



(a) Optimal amount sponsored. (b) CP-to-user utility ratios. (c) CP and user utilities.

Fig. 9: (a) Optimal amount sponsored and the resulting distributions over all user-CP pairs of (b) CP-to-user utility ratios and (c) CP and user utilities with parameters from trial data (Figure 8; $d_{i,j} = 6.295$, 0.607, 3 for the three promotion CPs).



(a) User demands. (b) Fairness of CP demands. (c) Fairness of user demands.

Fig. 10: (a) Demand for all user-CP pairs and the resulting fairness of demands for (a) each CP and (b) each user (parameters from Figures 8 and 9). CP types from 1 to 6 are social networking (revenue CP), browsing, video, social networking (promotion CP), downloads, email.

and CP and user utilities. Our work is necessarily an idealized approximation to user, CP, and ISP behavior, but nevertheless provides a guide towards understanding these implications.

REFERENCES

[1] S. Sen, C. Joe-Wong, S. Ha, and M. Chiang, "A survey of smart data pricing: Past proposals, current plans, and future trends," *ACM Computing Surveys*, vol. 46, no. 2, p. 15, 2013.

[2] ——, "Smart data pricing (SDP): Economic solutions to network congestion," in *Recent Advances in Networking*, H. Haddadi and O. Bonaventure, Eds. ACM SIGCOMM, 2013, pp. 221–274.

[3] S. Murlidhar, "Fast and free Facebook mobile access with 0.facebook.com," 2010. [Online]. Available: https://www.facebook.com/notes/facebook/fast-and-free-facebook-mobile-access-with-0facebookcom/391295167130

[4] AT&T, "AT&T sponsored data," 2014. [Online]. Available: http://www.att.com/att/sponsoreddata/en/index.html#fbid=fNjHshoHkg_

[5] I. Lunden, "FreedomPop to offer app-sized data plans, free use of sponsored apps," TechCrunch, 2014. [Online]. Available: http://techcrunch.com/2014/06/24/freedompop-to-offer-app-sized-data-plans-free-use-of-sponsored-apps/

[6] M. Reardon, "AT&T says 'sponsored data' does not violate net neutrality," CNet, 2014. [Online]. Available: http://www.cnet.com/news/at-t-says-sponsored-data-does-not-violate-net-neutrality/

[7] Apple, "iOS 7 and business," July 2014. [Online]. Available: https://www.apple.com/ios/business/

[8] S. M. Kerner, "Juniper debuts new per-app VPN tech," Enterprise Networking Planet, 2013. [Online]. Available: http://www.enterprisenetworkingplanet.com/netsecur/juniper-debuts-new-per-app-vpn-tech.html

[9] P. Hande, M. Chiang, R. Calderbank, and S. Rangan, "Network pricing and rate allocation with content provider participation," in *Proc. of IEEE INFOCOM*. IEEE, 2009, pp. 990–998.

[10] Y. Wu, H. Kim, P. H. Hande, M. Chiang, and D. H. Tsang, "Revenue sharing among ISPs in two-sided markets," in *Proc. of IEEE INFOCOM*. IEEE, 2011, pp. 596–600.

[11] L. Zhang and D. Wang, "Sponsoring content: Motivation and pitfalls for content service providers," in *Proc. of the SDP Workshop*. IEEE, 2014.

[12] M. Andrews, U. Ozen, M. I. Reiman, and Q. Wang, "Economic models of sponsored content in wireless networks with uncertain demand," in *Proc. of the SDP Workshop*. IEEE, 2013, pp. 3213–3218.

[13] App Annie, "App Annie & IDC mobile app advertising and monetization trends 2012-2017: The economics of free," 2014. [Online]. Available: http://inbound.appannie.com/app-annie-idc-mobile-app-advertising-and-monetization-trends-2012-2017

[14] Pandora, "SEC Form 10-K." [Online]. Available: http://investor.pandora.com/phoenix.zhtml?c=227956&p=proxy

[15] J. Constine, "Facebook beats in Q1 with $2.5B in revenue, 59% of ad revenue from mobile, 1.28B users," TechCrunch, 2014. [Online]. Available: http://techcrunch.com/2014/04/23/facebook-q1-2014-earnings/

[16] N. Economides and J. Tåg, "Network neutrality on the Internet: A two-sided market analysis," *Information Economics and Policy*, vol. 24, no. 2, pp. 91–104, 2012.

[17] E. Altman, J. Rojas, S. Wong, M. K. Hanawal, and Y. Xu, "Net neutrality and quality of service," in *Game Theory for Networks*. Springer, 2012, pp. 137–152.

[18] S. Caron, G. Kesidis, and E. Altman, "Application neutrality and a paradox of side payments," in *Proc. of the Re-Arch Workshop*. ACM, 2010, p. 9.

[19] M. Andrews, "Implementing sponsored content in wireless data networks," in *Proc. of Allerton*. IEEE, 2013, pp. 1208–1212.

[20] S. E. Ante and R. Knutson, "Good luck leaving your wireless phone plan," Wall Street Journal, July 2013. [Online]. Available: http://online.wsj.com/news/articles/SB10001424127887323971204578630202425653338

[21] AT&T, "Family cell phone plans," July 2014. [Online]. Available: http://www.att.com/shop/wireless/data-plans.html#fbid=bOFEaeX6Gbp

[22] B. Osman, "Digital advertising: Click-through rates," Coull.com, 2014. [Online]. Available: http://coull.com/our-blog/digital-advertising-click-through-rates

[23] J. Lee, "Mobile cost per click down for PPC in retail, business services," Search Engine Watch, 2014. [Online]. Available: http://searchenginewatch.com/article/2325359/Mobile-Cost-Per-Click-Down-for-PPC-in-Retail-Business-Services

[24] T. Lan, D. Kao, M. Chiang, and A. Sabharwal, "An axiomatic theory of fairness in network resource allocation," *Proc. of IEEE INFOCOM*, pp. 1–9, 2010.

[25] H. Kellerer, U. Pferschy, and D. Pisinger, *Knapsack Problems*. Springer, 2004.

Appendix A
Proofs

A. Proof of Proposition 1

We first note that since (5–6) is additively separable with respect to the $\gamma_{i,j}$ variables, we can simply choose each $\gamma_{i,j}$ so as to maximize $W_{i,j}(\gamma_{i,j})$. Moreover, $W_{i,j}$ is maximized at one of three possible values: $\gamma_{i,j} = 0$, $\gamma_{i,j} = 1 + s_j$, or a critical point of $W_{i,j}(\gamma_{i,j})$. Since we assume that $d_{i,j} < p_c(1+s_j)$, taking $\gamma_{i,j} = 1+s_j$ yields a negative profit for the CP. Thus, either the optimal value of $\gamma_{i,j}$, which we denote as $\gamma^*_{i,j}$, is either 0 or a critical point of $W_{i,j}$.

We find a unique critical point of $W_{i,j}(\gamma_{i,j})$ by taking the first derivative of (5) and setting it equal to zero:

$$\begin{aligned} &p_u\left(\frac{d_{i,j}-p_c\gamma_{i,j}}{1+r_{i,j}s_j}\right)\left(p_u\left(1-\gamma_{i,j}+s_j\right)\right)^{\frac{-1}{\alpha_{i,j}}-1} \\ &-\alpha_{i,j}\left(\frac{p_c}{1+r_{i,j}s_j}\right)\left(p_u\left(1-\gamma_{i,j}+s_j\right)\right)^{\frac{-1}{\alpha_{i,j}}} = 0. \end{aligned} \tag{11}$$

We then solve (11) for

$$\gamma^*_{i,j} = \frac{d_{i,j}}{p_c(1-\alpha_{i,j})} - \frac{\alpha_{i,j}(1+s_j)}{1-\alpha_{i,j}}. \tag{12}$$

Note that (12) satisfies the constraint $\gamma^*_{i,j} \leq 1+s_j$ if and only if $d_{i,j} \leq 1+s_j$, which is true by assumption.

