Abstract—In January 2014, AT&T introduced sponsored data to the U.S. mobile data market, allowing content providers (CPs) to subsidize users’ cost of mobile data. As sponsored data gains traction in industry, it is important to understand its implications. This work considers CPs’ choice of how much content to sponsor and the implications for users, CPs, and ISPs (Internet service providers). We first formulate a model of user, CP, and ISP interaction for heterogeneous users and CPs and derive their optimal behaviors. We then show that these behaviors can reverse our intuition as to how user demand and utility change with different user and CP characteristics. While all three parties can benefit from sponsored data, we find that sponsorship disproportionately favors less cost-constrained CPs and more cost-constrained users, exacerbating CP inequalities but making user demand more even. We also show that users’ utilities increase more than CPs’ with sponsored data. We finally illustrate these results in practice through numerical simulations with data from a commercial pricing trial and introduce a framework for CPs to decide which, in addition to how much, content to sponsor.

I. INTRODUCTION

As demand for mobile data grows, ISPs (Internet service providers) are turning to new types of smart data pricing (SDP) to generate revenue for expanding their network capacity [1]. Though much SDP research studies different ways to charge users for data [2], sponsored data instead introduces a new party to data pricing: content providers (CPs). Facebook has long sponsored data for its mobile site in developing markets [3], and AT&T began offering sponsored data in January 2014 [4]. The ISP FreedomPop has introduced free, 100% sponsored data plans [5], and the startups Syntonic Wireless and DataMi† offer marketplaces for CPs to sponsor data for different apps.‡

A. Sponsoring Mobile Data

Under AT&T’s sponsored data plan, CPs can split the cost of transferring mobile data traffic over ISP networks with end users. Figure 1 illustrates the data and payment flows between end users, CPs, and ISPs. Sponsored content is represented by the dashed arrow showing payments from CPs to the ISP, while the solid arrow represents user payments.

Content sponsorship has the potential to benefit users, CPs, and ISPs: users are charged lower prices due to CP subsidies, CPs can attract more traffic as users increase their demand, and ISPs can generate more revenue. Yet such plans have raised concerns over the possible advantage they give to larger CPs that can better afford to sponsor data [6]. These concerns echo the controversy over CPs paying for higher Quality-of-Service (QoS) for their users. In this paper, we do not consider QoS: we show that even without a QoS component, sponsored data can exacerbate profit and demand unevenness among different CPs, but can also even out demand among different types of users and benefit users more than CPs.

In current sponsored data implementations, CPs decide which content to sponsor and mark it as such in the app, as in Figure 2’s example of sponsored news stories on Trove. The CP app routes all sponsored traffic through per-app proxies or VPNs implemented on users’ devices, allowing ISPs to identify sponsored traffic and bill the CP. With this architecture, CPs can easily sponsor different content for different users. Though not trivial to implement, per-app VPN support is becoming more widespread in iOS [7] and Android [8].

In our sponsored data model, CPs decide how much data to sponsor for each user, unlike previous works in which ISPs choose the amount sponsored [9], [10]. While ISPs can influence CPs by changing their data prices, sponsored data’s effect on the mobile data market will be determined by how much data CPs decide to sponsor. It is this decision, and the impact on users and CPs, that we examine in this paper.

B. Modeling Content Provider Behavior

Since CPs decide how much data to sponsor for different users, an accurate model of their behavior must include the full heterogeneity of CP and user data valuations. We therefore explicitly model not only differences in user price sensitivity,
as was done in [11], but also the different benefits that CPs receive from greater user demand, as shown in Table I. While some CPs take in ad revenue, as considered in [12], others rely on subscription revenue, and still other CPs benefit from user goodwill or usage itself. We thus classify CPs as “revenue CPs,” which benefit from usage as it contributes to revenue, and “promotion CPs,” which directly benefit from usage.

Most revenue CPs rely on either ads or freemium subscriptions to make money [13]. For apps that rely on ads like Pandora or Facebook [14], [15], revenue grows linearly with usage, as the number of ads shown is often proportional to the amount of content consumed (e.g., ads at regular intervals between songs or news stories). Other apps charge users per unit of content consumed, e.g., Vimeo’s per-video viewing fees. Still others, like Netflix or freemium apps, offer flat-fee subscriptions that do not depend on usage volume. These CPs are arguably less likely to sponsor data to increase subscriptions, since non-subscribers likely derive little utility from the content itself and would require high subsidies.