We now show that taking $\gamma^*_{i,j} = 0$ yields a higher CP profit than (12) if and only if (12) is negative, i.e., $d_{i,j} < \alpha_{i,j}p_c(1+s_j)$. We find the ratio of CP utility with $\gamma^*_{i,j}$ as in (12) to that with $\gamma^*_{i,j} = 0$ to be

$$\sigma_{i,j} = \frac{\alpha_{i,j}}{1-\alpha_{i,j}}\left(\mu_{i,j}-1\right)\left(\frac{1}{1-\alpha_{i,j}} - \frac{1}{\mu_{i,j}\left(1-\alpha_{i,j}\right)}\right)^{\frac{-1}{\alpha_{i,j}}}$$

where for notational convenience we define $\mu_{i,j} = p_c(1+s_j)/(d_{i,j}) > 1$. Setting $\sigma_{i,j} \geq 1$, we find the equivalent condition

$$\left(\mu_{i,j}-1\right)^{1-\frac{1}{\alpha_{i,j}}} \geq \frac{\mu_{i,j}^{\frac{-1}{\alpha_{i,j}}}}{\alpha_{i,j}}\left(1-\alpha_{i,j}\right)^{1-\frac{1}{\alpha_{i,j}}}.$$

Raising both sides to the power $\alpha_{i,j}/(\alpha_{i,j}-1)$, we find the equation

$$\alpha_{i,j}^{\frac{\alpha_{i,j}}{1-\alpha_{i,j}}}\left(1-\alpha_{i,j}\right)\mu_{i,j}^{\frac{1}{1-\alpha_{i,j}}} - \mu_{i,j} + 1 \geq 0.$$

At $\mu_{i,j} = \alpha_{i,j}^{-1}$, we see by inspection that this inequality is satisfied with equality. Taking the derivative with respect to $\mu_{i,j}$, we find the expression

$$\alpha_{i,j}^{\frac{\alpha_{i,j}}{1-\alpha_{i,j}}}\mu_{i,j}^{\frac{\alpha_{i,j}}{1-\alpha_{i,j}}} - 1,$$

which is positive if and only if $\mu_{i,j} \geq \alpha_{i,j}^{-1}$. Thus, $\sigma_{i,j} \geq 1$ and $\gamma_{i,j} = 0$ yields lower profit than (12) for all $\mu_{i,j} \geq \alpha_{i,j}^{-1}$, or equivalently, $\alpha_{i,j}p_c(1+s_j) \geq d_{i,j}$. ■

B. Proof of Proposition 2

As in Prop. 1, we first observe that it suffices to consider the $\gamma_{i,j}$ separately, and that the optimal value of each $\gamma_{i,j}$ is either 0, $1+s_j$, or a critical point of $W_{i,j}(\gamma_{i,j})$. We now observe that

$$\begin{aligned} \lim_{\gamma_{i,j}\to 1+s_j} W_{i,j}(\gamma_{i,j}) = &\frac{d_{i,j}}{1-\beta_{i,j}}\left(\frac{p_u(1-\gamma_{i,j}+s_j)}{c_{i,j}}\right)^{\frac{\beta_{i,j}-1}{\alpha_{i,j}}} \\ &- p_c\gamma_{i,j}\left(\frac{p_u(1-\gamma_{i,j}+s_j)}{c_{i,j}}\right)^{\frac{-1}{\alpha_{i,j}}} \\ &\to -\infty. \end{aligned}$$

Thus, we first find the unique critical point of $W_{i,j}$ and then derive conditions under which this critical point yields greater CP utility than taking $\gamma_{i,j} = 0$. We thus take the derivative and find

$$\begin{aligned} \frac{dW_{i,j}}{d\gamma_{i,j}} \propto &\frac{d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}} - p_c\alpha_{i,j}c_{i,j}}{\alpha_{i,j}c_{i,j}\left(1+r_{i,j}s_j\right)} \\ &- \frac{p_cp_u\gamma_{i,j}}{\alpha_{i,j}c_{i,j}\left(1+r_{i,j}s_j\right)}\left(\frac{p_u(1-\gamma_{i,j}+s_j)}{c_{i,j}}\right)^{-1}. \end{aligned}$$

After multiplying through by common factors, we find the equation

$$\begin{aligned} &p_u\left(1+s_j\right)\left(\frac{d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}}}{\alpha_{i,j}c_{i,j}} - p_c\right) \\ &= \gamma_{i,j}\left(\frac{p_up_c}{\alpha_{i,j}} + p_u\left(\frac{d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}}}{\alpha_{i,j}c_{i,j}} - p_c\right)\right), \end{aligned}$$

which simplifies to

$$\gamma^*_{i,j} = \frac{d_{i,j}}{p_c(1-\alpha_{i,j})} - \frac{\alpha_{i,j}\left(1+s_j\right)}{1-\alpha_{i,j}}. \tag{13}$$

It is easy to see that $\gamma^*_{i,j} < 1+s_j$.

We now find the ratio of CP utility with (13) to that with $\gamma_{i,j} = 0$, showing that taking $\gamma^*_{i,j}$ as in (13) yields higher utility when (13) is positive. We find the ratio

$$\begin{aligned} \sigma_{i,j} = &\left(1-\left(1-\alpha_{i,j}\right)\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}-1} + \frac{c_{i,j}p_c\alpha_{i,j}(1-\alpha_{i,j})}{d_{i,j}p_u(1+r_{i,j}s_j)}\right) \\ &\times\left(1-\alpha_{i,j} + \frac{d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}}}{c_{i,j}p_c}\right)^{\frac{1}{\alpha_{i,j}}-1}. \end{aligned}$$

For notational convenience, we define $\nu_{i,j} = d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}}/\left(\alpha_{i,j}c_{i,j}p_c\right)$ and rewrite $\sigma_{i,j} \geq 1$ as

$$\begin{aligned} &\left(1+\left(1-\alpha_{i,j}\right)\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}-1}\left(\frac{1}{\nu_{i,j}}-1\right)\right) \\ &\times\left(1+\alpha_{i,j}\left(\nu_{i,j}-1\right)\right)^{\frac{1}{\alpha_{i,j}}-1} \geq 1. \end{aligned}$$

Simplifying and multiplying through by $\nu_{i,j}$, we find the condition

$$\begin{aligned} &\left(\nu_{i,j}-\left(1-\alpha_{i,j}\right)\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}-1}\left(\nu_{i,j}-1\right)\right) \\ &\times\left(1+\alpha_{i,j}\left(\nu_{i,j}-1\right)\right)^{\frac{1}{\alpha_{i,j}}-1} \geq \nu_{i,j}. \end{aligned}$$

We now show that

$$\nu_{i,j} - (1-\alpha_{i,j})\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}-1}(\nu_{i,j}-1) > 1-\alpha_{i,j}(\nu_{i,j}-1),$$

which is equivalent to

$$(\nu_{i,j}-1)(1-\alpha_{i,j})\left(1-\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}-1}\right) > 0.$$

Since $\nu_{i,j} > 1$ when (13) is positive, this inequality holds and $\sigma_{i,j} \geq 1$ becomes

$$0 \geq \nu_{i,j}^{\alpha_{i,j}} - \alpha_{i,j}(\nu_{i,j}-1) - 1.$$

At $\nu_{i,j} = 1$, this inequality holds by inspection; we see that the derivative of the right-hand side with respect to $\nu_{i,j}$, $\alpha_{i,j}\nu_{i,j}^{\alpha_{i,j}-1} - \alpha_{i,j}$, is negative if $\nu \geq 1$. Thus, $\sigma_{i,j} \geq 1$ for all $\nu_{i,j} \geq 1$, i.e., (13) positive. ■

C. Proof of Proposition 3

We first note that since each CP optimally chooses $\gamma_{i,j}$, $dW_{i,j}(\gamma_{i,j})/d\gamma_{i,j}$ must equal zero if $\gamma_{i,j} \in (0, 1+s_j)$, where we compute

$$\begin{aligned}\frac{dW_{i,j}}{d\gamma_{i,j}} =& \frac{d_{i,j}p_u}{\alpha_{i,j}c_{i,j}\left(1+r_{i,j}s_j\right)^{1-\beta_{i,j}}}\left(\frac{p_u(1-\gamma_{i,j}+s_j)}{c_{i,j}}\right)^{\frac{\beta_{i,j}-1}{\alpha_{i,j}}-1} \\ &- \frac{p_c}{1+r_{i,j}s_j}\left(\frac{p_u(1-\gamma_{i,j}+s_j)}{c_{i,j}}\right)^{\frac{-1}{\alpha_{i,j}}} \\ &- \frac{p_c p_u \gamma_{i,j}}{\alpha_{i,j}c_{i,j}\left(1+r_{i,j}s_j\right)}\left(\frac{p_u(1-\gamma_{i,j}+s_j)}{c_{i,j}}\right)^{\frac{-1}{\alpha_{i,j}}-1}.\end{aligned} \tag{14}$$

Thus, by taking $\gamma_{i,j}^*$ as a function of p_c and taking the total derivative of (14) with respect to p_c, we find that

$$\frac{d\gamma_{i,j}^*}{dp_c} = \frac{-\alpha_{i,j}p_c(1+s_j) - \gamma_{i,j}^* p_c(1-\alpha_{i,j})}{\frac{p_u\beta_{i,j}d_{i,j}(1+r_{i,j}s_j){x_{ij}^*}^{\alpha_{i,j}-\beta_{i,j}}}{\alpha_{i,j}c_{i,j}} + p_c(1-\alpha_{i,j})} < 0.$$

We now find that

$$\frac{dx_{i,j}^*}{dp_c} = \frac{p_u}{\alpha_{i,j}c_{i,j}}\left(\frac{p_u\left(1-\gamma_{i,j}^*+s_j\right)}{c_{i,j}}\right)^{\frac{-1}{\alpha_{i,j}}-1}\frac{d\gamma_{i,j}^*}{dp_c} < 0.$$

We now define $R_{i,j} = \left(\pi_{i,j}^* + p_c\gamma_{i,j}^*\right)x_{i,j}^*$ and compute

$$\begin{aligned}\frac{dR_{i,j}}{dp_c} =& \left((1-\alpha_{i,j})c_{i,j}{x_{i,j}^*}^{-\alpha_{i,j}} + p_c\gamma_{i,j}^*\right)\frac{\partial x_{i,j}^*}{\partial p_c} \\ &+ \gamma_{i,j}^* x_{i,j}^* + p_c x_{i,j}^*\frac{\partial\gamma_{i,j}^*}{\partial p_c}\end{aligned} \tag{15}$$

$$\begin{aligned}&\propto \alpha_{i,j}\gamma_{i,j}^* x_{i,j}^* + \Bigg(\left(\alpha_{i,j}p_c + p_u(1-\alpha_{ij})\right)x_{i,j}^* \\ &\quad + \frac{p_u p_c\gamma_{i,j}^*}{c}{x_{i,j}^*}^{1+\alpha_{i,j}}\Bigg)\frac{\partial\gamma_{i,j}^*}{\partial p_c} \\ &\propto \gamma_{i,j}^* p_u\beta_{i,j}d_{i,j}\left(1+r_{i,j}s_j\right)^{\beta_{i,j}}{x_{i,j}^*}^{\alpha_{i,j}-\beta_{i,j}} \\ &\quad - \alpha_{i,j}^2 c_{i,j}p_c(1+s_j) \\ &\quad - p_u c_{i,j}(1-\alpha_{i,j})\left(\alpha_{i,j}(1+s_j) + \gamma_{i,j}^*(1-\alpha_{i,j})\right)\end{aligned} \tag{16}$$

We now use the fact that (14) equals zero at $\gamma_{i,j}^*$ to solve for

$$d_{i,j}\left(1+r_{i,j}s_j\right)^{\beta_{i,j}}{x_{i,j}^*}^{-\beta_{i,j}} = \gamma_{i,j}^* p_c(1-\alpha_{i,j}) + \alpha_{i,j}p_c(1+s_j),$$

and substituting into (16), we find

$$\begin{aligned}\frac{dR_{i,j}}{dp_c} \propto& -\alpha_{i,j}^2 c_{i,j}p_c(1+s_j) \\ &- p_u c_{i,j}(1-\alpha_{i,j})\left(\alpha_{i,j}(1+s_j) + \gamma_{i,j}(1-\alpha_{i,j})\right) \\ &+ (\beta_{i,j}-1)p_u p_c\gamma_{i,j}{x_{i,j}^*}^{\alpha_{i,j}} \\ &\times\left(\gamma_{i,j}(1-\alpha_{i,j}) + \alpha_{i,j}(1+s_j)\right) \\ &< 0.\end{aligned}$$

If (14) cannot be exactly satisfied by $\gamma \in [0, 1+s_j]$, then either $\gamma_{i,j} = 0$ or $\gamma_{i,j} = 1+s_j$. We first suppose that $\gamma_{i,j}^* = 0$; it is easy to see by inspection that $x_{i,j}^*$ is then independent of p_c, as is $R_{i,j} = \pi_{i,j}^* x_{i,j}^* + p_c\gamma_{i,j}x_{i,j}^*$. If $\gamma_{i,j}^* = 1+s_j$, then $\pi_{i,j}^* = 0$ and $x_{i,j}^* \to \infty$, which violates the constraint (10).

We now see that for a given p_u, since (9) is decreasing in p_c, the optimal value of p_c is the minimal one for which the constraints are all satisfied. Neglecting the constraints $d_{i,j} < p_c(1+s_j)$ from revenue CPs, this optimal value of p_c is the unique one for which (10) is satisfied with equality: for all higher values of p_c, (10) is satisfied without equality since $x_{i,j}^*$ is decreasing in p_c. If we include the constraints from revenue CPs, we may need to increase p_c until all constraints are satisfied, and one is satisfied with equality.

It remains to show that we can perform a line search for the optimal value of p_u on a bounded range. Clearly, we have a lower bound of $p_u \geq 0$. We see that there is an upper bound by noting that $\gamma_{i,j}^*$ is increasing with p_u if (14) equals zero:

$$\frac{d\gamma_{i,j}^*}{dp_u} = \frac{d_{i,j}\left(1+r_{i,j}s_j\right){x_{i,j}^*}^{\frac{\beta_{i,j}}{\alpha_{i,j}}-1}\beta_{i,j}\left(1-\gamma_{i,j}^*+s_j\right)}{d_{i,j}p_u\left(1+r_{i,j}s_j\right){x_{i,j}^*}^{\frac{\beta_{i,j}}{\alpha_{i,j}}-1}\beta_{i,j} + \alpha_{i,j}c_{i,j}p_c(1-\alpha_{i,j})}.$$

Since $\gamma_{i,j}^*$ is upper-bounded by $1+s_j$, $\gamma_{i,j}^*(p_u)$ must converge as $p_u \to \infty$. As $\gamma_{i,j}^*$ converges, we then have

$$\begin{aligned}\frac{dR_{i,j}}{dp_u} &= \left((1-\alpha_{i,j})c_{i,j}{x_{i,j}^*}^{-\alpha_{i,j}} + p_c\gamma_{i,j}^*\right)\frac{\partial x_{i,j}^*}{\partial p_u} + p_c x_{i,j}^*\frac{\partial\gamma_{i,j}^*}{\partial p_u} \\ &\approx \left((1-\alpha_{i,j})c_{i,j}{x_{i,j}^*}^{-\alpha_{i,j}} + p_c\gamma_{i,j}^*\right) \\ &\quad\times\frac{{x_{i,j}^*}^{1+\alpha_{i,j}}\left(\gamma_{i,j}^*-1-s_j\right)}{c_{i,j}\alpha_{i,j}} \\ &< 0,\end{aligned}$$

since

$$\begin{aligned}\frac{\partial x_{i,j}^*}{\partial p_u} =& \frac{1}{\alpha_{i,j}}\left(\frac{p_u(1-\gamma_{i,j}+s_j)}{c_{i,j}}\right)^{\frac{-1}{\alpha_{i,j}}-1} \\ &\times\left(\frac{\gamma_{i,j}-1-s_j}{c_{i,j}} + \frac{p_u}{c_{i,j}}\frac{\partial\gamma_{i,j}}{\partial p_u}\right).\end{aligned}$$ ■

D. Proof of Proposition 4

The first part of the proposition follows directly from observing that $x_{i,j}^*\left(p_u(1+\gamma_{i,j}-s_j)\right) = c_{i,j}^{1/\alpha_{i,j}}/(1+r_{i,j}s_j)\left(p_u\left(1-\gamma_{i,j}+s_j\right)\right)^{-1/\alpha_{i,j}}$.