Promotion CPs benefit directly from increased usage. For instance, a new photo-sharing app may sponsor data to attract users and usage in the early stages of its release. Another type of CP, enterprises, might subsidize employee usage of company apps, encouraging them to work more while out of the office. In both cases, the CP’s benefit from increased demand is concave rather than linear, mirroring users’ diminishing marginal utility from more data usage.

### C. Implications of Sponsored Data

By fully accounting for user and CP heterogeneity, we find that content sponsorship can:

- Reverse our intuition as to how user demand changes with different user and CP characteristics. For instance, while we would expect user utility to decrease as CPs show more ads, with sponsorship revenue CPs may sponsor more data if they show more ads, increasing users’ utility.
- Disproportionately benefit more cost-aware users and less cost-aware CPs$^3$. We thus justify the concern that sponsored data will exacerbate the advantage of larger, less cost-aware CPs [6]. However, sponsored content’s benefit to more cost-conscious users implies that it evens out the distributions of demand and utility across users.
- Benefit users more than CPs. User utility increases proportionally more than a CP’s when the CP chooses the amount of data sponsored so as to maximize its utility.

After briefly reviewing related work in Section II, we derive the optimal behavior for users, CPs, and ISPs in Section III and analyze the implications in Section IV. In Section V, we consider sponsored data in practice, illustrating our results with data from a commercial pricing trial and proposing a framework for CPs to decide which pieces of content to sponsor. We conclude in Section VI. All proofs are in Appendix A.

### II. RELATED WORK

Much of the prior work on sponsored content either focuses on ISPs’ optimal actions in splitting costs between CPs and users or includes QoS prioritization and examines the implications for network neutrality. For instance, CPs might explicitly pay ISPs extra fees for higher QoS [16]. Many such works use game theory to identify ISPs’ and CPs’ optimal actions and the consequences for network neutrality [17]. When user and CP demand are defined in terms of bandwidth speed (i.e., QoS), [9] considers monopolistic and perfectly competitive ISPs’ optimal amount to charge CPs and end users, while [10] proposes a similar model that includes transit and user-facing ISPs. However, these works allow ISPs, instead of CPs, to determine the amount of sponsorship. They therefore do not reflect the full heterogeneity of CP behavior.

Other works study the effects of sponsored data on CPs and end users, e.g., [11] and [18] examine CPs’ market share when CPs sponsor data for homogeneous users or pay fixed side payments to ISPs respectively. CP sponsorship of advertisements is explicitly addressed in [12], which accounts for users’ probability of viewing different ads and monthly data caps. Implementation challenges are discussed in [19].

### III. SPONSORED DATA MODEL

We consider three players in the sponsored data ecosystem: users, CPs, and ISPs. They make decisions in three stages, as shown in Figure 3: first, the ISP chooses the prices to charge users and CPs. CPs then decide how much content to sponsor, and finally users choose how much content to consume from each CP, depending on the amount sponsored and ISP price. CPs can sponsor different amounts of data for different types of users. Each party selfishly maximizes its own utility subject to others’ decisions. We assume a monopolistic ISP; while many ISPs, e.g., in the US, are oligopolists, due to low churn rates they are often effective monopolies [20].

#### A. End Users’ Decisions

Suppose that $N$ users and $M$ CPs exchange traffic over the ISP’s network. Each user $i$ receives a utility $V_{i,j}(x_{i,j}, p_u, \gamma_{i,j})$
from CP $j$, where $x_{i,j}$ is the monthly volume of content that user $i$ consumes from CP $j$, $p_u$ is the unit price of data that the ISP charges users, and $\gamma_{i,j}$ is the fraction of content sponsored by CP $j$ for user $i$.\footnote{In practice, a user $i$ can represent a group of users with similar behavior.} We assume that users incur a linear data cost, as is commonly done \cite{9, 11, 12}. For instance, users can choose one of several data plans that charge different amounts for different monthly data caps \cite{21}.