We now consider revenue CPs and compute

$$x^*_{i,j}(\pi^*_{i,j}) = \frac{1}{1+r_{i,j}s_j}\left(\frac{p_c p_u(1+s_j) - p_u d_{i,j}}{c_{i,j}p_c(1-\alpha_{i,j})}\right)^{\frac{-1}{\alpha_{i,j}}}. \quad (17)$$

Taking the derivative of (17) with respect to $\alpha_{i,j}$ when $\gamma^*_{i,j} > 0$, obtaining the expression

$$\frac{\partial x^*_{i,j}}{\partial \alpha_{i,j}} \propto \frac{1}{\alpha^2_{i,j}} \ln\left(\frac{p_u\left(p_c(1+s_j) - d_{i,j}\right)}{p_c c_{i,j}(1-\alpha_{i,j})}\right) - \frac{1}{\alpha_{i,j}(1-\alpha_{i,j})}.$$

This derivative is negative if

$$p_u p_c\left(1+s_j\right) - p_u d_{i,j} < c_{i,j}p_c(1-\alpha_{i,j})\exp\left(\frac{\alpha_{i,j}}{1-\alpha_{i,j}}\right).$$

Since $(1-\alpha_{i,j})\exp\left(\alpha_{i,j}/(1-\alpha_{i,j})\right)$ is minimized at $\alpha = 0$, $x^*_{i,j}$ is always decreasing in $\alpha_{i,j} \in (0,1)$ if $p_u\left(p_c(1+s_j) - d_{i,j}\right) < p_c c_j$.

Finally, we consider promotion CPs when $\gamma^*_{i,j} > 0$ and compute

$$x^*_{i,j}(\pi^*_{i,j}) = \left(\frac{p_u p_c(1+s_j)(1+r_{i,j}s_j)^{\alpha_{i,j}}}{(1-\alpha_{i,j})c_{i,j}p_c + d_{i,j}p_u(1+r_{i,j}s_j)^{\alpha_{i,j}}}\right)^{\frac{-1}{\alpha_{i,j}}}. \quad (18)$$

We then find that

$$\frac{\partial x^*_{i,j}(\pi^*_{i,j})}{d\alpha_{i,j}} \propto \frac{\alpha_{i,j}\left(d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}}\ln\left(1+r_{i,j}s_j\right) - c_{i,j}p_c\right)}{(1-\alpha_{i,j})c_{i,j}p_c + d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}}} + \ln\left(\frac{p_u p_c(1+s_j)}{(1-\alpha_{i,j})c_{i,j}p_c + d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}}}\right)$$

The second term is always negative by assumption, so it suffices to derive conditions under which the first term is always negative. We have the criterion $p_u p_c(1+s_j) < (1-\alpha_{i,j})c_{i,j}p_c + d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}}$, so it is sufficient for

$$p_u\left(p_c(1+s_j) - d_{i,j}\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}}\right) < c_{i,j}p_c,$$

the left-hand side of which is maximized at $\alpha_{i,j} = 0$.■

E. Proof of Proposition 5

We first find that user i's utility for a revenue CP j, $V^*_{i,j}$, is

$$\frac{c_{i,j}\left(\alpha_{i,j} + r_{i,j}s_j\right)}{(1-\alpha_{i,j})(1+r_{i,j}s_j)}\left(\frac{p_u p_c(1+s_j) - p_u a r_{i,j}s_j}{c_{i,j}p_c(1-\alpha_{i,j})}\right)^{1-\frac{1}{\alpha_{i,j}}}. \quad (19)$$

when $\gamma^*_{i,j} > 0$. We can then see that the second term,

$$\left(\frac{p_u p_c(1+s_j) - p_u a r_{i,j}s_j}{c_{i,j}p_c(1-\alpha_{i,j})}\right)^{1-\frac{1}{\alpha_{i,j}}},$$

is increasing in s_j if $ar_{i,j} > p_c$. It then suffices to show that

$$\frac{\partial}{\partial s_j}\left(\frac{c_{i,j}\left(\alpha_{i,j} + r_{i,j}s_j\right)}{(1-\alpha_{i,j})(1+r_{i,j}s_j)}\right) = \frac{1-\alpha_{i,j}}{\left(1+r_{i,j}s_j\right)^2} > 0.$$

Each term of $V^*_{i,j}$ is then increasing in s_j. ■

F. Proof of Lemma 1

Since F is Schur-concave and homogeneous, it is sufficient to show that $\vec{x}/\left|\vec{x}\right|$ can be obtained from $\vec{y}/\left|\vec{y}\right|$ via a finite set of Robin-Hood operations. We wish to find a threshold index k^* such that for $l < k^*$, $\vec{y}_l/\left|\vec{y}\right| \leq \vec{x}_l/\left|\vec{x}\right|$, and for $l \geq k^*$, $\vec{y}_l/\left|\vec{y}\right| \geq \vec{x}_l/\left|\vec{x}\right|$. If such a k^* exists, we can easily obtain $\vec{x}/\left|\vec{x}\right|$ from $\vec{y}/\left|\vec{y}\right|$ by noting that for $l \geq k^*$, we need to reduce $\vec{y}_l/\left|\vec{y}\right|$, and for $l < k^*$, we must increase $\vec{y}_l/\left|\vec{y}\right|$. We do so by starting from $\vec{y}_K$ (the largest element) and reducing it by the required amount, distributing this amount to $\vec{y}_l$ with $l < k^*$. Since $\vec{y}/\left|\vec{y}\right| = \vec{x}/\left|\vec{x}\right| = 1$, this procedure will recover $\vec{x}/\left|\vec{x}\right|$.

We thus need to show that such a threshold k^* exists from our assumption that $\vec{y}_k/\vec{x}_k \leq \vec{y}_{k+1}/\vec{x}_{k+1}$ for all k. It suffices to show that $\vec{y}_1/\vec{x}_1 \leq \left|\vec{y}\right|/\left|\vec{x}\right| \leq \vec{y}_K/\vec{x}_K$. We do so by induction on K: clearly, the assertion is true for $K = 1$. Assuming it is true for $K = n$, we suppose that $\vec{y}_n/\vec{x}_n \leq \vec{y}_{n+1}/\vec{x}_{n+1}$. Then

$$\frac{\sum_{k\leq n}\vec{x}^*_k + \vec{x}^*_{n+1}}{\sum_{k\leq n}\vec{x}_k + \vec{x}_{n+1}} \leq \frac{\vec{x}^*_{n+1}}{\vec{x}_{n+1}}$$

which we find by multiplying out the terms and using the fact that $\sum_{k\leq n}\vec{y}_k/\sum_{k\leq n}\vec{x}_n \leq \vec{y}_n/\vec{x}_n \leq \vec{y}_{n+1}/\vec{x}_{n+1}$. Similarly, $\left(\sum_{k\leq n}\vec{x}^*_k + \vec{x}^*_{n+1}\right)/\left(\sum_{k\leq n}\vec{x}_k + \vec{x}_{n+1}\right) \geq \sum_{k\leq n}\vec{x}^*_k/\sum_{k\leq n}\vec{x}_k \geq \vec{x}^*_1/\vec{x}_1$. ■

G. Proof of Proposition 6

1) Revenue CPs: We first show that users with lower $\alpha_{i,j}$ receive a proportionally greater increase in demand with content sponsorship. Taking the derivative of $x^*_{i,j}(\pi^*_{i,j})/x^*_{i,j}(p_u(1+s_j))$ with respect to $\alpha_{i,j}$ if $\gamma^*_{i,j} > 0$, we find that it is proportional to

$$\frac{1}{\alpha^2_{i,j} - \alpha_{i,j}} + \frac{1}{\alpha^2_{i,j}}\ln\left(\left(\frac{1}{1-\alpha_{i,j}}\right)\left(1 - \frac{d_{i,j}}{p_c(1+s_j)}\right)\right),$$

which is always negative if $\gamma^*_{i,j} > 0$. and 0 at $d_{i,j} = \alpha_{i,j}p_c(1+s_j)$. For $d_{i,j} < \alpha_{i,j}p_c(1+s_j)$, $\gamma^*_{i,j} = 0$, so relative demand equals 1; thus, relative demand can never increase as $\alpha_{i,j}$ increases.

We now show that if $p_u\left(1+s_j\right) < c_{i,j}$, then users with lower $\alpha_{i,j}$ have higher demands if no content is sponsored. A user's demand for CP j without content sponsorship is $\left(1/(1+r_{i,j}s_j)\right)\left(p_u\left(1+s_j\right)/c_{i,j}\right)^{-1/\alpha_{i,j}}$, which is decreasing with $\alpha_{i,j}$ if and only if $p_u\left(1+s_j\right) < c_{i,j}$.

Given this result, we now let $\vec{x}$ denote a K-element vector containing the demands before sponsorship and $\vec{x}^*$ that after sponsorship, both sorted in increasing order (i.e., decreasing $\alpha_{i,j}$). We have shown that $\vec{x}^*_k/\vec{x}_k \leq \vec{x}^*_{k+1}/\vec{x}_{k+1}$ for all k, so the result follows from Lemma 1.

To show the second part of the proposition, we use the fact that, relative to the case with sponsorship, users with higher $\alpha_{i,j}$ experience proportionally higher increases of demand without content sponsorship. Thus, by Lemma 1, it suffices to show that users with higher $\alpha_{i,j}$ also experience higher levels of demand with content sponsorship under the conditions of

S_1. Taking the derivative of $x^*_{i,j}\left(\pi^*_{i,j}\right)$ with respect to $\alpha_{i,j}$ when $\gamma^*_{i,j} > 0$, we obtain the condition

$$\frac{1}{\alpha_{i,j}^2}\ln\left(\frac{p_u\left(p_c(1+s_j)-d_{i,j}\right)}{p_c c_{i,j}(1-\alpha_{i,j})}\right) \geq \frac{1}{\alpha_{i,j}(1-\alpha_{i,j})}.$$

Dividing by common factors and exponentiating, we find the condition

$$p_u\left(p_c(1+s_j)-d_{i,j}\right) \geq c_{i,j}p_c\left(1-\alpha_{i,j}\right)\exp\left(\frac{\alpha_{i,j}}{1-\alpha_{i,j}}\right). \tag{20}$$

We now note that $(1-\alpha_{i,j})\exp\left(\alpha_{i,j}/(1-\alpha_{i,j})\right)$ is increasing in $\alpha_{i,j}$:

$$\begin{aligned}&\frac{\partial}{\partial\alpha_{i,j}}\left(1-\alpha_{i,j}\right)\exp\left(\frac{\alpha_{i,j}}{1-\alpha_{i,j}}\right)\\ &= \exp\left(\frac{\alpha_{i,j}}{1-\alpha_{i,j}}\right)\left(\frac{1}{1-\alpha_{i,j}}-1\right) > 0.\end{aligned}$$

Thus, it suffices to show that (20) holds for the maximum value of $\alpha_{i,j}$, i.e., $d/(p_c(1+s_j))$. We then derive the condition

$$\begin{aligned}&p_u\left(p_c(1+s_j)-d_{i,j}\right)\\ &> c_{i,j}p_c\left(\frac{p_c(1+s_j)-d_{i,j}}{p_c(1+s_j)}\right)\exp\left(\frac{d_{i,j}}{p_c(1+s_j)-d_{i,j}}\right),\end{aligned}$$

which yields the condition in the proposition upon canceling common factors. We now show that when $\gamma^*_{i,j} = 0$, $x^*_{i,j}$ is also increasing in $\alpha_{i,j}$; then by continuity of $x^*_{i,j}$ as a function of $\alpha_{i,j}$, $x^*_{i,j}$ is always increasing in $\alpha_{i,j}$. For $\gamma^*_{i,j} = 0$, we have $x^*_{i,j} = \left(p_u\left(1+s_j\right)/c_{i,j}\right)^{-1/\alpha_{i,j}}/(1+r_{i,j}s_j)$, which is increasing with $\alpha_{i,j}$ if $p_u(1+s_j) > c_{i,j}$. Since $\exp\left(d_{i,j}/(p_c(1+s_j)-d_{i,j})\right) > 1$, this is always the case. ■

2) Promotion CPs: We first take the derivative of $\partial\ln\left(x^*_{i,j}(\pi^*_{i,j})/x^*_{i,j}(p_u(1+s_j))\right)/\partial\alpha_{i,j}$ when $\gamma^*_{i,j} > 0$ to find the expression

$$\begin{aligned}&\frac{-1}{\alpha_{i,j}^2}\ln\left(1-\alpha_{i,j}+\frac{d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}}}{c_{i,j}p_c}\right)\\ &+\frac{d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}}\ln\left(1+r_{i,j}s_j\right)-c_{i,j}p_c}{\alpha_{i,j}c_{i,j}p_c\left(1-\alpha_{i,j}\right)+\alpha_{i,j}d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}}}.\end{aligned} \tag{21}$$

The first term is negative if

$$1-\alpha_{i,j}+\frac{d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}}}{c_{i,j}p_c} > 1,$$

which is always the case since $d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}} > \alpha_{i,j}c_{i,j}p_c$ if $\gamma^*_{i,j} > 0$. The second term is negative if

$$\frac{d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}}\ln\left(1+r_{i,j}s_j\right)}{c_{i,j}p_c} < 1,$$

which is true by assumption. Thus, relative demand always decreases as $\alpha_{i,j}$ increases if $\gamma^*_{i,j} > 0$. At $d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}} = \alpha_{i,j}c_{i,j}p_c$, $\gamma^*_{i,j} = 0$ and (21) is still negative. If $d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}} < \alpha_{i,j}c_{i,j}p_c$, then $\gamma^*_{i,j} = 0$ and relative demand is constant at 1. Thus, relative demand is always non-increasing as $\alpha_{i,j}$ increases.

To show that fairness decreases for S_1, note that if $p_u(1+s_j) < c_{i,j}$, then demand without sponsorship $\left(\left(p_u(1+s_j)/c_{i,j}\right)^{-1/\alpha_{i,j}}/(1+r_{i,j}s_j)\right)$ is decreasing with $\alpha_{i,j}$. Since relative demand also decreases with $\alpha_{i,j}$, we see that fairness decreases with sponsorship using the same argument as in Prop. 6.

To show the second part of the proposition, it suffices to show that when $\alpha_{i,j}p_c(1+s_j) > \exp(\alpha_{i,j})d_{i,j}(1+r_{i,j}s_j)^{\alpha_{i,j}}$ and $p_u(1+s_j) > c_{i,j}$, demand with sponsorship increases with $\alpha_{i,j}$. From the proof of Prop. 4, we find that optimal demand increases with $\alpha_{i,j}$ when $\gamma^*_{i,j} > 0$ if and only if

$$\begin{aligned}&\frac{1}{\alpha_{i,j}^2}\ln\left(\frac{p_u p_c(1+s_j)}{(1-\alpha_{i,j})c_{i,j}p_c+d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}}}\right)\\ &+\frac{d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}}\ln\left(1+r_{i,j}s_j\right)-c_{i,j}p_c}{\alpha_{i,j}\left((1-\alpha_{i,j})c_{i,j}p_c+d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}}\right)} > 0.\end{aligned}$$

Using the fact that $c_{ij}p_c\alpha_{i,j} \leq d_{i,j}p_u(1+r_{i,j}s_j)^{\alpha_{i,j}}$, we find the sufficient condition

$$\frac{1}{\alpha_{i,j}^2}\ln\left(\frac{p_u p_c(1+s_j)}{(1-\alpha_{i,j})c_{i,j}p_c+d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}}}\right) > \frac{1}{\alpha_{i,j}}.$$

Exponentiating and substituting $(1/\alpha_{i,j}-1)d_{i,j}p_u(1+r_{i,j}s_j)^{\alpha_{i,j}}$ for $(1-\alpha_{i,j})c_{i,j}p_c$, we find the condition

$$\frac{\alpha_{i,j}p_c(1+s_j)}{d_{i,j}(1+r_{i,j}s_j)^{\alpha_{i,j}}} \geq \exp\left(\alpha_{i,j}\right),$$

which is exactly that given in the proposition. We now show that when $\gamma_{i,j} = 0$, demand with sponsorship is still increasing with $\alpha_{i,j}$. Equivalently, if $d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}} < \alpha_{i,j}c_{i,j}p_c$, $\gamma^*_{i,j} = 0$ and, from the proof of Prop. 6, user demand is increasing with $\alpha_{i,j}$ if $p_u(1+s_j) > c_j$. ■

H. Proof of Proposition 7

We first consider a promotion CP and observe by inspection that user utility with optimal sponsorship, $U_{i,j}\left(x^*_{i,j}\left(\pi^*_{i,j}\right)\right)$, given by (23), is increasing in $c_{i,j}$ if $d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}} \geq \alpha_{i,j}c_{i,j}p_c$. If $d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}} < \alpha_{i,j}c_{i,j}p_c$, then user utility equals

$$c_{i,j}^{1/\alpha_{i,j}}\frac{\alpha_{i,j}+r_{i,j}s_j}{(1-\alpha_{i,j})(1+r_{i,j}s_j)}\left(p_u(1+s_j)\right)^{1-1/\alpha_{i,j}},$$

which is clearly increasing in $c_{i,j}$.