Users are affected by two variables chosen by the CP: the volume of ads per volume of content, $s_{j}$, and $\gamma_{i,j}$. We assume that $s_{j}$ is constant over all of the CP’s content; for instance, Pandora plays ads at regular intervals between songs. While in theory $s_{j}$ can take any positive value, we assume $s_{j} \in [0, 1)$ since most CPs provide more content than ads. We note that $\gamma_{i,j}$ represents the aggregate fraction of data sponsored over one month, including sponsorship of ads and content; thus, $\gamma_{i,j} \in [0, 1]$. The scenario before sponsored data corresponds to taking $\gamma_{i,j} \equiv 0$ for all users $i$ and CPs $j$. With sponsored data, the user pays for $(1 - \gamma_{i,j} + s_{j}) x_{i,j}$ amount of data; thus, her data cost is $p_u (1 - \gamma_{i,j} + s_{j}) x_{i,j}$.

We suppose that, absent the data cost, a user derives utility $V_{i,j}(x_{i,j}(1 + r_{i,j}s_{j}))$ from consuming $x_{i,j}$ amount of content from CP $j$, where $U_{i,j}$ is a concave utility function. The factor $r_{i,j}$ represents user $i$’s ad-click-through rate for CP $j$, i.e., the fraction of ads contributing to user utility. While users rarely derive utility from ads, clicking on an ad indicates that they find it entertaining or useful. We assume that $r_{i,j}$ is independent of sponsorship (the bottleneck to users’ clicking on ads is likely their lack of interest, not the data cost). Each user $i$’s utility function for CP $j$ is then

$$V_{i,j} = c_{i,j} (x_{i,j}(1 + r_{i,j}s_{j}))^{1 - \alpha_{i,j}} - p_u (1 - \gamma_{i,j} + s_{j}) x_{i,j},$$

where we take $U_{i,j}(x) = c_{i,j} x^{1 - \alpha_{i,j}}/(1 - \alpha_{i,j})$, the $\alpha$-fair utility function with $\alpha_{i,j} \in [0, 1]$ and a scaling factor $c_{i,j} > 0$. A user’s total utility is the sum of his utilities from each CP: usage of one CP does not affect the utility from others, e.g., browsing Facebook does not affect the utility of watching Hulu. User $i$’s optimal demand for data from each CP $j$ is then

$$x_{i,j}^* (p_u (1 - \gamma_{i,j} + s_{j})) = \left( p_u (1 - \gamma_{i,j} + s_{j}) \right) / c_{i,j} (1 + r_{i,j}s_{j})^{1 - \alpha_{i,j}}.$$

B. Content Provider Sponsorship

As discussed in Section I-B, the amount of data that a CP sponsors depends on the CP’s benefit from user demand, i.e., whether the CP is a “revenue” or “promotion” CP. These two scenarios can be viewed as special cases of a general CP utility model. As with end users, we suppose that CPs’ utility functions include a utility and a cost component. We use $W_{i,j}$ to denote CP $j$’s overall utility function for user $i$:

$$W_{i,j} (\gamma_{i,j}) = U_{i,j}(x_{i,j}^*) - p_c \gamma_{i,j} x_{i,j}^*.$$

where $U_{i,j}(x) = d_{i,j} x^{1 - \beta_{i,j}}/(1 - \beta_{i,j})$, with $\beta_{i,j} \in [0, 1]$ and $d_{i,j}$ a positive scaling factor, specifies the CP’s utility from data usage. Here $x_{i,j}^*$ is the user demand (2). The term $p_c \gamma_{i,j} x_{i,j}^*$ represents the cost of delivering content for each user $i$; $p_c$ is the unit data price that ISPs charge CPs. Substituting (2) into (3), we now find that (3) equals

$$W_{i,j} (\gamma_{i,j}) = \frac{d_{i,j}}{1 - \beta_{i,j}} \left( \frac{p_u (1 - \gamma_{i,j} + s_{j})}{c_{i,j} (1 + r_{i,j}s_{j})^{-\alpha_{i,j}}} \right)^{1 - \beta_{i,j}} - \frac{p_c \gamma_{i,j}}{1 + r_{i,j}s_{j}} \left( \frac{p_u (1 - \gamma_{i,j} + s_{j})}{c_{i,j}} \right)^{1 - \beta_{i,j}}.$$

We assume that $s_{j}$, the fraction of ads per content volume, does not change with sponsorship (e.g., to avoid disrupting users’ experience). The CP chooses $\gamma_{i,j}$ to maximize $\sum_{i=1}^{N} W_{i,j}$, with the constraint $\gamma_{i,j} \in [0, 1 + s_{j}]$. If the CP must sponsor the same amount of content for all users $i$ (e.g., due to lack of information on different users’ behavior), $\gamma_{i,j} \equiv \gamma$ for all $i$, turning the optimization into a bounded line search.