From (18), $x^*_{i,j}(\pi^*_{i,j})$ is also increasing in $c_{i,j}$ if $d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}} \geq \alpha_{i,j}c_{i,j}p_c$; if $d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}} < \alpha_{i,j}c_{i,j}p_c$, user demand is that before sponsorship and is also increasing in $c_{i,j}$.

We then observe from Table III that relative user benefit and relative demand decrease with $c_{i,j}$ when $d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}} \geq \alpha_{i,j}c_{i,j}p_c$; if $d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}} \geq \alpha_{i,j}c_{i,j}p_c$, $\gamma^*_{i,j} = 0$ and relative user benefit and relative demand equal 1. Thus, both relative user benefit and relative demand are non-increasing in $c_{i,j}$. Using an analogous argument to that in Prop. 6, we see that fairness of the utilities and demands increases with content sponsorship.

We now observe that since $\gamma_{i,j}^*$ is independent of $c_{i,j}$ for revenue CPs (Prop. 1), either $\gamma_{i,j}^* = 0$ for all users or $\gamma_{ij}^* > 0$ for all users. Since relative demand and relative user benefit is independent of $c_{i,j}$ for revenue CPs if $\gamma_{i,j}^* > 0$ (Table III), content sponsorship merely multiplies users' demands and utilities by a constant. Since our fairness function F is homogeneous, fairness does not change. ■

I. Proof of Proposition 8

1) Relative Demands: We first note that, from Table III, relative demand for revenue and promotion CPs increases with $d_{i,j}$ if $d_{i,j} \geq \alpha_{i,j}p_c(1+s_j)$ (for revenue CPs) or $d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}} \geq \alpha_{i,j}c_{i,j}p_c$ (for promotion CPs). If $d_{i,j} < \alpha_{i,j}p_c(1+s_j)$ (revenue CPs) or $d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}} < \alpha_{i,j}c_{i,j}p_c$ (promotion CPs), $\gamma_{i,j}^* = 0$ and relative demand is a constant 1. Thus, relative demand is nondecreasing in $d_{i,j}$.

Following the method used in the proof of Prop. 6, it now suffices to show that user demand without sponsorship also increases in $d_{i,j}$. Since user demand is independent of $d_{i,j}$ this follows immediately.

2) Relative Benefits: We first consider revenue CPs. CP utility before sponsorship is then $d_{i,j}x_{i,j}^*\left(p_u(1+s_j)\right)$, which is clearly increasing in $d_{i,j}$.

We find that the CP's relative benefit when $d_{i,j} \geq \alpha_{i,j}p_c(1+s_j)$ (then $\gamma_{i,j}^* > 0$ unless $d_{i,j} = \alpha_{i,j}p_c(1+s_j)$, from Corollary 1) is

$$\frac{\alpha_{i,j}}{1-\alpha_{i,j}}\left(\frac{p_c(1+s_j)}{d_{i,j}}-1\right)\left(\frac{1}{1-\alpha_{i,j}}\left(1-\frac{d_{i,j}}{p_c(1+s_j)}\right)\right)^{\frac{-1}{\alpha_{i,j}}}$$

whose derivative with respect to $d_{i,j}$ is proportional to

$$\frac{-p_c(1+s_j)}{\left(d_{i,j}\right)^2\left(1-\alpha_{i,j}\right.}\left(1-\frac{d_{i,j}}{p_c(1+s_j)}\right) + \left(\frac{p_c(1+s_j)}{d_{i,j}}-1\right)\left(\frac{1}{\alpha_{i,j}p_c(1+s_j)(1-\alpha_{i,j})}\right),$$

which is positive if

$$\left(1+\alpha_{i,j}\right)d_{i,j}p_c(1+s_j) > p_c^2\left(1+s_j\right)^2\alpha_{i,j} + \left(d_{i,j}\right)^2.$$

Gathering terms and factoring, we find the condition $\left(d_{i,j}-\alpha_{i,j}p_c(1+s_j)\right)\left(p_c(1+s_j)-d_{i,j}\right) < 0$, which holds by assumption. Again, if $\gamma_{i,j} = 0$, then relative benefit is 1, and independent of $d_{i,j}$. Thus, both CP utility before sponsorship and relative benefit are nondecreasing in $d_{i,j}$, which by the same argument in Prop. 4 proves the desired fairness result.

We now consider promotion CPs. CP utility before sponsorship, $d_{i,j}x_{i,j}^*\left(p_u(1+s_j)\right)^{1-\alpha_{i,j}}/(1-\alpha_{i,j})$, is increasing in $d_{i,j}$; thus, it suffices to show that CPs' relative benefit increases with $d_{i,j}$. We take the derivative of the relative benefit (Table III if $d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}} \geq \alpha_{i,j}c_{i,j}p_c$) with respect to $d_{i,j}$ to find that it is proportional to

$$\frac{-c_{i,j}p_c\alpha_{i,j}}{d_{i,j}^2p_u(1+r_{i,j}s_j)}\left(1-\alpha_{i,j}+\frac{d_{i,j}p_u(1+r_{i,j}s_j)^{\alpha_{i,j}}}{c_{i,j}p_c}\right) + \frac{p_u(1+r_{i,j}s_j)^{\alpha_{i,j}}}{c_{i,j}p_c\alpha_{i,j}}\left(1-\left(1-\alpha_{i,j}\right)\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}-1} + \frac{c_{i,j}p_c\alpha_{i,j}(1-\alpha_{i,j})}{d_{i,j}p_u(1+r_{i,j}s_j)}\right).$$

Upon multiplying out this expression, we find the equivalent condition

$$\begin{aligned}
&c_{i,j}^2\alpha_{i,j}^2\left(1-\alpha_{i,j}\right)p_c^2 + d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}} \\
&\qquad\times\left(\alpha_{i,j}^2c_{i,j}p_c + d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}}\left(1-\alpha_{i,j}\right)\right) \\
&< d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}} \\
&\qquad\times\left(d_{i,j}p_u\left(1+r_{i,j}s_j\right) + (1-\alpha_{i,j})c_{i,j}p_c\alpha_{i,j}\right).
\end{aligned}$$

We now observe that $\left(1+r_{i,j}s_j\right)^{1+\alpha_{i,j}} > \left(1+r_{i,j}s_j\right)^{2\alpha_{i,j}}$ to find the sufficient condition

$$\begin{aligned}
&\alpha_{i,j}\left(1-\alpha_{i,j}\right)c_{i,j}^2p_c^2 + \left(2\alpha_{i,j}-1\right)d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}}c_{i,j}p_c \\
&-\alpha_{i,j}d_{i,j}^2p_u^2\left(1+r_{i,j}s_j\right)^{2\alpha_{i,j}} < 0.
\end{aligned}$$

Factoring the left-hand side of the equation, we obtain

$$\begin{aligned}
&\left(-d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}} - (1-\alpha_{i,j})c_{i,j}p_c\right) \\
&\times\left(d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}} - \alpha_{i,j}c_{i,j}p_c\right) < 0.
\end{aligned}$$

The first term in this product is always negative, and the second is always positive when $\gamma_{i,j}^* > 0$ from (8). Thus, if $d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}} \geq \alpha_{i,j}c_{i,j}p_c$ relative benefit is always decreasing in $d_{i,j}$; if $d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}} = \alpha_{i,j}c_{i,j}p_c$, relative benefit is constant, so the relative benefit is always non-decreasing with $d_{i,j}$. ■

J. Proof of Lemma 2

Suppose that CP j sponsors $\gamma_{i,j}$ amount of content for user i. Then users' utility is

$$\left(\frac{c_{i,j}^{\frac{1}{\alpha_{i,j}}}\left(r_{i,j}s_j+\alpha_{i,j}\right)}{(1-\alpha_{i,j})(1+r_{i,j}s_j)}\right)\left(p_u\left(1-\gamma_{i,j}+s_j\right)\right)^{1-\frac{1}{\alpha_{i,j}}}. \quad (22)$$

Since $\alpha_{i,j} < 1$, we see that (22) is increasing with $\gamma_{i,j}$. Thus, the user will always benefit if a CP decides to sponsor some data. ■

K. Proof of Proposition 9

We first show that ISPs benefit relative to before sponsorship. Suppose that the ISP chooses $p_u^* = p_c^*$ to be the same as p_u before sponsorship. Then the ISP price per unit of content is independent of the $\gamma_{i,j}^*$. Moreover, since user demand can only increase with sponsorship (Prop. 4), ISP revenue can only increase. If the constraint on total demand $\sum_{i,j}x_{i,j}^* \leq X$ does not hold with $p_u^* = p_c^*$ equal to the price before sponsorship, then the ISP can increase p_c until it holds. Its marginal price $p_u(1+s_j) + \gamma_{i,j}^*\left(p_c - p_u\right)$ must then increase relative to that before sponsorship, while its demand is the maximum possible and therefore at least that before sponsorship; its total revenue is then at least that before sponsorship. Without the constraint $p_u^* = p_c^*$, ISPs have more flexibility to choose their prices, so their revenue can only increase further.

We now consider users and CPs. If p_u^* decreases relative to that before sponsorship, users' effective data prices $p_u(1-\gamma_{i,j}^* + s_j)$ must decrease (for any $\gamma_{i,j}^* > 0$). Lemma 2 then shows the result for end users. Finally, we show that CPs benefit since they optimally choose $\gamma_{i,j}^*$. Thus, their utilities must be at least as large as that obtained by choosing $\gamma_{i,j} = 0$, i.e., no sponsorship. If CPs choose not to sponsor data, their cost of sponsorship is zero, as it is before sponsorship. However, the data component $\overline{U}_{i,j}$ of their utility function (3) increases since user demand increases due to lower p_u^*. Thus, CP utility also increases relative to that without sponsorship. ■

L. *Proof of Proposition 10*

We first compute promotion CP utility before sponsorship, which is $d_{i,j}\left(p_u(1+s_j)/c_{i,j}\right)^{\frac{-1}{\alpha_{i,j}}+1}/(1-\alpha_{i,j})$. With sponsorship, promotion CP utility $W_{i,j}\left(\gamma_{i,j}^*\right)$ is

$$\frac{(1+s_j)p_c}{1-\alpha_{i,j}}\left(\frac{p_u p_c(1+s_j)}{(1-\alpha_{i,j})c_{i,j}p_c + d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}}}\right)^{\frac{-1}{\alpha_{i,j}}}$$
$$\left(\frac{d_{i,j}p_u\left(1+r_{i,j}s_j - (1-\alpha_{i,j})\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}}\right)}{(1+r_{i,j}s_j)\left((1-\alpha_{i,j})c_{i,j}p_c + d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}}\right)}\right.$$
$$\left.+\frac{\alpha_{i,j}(1-\alpha_{i,j})c_{i,j}p_c}{(1+r_{i,j}s_j)\left((1-\alpha_{i,j})c_{i,j}p_c + d_{i,j}p_u\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}}\right)}\right).$$

We can then compute the desired ratios upon noting that user utility without sponsorship is

$$\frac{c_{i,j}\left(\alpha_{i,j}+r_{i,j}s_j\right)}{(1-\alpha_{i,j})(1+r_{i,j}s_j)}\left(\frac{p_u(1+s_j)}{c_{i,j}}\right)^{\frac{-1}{\alpha_{i,j}}+1}$$

and that with sponsorship is

$$\frac{c_{i,j}\left(\alpha_{i,j}+r_{i,j}s_j\right)}{(1-\alpha_{i,j})\left(1+r_{i,j}s_j\right)}$$
$$\times\left(\frac{p_u p_c(1+s_j)}{(1-\alpha_{i,j})c_{i,j}p_c + d_{i,j}p_u(1+r_{i,j}s_j)^{\alpha_{i,j}}}\right)^{\frac{-1}{\alpha_{i,j}}+1}. \quad (23)$$

Similarly, we find that revenue CP utility before sponsorship is $d_{i,j}\left(p_u(1+s_j)/c_{i,j}\right)^{\frac{-1}{\alpha_{i,j}}}/(1+r_{i,j}s_j)$, while that after sponsorship is given by

$$\frac{\alpha_{i,j}\left(p_c(1+s_j)-d_{i,j}\right)}{(1-\alpha_{i,j})(1+r_{i,j}s_j)}\left(\frac{p_u p_c(1+s_j)-p_u d_{i,j}}{c_{i,j}p_c(1-\alpha_{i,j})}\right)^{\frac{-1}{\alpha_{i,j}}}.$$

We can then compute the desired ratios with user utility, which after sponsorship is

$$\frac{c_{i,j}\left(\alpha_{i,j}+r_{i,j}s_j\right)}{(1-\alpha_{i,j})(1+r_{i,j}s_j)}\left(\frac{p_u p_c(1+s_j)-p_u d_{i,j}}{c_{i,j}p_c(1-\alpha_{i,j})}\right)^{1-\frac{1}{\alpha_{i,j}}}.$$

To see that the CP-to-user utility ratio for revenue CPs always decreases with sponsorship, we set

$$\frac{(1-\alpha_{i,j})d_{i,j}}{\left(r_{i,j}s_j+\alpha_{i,j}\right)p_u\left(1+s_j\right)} \geq \frac{p_c\alpha_{i,j}\left(1-\alpha_{i,j}\right)}{p_u\left(r_{i,j}s_j+\alpha_{i,j}\right)},$$

or equivalently $d_{i,j} \geq p_c\alpha_{i,j}(1+s_j)$, which is always the case when $\gamma_{i,j}^* > 0$. With promotion CPs, we have

$$\frac{d_{i,j}p_u\left(1+r_{i,j}s_j-(1-\alpha_{i,j})\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}}\right)}{p_u c_{i,j}\left(\alpha_{i,j}+r_{i,j}s_j\right)}$$
$$+\frac{\alpha_{i,j}(1-\alpha_{i,j})c_{i,j}p_c}{p_u c_{i,j}\left(\alpha_{i,j}+r_{i,j}s_j\right)}$$
$$\leq \frac{d_{i,j}(1+r_{i,j}s_j)}{\left(\alpha_{i,j}+r_{i,j}s_j\right)c_{i,j}},$$

which simplifies to

$$\alpha_{i,j}(1-\alpha_{i,j})c_{i,j}p_c \leq d_{i,j}p_u(1-\alpha_{i,j})\left(1+r_{i,j}s_j\right)^{\alpha_{i,j}},$$

which is always the case when $\gamma_{i,j}^* > 0$. ■

Appendix B
Additional Simulations

In this appendix, we present numerical evidence that our conclusions in Section IV hold for $\beta_{i,j} \neq \alpha_{i,j}$ in the CP utility function (3). In particular, we plot analogues of Figures 4–6, as well as evaluating the fairness of the demand, user utility, and CP utility distributions when $\beta_{i,j}$ is fixed at 0.5 (Table II).

Figure 11 shows analogues of Figures 4 and 5 for $\beta_{i,j} \neq \alpha_{i,j}$ in the CP utility function (3). We see in Figure 11a that when $\beta_{i,j} = 0.2$, user demand can both increase and decrease with $\alpha_{i,j}$, just as it does when $\beta_{i,j} = \alpha_{i,j}$ in Figure 4b (Prop. 4). Similarly, when the amount of ads s_j decreases, user utility for promotion CPs decreases for $\beta_{i,j} = 0.4$ and for $\beta_{i,j} = \alpha_{i,j}$ in Figure 11b. We see a similar trend in CP utility in Figure 11c. Finally, we observe that as $\beta_{i,j}$ varies in Figure 11d, we see qualitatively similar user demands for $\beta_{i,j} = \alpha_{i,j}$ and $\alpha_{i,j}$ fixed at 0.4 as $\beta_{i,j}$ varies. For reference, we also plot demand when $\beta_{i,j} = \alpha_{i,j}$ is fixed at 0.4; the demand then does not change with $\beta_{i,j}$, but is of similar magnitude.