We now consider the CP’s optimization problem for Section I-B’s revenue and promotion CPs:

1) Revenue CPs: We first consider a CP whose utility $U_{i,j}$ is its revenue. We assume that the CP’s revenue is proportional to user demand, as discussed in Section I-B and Table I.\footnote{CPs with flat subscription fees would choose the minimum $\gamma_{i,j}$ such that each user $i$’s utility is higher than the cost of subscription. As discussed in Section I-B, these CPs will be less likely to sponsor data.} Aside from transport costs paid to the ISP, we do not explicitly consider CP data costs; these may be included by reducing the revenue per unit volume by a constant marginal cost.

We now take $\beta_{i,j} = 0$ and $d_{i,j}$ to be the marginal revenue per unit of content in (3), making (3) the CP’s revenue less the cost of sponsoring data. For instance, CPs deriving revenue from a cost-per-click advertising model would take $d_{i,j} = a r_{i,j}s_{j}$, where $a$ is the revenue per volume of ads clicked on. We can calculate $a$ from the cost per click and average ad volume.\footnote{Similarly, $d_{i,j} = a s_{j}$ if the CP receives cost-per-mille ad revenue.} Thus, we find the CP utility function

$$W_{i,j} (\gamma_{i,j}) = \frac{d_{i,j} - p_c \gamma_{i,j}}{1 + r_{i,j}s_{j}} \left( \frac{p_u (1 - \gamma_{i,j} + s_{j})}{c_{i,j}} \right)^{1 - \beta_{i,j}},$$

yielding the optimization problem

$$\max_{\gamma_{i,j}} \sum_{i=1}^{N} \left( \frac{d_{i,j} - p_c \gamma_{i,j}}{1 + r_{i,j}s_{j}} \right) \left( \frac{p_u (1 - \gamma_{i,j} + s_{j})}{c_{i,j}} \right)^{1 - \beta_{i,j}} s.t. \gamma_{i,j} \in [0, 1 + s_{j}].$$

Proposition 1: Suppose that $d_{i,j} < p_c (1 + s_{j})$ for all users $i$. Then (5–6) has the optimal solution

$$\gamma_{i,j} (p_c, p_u) = \max \left\{ 0, \frac{d_{i,j}}{p_c (1 - \alpha_{i,j}) - (1 + s_{j}) \alpha_{i,j}} \right\}.$$

For instance, if $d_{i,j} = a r_{i,j}s_{j}$, users’ click-through rates $r_{i,j}$ are generally small (< 5% \cite{22}) and Prop. 1’s condition
\[ d_{i,j} < p_c(1 + s_j) \] is reasonable. By setting \( \gamma_{i,j}^* > 0 \) in (7), we can find conditions under which a CP wishes to sponsor data:

**Corollary 1:** CP \( j \) sponsors data for user \( i \) (\( \gamma_{i,j}^* > 0 \)) if and only if \( d_{i,j} > \alpha_{i,j}p_c(1 + s_j) \).

Thus, if a CP's marginal revenue \( d_{i,j} \) is sufficiently small compared to its data price \( p_c \), it will not sponsor any data. As user utility becomes more concave (\( \alpha_{i,j} \) increases), the CP becomes less likely to sponsor data: its sponsorhip has less effect on user demand and is thus less profitable. However, \( \gamma_{i,j}^* \) is independent of \( c_{i,j} \), indicating that revenue CPs will not discriminate among users with different utility "levels."

2) **Promotion CPs:** Promotion CPs benefit directly from usage. These CPs can try to attract users by optimizing their users’ utilities, e.g., in order to promote their brand or app. Similarly, enterprise CPs will have the same utility function as their employees when sponsoring company apps: the benefit of increased usage for both is its contribution to employee productivity. Though promotion CPs can sponsor ads, we assume that their ad revenue is negligible compared to their utility from usage (e.g., recently released apps may earn very little revenue per ad since advertisers are unfamiliar with their value). Enterprise CPs will likely not show ads.

For promotion CPs, we thus take \( \beta_{i,j} = \alpha_{i,j} \) in (3): user and CP utility components have the same shape.\(^7\) By taking the CP and user scaling factors \( d_{i,j} \) and \( c_{i,j} \) to be different, we can introduce different weights on the utility, e.g., if CPs care less about cost relative to gaining demand than users care about their data cost. We then solve for the optimal \( \gamma_{i,j}^* \).