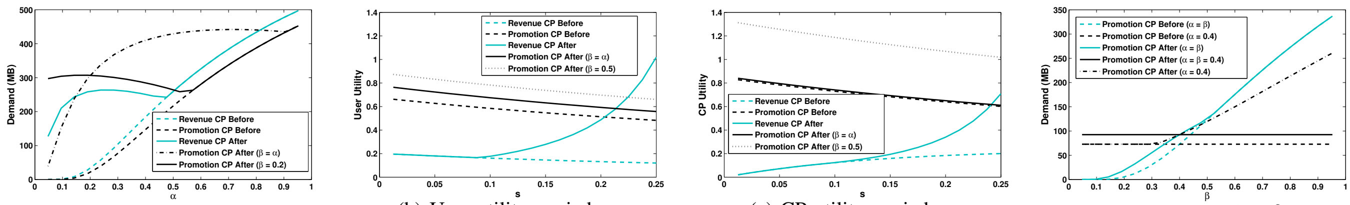
Figure 12 expands Figure 6's results for a fixed $\beta_{i,j} = 0.3$. User utility (Figure 12b) and CP utility (Figure 12d) show qualitatively similar trends as their cost awareness varies for both $\beta_{i,j} = \alpha_{i,j}$ and $\beta_{i,j} = 0.3$, as do the demands (Figures 12a and 12c). However, $\gamma_{i,j}^* = 0$, i.e., the CP does not sponsor content, for lower values of $c_{i,j}$ and higher values of $d_{i,j}$ with $\beta_{i,j} = 0.3$ compared to $\beta_{i,j} = \alpha_{i,j}$. Intuitively, for $\beta_{i,j} < \alpha_{i,j}$, the CP's utility function is less concave, so as users' cost awareness shrinks, the CP benefits more from the resulting increase in demand and need not sponsor as much data. Similarly, as the CP's cost awareness decreases, a less concave utility function means that CPs already derive more utility from user demand and can sponsor less data.

Table V shows that the fairness trends observed in Section IV-B still hold for fixed β. As we vary $\alpha_{i,j}$ and hold $\beta_{i,j} = 0.5$ constant, the fairness of user demand and user utility increases with sponsorship, just as it does with $\beta_{i,j} = \alpha_{i,j}$[11]. Similarly, as user cost awareness $c_{i,j}$ varies, the distributions of user

[11] Note that we cannot directly compare the fairness of CP utilities with fixed $\beta_{i,j}$ after sponsorship with that before sponsorship, as the fairness before sponsorship depends on $\beta_{i,j}$.

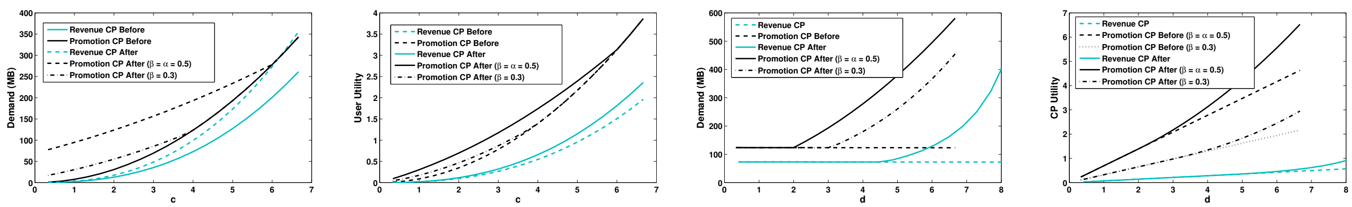
TABLE V: Jain's fairness index before and after sponsorship over all CP-user pairs. Index values are reported for distributions of (demands, user utilities, CP utilities).

Variable	Before (Revenue CP)	After (Revenue CP)	Before (Promotion CP)	After (Promotion CP)	After (Promotion CP, $\beta \neq \alpha$)
$\alpha_{i,j}$ (Figure 11a)	(0.674, 0.214, 0.674)	(0.912, 0.219, 0.719)	(0.65, 0.21, 0.256)	(0.922, 0.228, 0.267)	(0.971, 0.217, 0.82)
$c_{i,j}$ (Figures 12a,12b)	(0.503, 0.503, 0.503)	(0.503, 0.503, 0.503)	(0.57, 0.57, 0.768)	(0.856, 0.676, 0.868)	(0.649, 0.6, 0.698)
$d_{i,j}$ (Figures 12c, 12d)	(1, 1, 0.769)	(0.643, 0.859, 0.673)	(1, 1, 0.769)	(0.769, 0.927, 0.705)	(0.793, 0.943, 0.709)



(a) User demand, varied $\alpha_{i,j}$. (b) User utility, varied s_j. (c) CP utility, varied s_j. (d) Demand, varied $\beta_{i,j}$.

Fig. 11: With sponsorship, (a) demand decreases as price elasticity decreases for users of a revenue ($c_{i,j} = 5.8$, $d_{i,j} = 5.4$) and promotion ($c_{i,j} = 5.3$, $d_{i,j} = 4.8$) CP; (b) user and (c) CP utility increases with ads for a user of revenue CPs ($\alpha_{i,j} = 0.3$, $c_{i,j} = 4$) and decreases for a user of promotion CPs ($\alpha_{i,j} = 0.4$, $c_{i,j} = 4$); (d) demand increases as CP price elasticity decreases for a user of revenue CPs ($\alpha_{i,j} = 0.3$, $c_{i,j} = 4$) or a user of promotion CPs ($\alpha_{i,j} = 0.4$, $c_{i,j} = 4$, $d_{i,j} = 2$).



(a) Demand, varied $c_{i,j}$. (b) User utility, varied $c_{i,j}$. (c) Demand, varied $d_{i,j}$. (d) CP utility, varied $d_{i,j}$.

Fig. 12: (a) User utility and (b) demand increases as user cost awareness decreases; similarly, (c) demand and (d) CP utility increases as CP cost awareness decreases. We use (a,b) one revenue ($\alpha_{i,j} = 0.4$) and one promotion ($\alpha_{i,j} = 0.5$, $d_{i,j} = 3$) CP; (c,d) one user of revenue ($\alpha_{i,j} = 0.4$, $c_{i,j} = 4$, $r_{i,j} \in [0.002, 0.03]$) and one of promotion CPs ($\alpha_{i,j} = 0.5$, $c_{i,j} = 4$, $d_{i,j} = 2$).

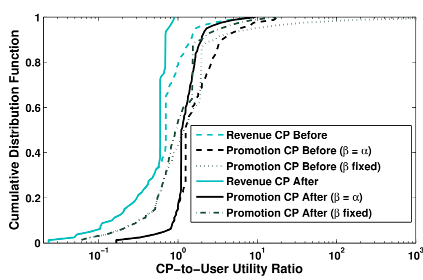


Fig. 13: Distribution of CP-to-user utility ratios over all CP-user pairs for all simulations in Figures 11a–11c and 12.

demand and utility become more fair, though to a lesser degree than for $\beta_{i,j} = \alpha_{i,j}$ (Prop. 7). As CP cost awareness varies, the distributions of demand and user utility both become more unfair for $\beta_{i,j} = \alpha_{i,j}$ and for $\beta_{i,j} = 0.3$. Before sponsorship, the fairness of CP utilities was 0.793 for $\beta_{i,j} = 0.3$, so we see that the distribution of CP utility becomes more unfair as well for both values of $\beta_{i,j}$ (Prop. 8).

Finally, in Figure 13, we show the ratios of CP to user utility before and after sponsorship for all simulations varying $\alpha_{i,j}$, s_j, $c_{i,j}$, and $d_{i,j}$ in Figures 11a–11c and 12. We see that for all types of CPs (revenue, promotion with $\beta_{i,j} = \alpha_{i,j}$, and promotion with $\beta_{i,j} \neq \alpha_{i,j}$), the utility ratio decreases with sponsorship (Prop. 10).