**Proposition 2:** Suppose that \( \beta_{i,j} = \alpha_{i,j} > 0 \). Then \( W_{i,j} \) is maximized with respect to \( \gamma_{i,j} \in [0, 1 + s_j] \) at

\[
\gamma_{i,j}^* = \max \left\{ 0, \frac{(1 + s_j) \left( d_{i,j} (1 + r_{i,j}s_j)^{\alpha_{i,j}}/p_u - \alpha_{i,j}c_{i,j}p_c \right)}{(1 - \alpha_{i,j})c_{i,j}p_c + d_{i,j} (1 + r_{i,j}s_j)^{\alpha_{i,j}}/p_u} \right\}
\]

Promotion CPs thus sponsor data only if the price \( p_u \) paid by users and the CP utility scaling \( d_{i,j} \) are sufficiently high:

**Corollary 2:** CP \( j \) sponsors data for user \( i \) if only if \( d_{i,j} (1 + r_{i,j}s_j)^{\alpha_{i,j}}/p_u > \alpha_{i,j}c_{i,j}p_c \). Intuitively, if users’ data is already inexpensive (small \( p_u \)) or the CP cares more about cost than promoting usage (small \( d_{i,j} \)), the CP has no incentive to sponsor data. In contrast to revenue CPs, however, \( \gamma_{i,j}^* \) depends on users’ utility scaling factors \( c_{i,j} \). Promotion CPs experience diminishing marginal utility with usage volume, so they will be less likely to sponsor data for users with higher \( c_{i,j} \), whose demand \( x_{i,j}^* \) is already large without sponsorship (cf. (2)).

**C. ISP Price Optimization**

Like the CPs, the ISP chooses the prices \( p_c \) and \( p_u \) so as to maximize its profit.\(^8\) We suppose that the ISP has a finite amount of available capacity in its network, e.g., LTE base stations can handle only a finite amount of traffic at any given time. We translate this instantaneous capacity into a maximum monthly demand for data, \( X \), by supposing that the peak demand over time is a function of the total demand for data. We thus introduce the constraint \( \sum_{i,j} (1 + s_j) x_{i,j}^* (\pi_{i,j}^*) \leq X \), where \( (1 + s_j) x_{i,j}^* \) is the total volume of data pushed over the ISP’s network by user \( i \) for CP \( j \), and \( \pi_{i,j}^* (p_c, p_u) = p_u (1 - \gamma_{i,j}^* (p_c, p_u) + s_j) \) denotes user \( i \)’s effective data price for each CP \( j \). The ISP then wishes to maximize its total profit subject to this capacity constraint, i.e., to solve

\[
\max_{p_c, p_u \geq 0} \sum_{i=1}^{N} \sum_{j=1}^{M} \left( \pi_{i,j}^* + p_c \gamma_{i,j}^* \right) x_{i,j}^* \left( \pi_{i,j}^* \right) \\
\text{s.t.} \sum_{i=1}^{N} \sum_{j=1}^{M} (1 + s_j) x_{i,j}^* (\pi_{i,j}^*) \leq X
\]

We can solve (9–10) by noting that both \( x_{i,j}^* (\pi_{i,j}^*) \) and \( (\pi_{i,j}^* + p_c \gamma_{i,j}^*) x_{i,j}^* (\pi_{i,j}^*) \) are decreasing in \( p_c \):

**Proposition 3:** If each CP optimally chooses \( \gamma_{i,j}^* \) so as to maximize (3), \( x_{i,j}^* (\pi_{i,j}^*) \) and (9) are both decreasing in \( p_c \). Thus, for any given \( p_u \), the optimal \( p_c \) is the unique minimum value of \( p_c \) for which either (10) is satisfied with equality or \( d_{i,j} = p_c(1 + s_j) \) for some revenue CP \( j \). The optimal value of \( p_u \) can be found by a bounded line search.

**IV. IMPACT ON USERS AND CPs**

We now consider the implications of user, CP, and ISP behavior. We show that sponsorship can reverse some trends of user behavior (Section IV-A) and that it favors more cost-aware users and less cost-aware CPs (Section IV-B). Section IV-C shows that users, CPs, and ISPs all benefit from content sponsorship, but users benefit more than CPs.

We primarily characterize users and CPs by two attributes:

**Definition 1 (Price elasticity):** User \( i \)’s price elasticity for CP \( j \) is defined in the usual economic manner as the % change in usage in response to a 1% change in price.

In our model, each user \( i \)’s price elasticity for CP \( j \) is a constant \( \alpha_{i,j}^{-1} \). As \( \alpha_{i,j} \) increases, users have lower price elasticity and their demands are less sensitive to price changes.

**Definition 2 (Cost awareness):** A user \( i \)’s cost awareness for CP \( j \) equals \( c_{i,j}^{-1} \), the reciprocal of the \( U_{i,j} \) scaling factor in users’ utility (1).\(^5\) Similarly, a CP’s cost awareness is \( d_{i,j}^{-1} \). As \( c_{i,j} \) or \( d_{i,j} \) increases, the user or CP becomes less cost-aware: the utility function’s cost term is weighted less compared to the utility from using data. For revenue CPs, “cost awareness” is simply the reciprocal of the marginal revenue per unit of content, e.g., \( d_{i,j} = ar_{i,j}s_j \) for advertising revenue.

We use the term “user-CP pair” to refer to a given user’s demand for and utility derived from a given CP, as well as

\(^{7}\)While this appears to be a restrictive assumption, in Appendix B we show numerically that our results qualitatively hold for arbitrary \( \beta_{i,j} \).

\(^{8}\)The ISP may face price constraints, e.g., to avoid user anger over nominally higher prices. We then constrain \( p_u \leq P_u \), the current user price.

\(^{5}\)Optimizing \( V_{i,j} \) is equivalent to optimizing \( V_{i,j}/c_{i,j} \), for which the utility scaling factor equals one and the cost term is scaled by \( c_{i,j}^{-1} \).
the CP’s utility from that user. We also make a reasonable technical assumption on promotion CPs’ cost awarenesses: $c_{i,j}p_c > d_{i,j}p_a (1 + r_{i,j} s_j)^{\alpha_{i,j}} \ln (1 + r_{i,j} s_j)$. We expect this to hold since users’ click-through rates $r_{i,j}$ are generally < 5% [22], so $\ln (1 + r_{i,j} s_j) \approx 0$. To focus on the effect of CP rather than ISP actions, in this section we assume that the ISP does not change users’ data price $p_a$ with sponsorship, e.g., to avoid user anger over (nominally) higher prices.

A. Variation in Demand and Utility

We first show some numerical examples of interesting user behaviors before deriving conditions under which they are observed. In all simulations in the paper, unless otherwise noted we use the following parameters: ISP prices are $p_a = p_c = $10/GB, which approximates current ISP data prices (e.g., AT&T’s data plans vary from $7.50/GB to $25/GB [21]). Revenue CPs are assumed to make money from advertising with $a = $1800 per GB of ads, based on a $2 revenue per ad click [23] and an 880 KB average ad size, e.g., a short video. We assume that CPs carry an additional 15% of ads per content volume ($s_j = 0.15$), e.g., a 30-second ad for a 200-second video, and that users’ ad click-through rate is 2% ($r_{i,j} = 0.02$) [22], [23]. Revenue CPs then have $d_{i,j} = ar_{i,j} s_j = 5.4$. In all figures, we use “before” and “after” to respectively denote the scenarios before (i.e., without) and after (with) sponsorship.

Without sponsorship, user demand $x^*_i,j = \left(\frac{p_a}{c_{i,j}}\right) (1 + s_j) / (\ln (1 + r_{i,j} s_j))$; thus, if $p_a (1 + s_j) > c_{i,j}$, users’ demands increase as they become less price-elastic (i.e., as $\alpha_{i,j}$ increases). However, as Figure 4a shows, with sponsorship demand can both increase ($\alpha_{i,j} \leq 0.238$ for revenue CPs, $\leq 0.714$ for promotion CPs) and decrease ($\alpha_{i,j} \in [0.238, 0.476]$ for revenue CPs, $\alpha_{i,j} \in [0.714, 0.905]$ for promotion CPs) as $\alpha_{i,j}$ increases even when $p_a (1 + s_j) c_{i,j}$. We can derive sufficient conditions under which demand decreases as price elasticity decreases for all $\alpha_{i,j}$ as in Figure 4b:

\textbf{Proposition 4:} User demand increases as the CP sponsors more data ($\gamma_{i,j}$ increases). Moreover, a CP with $\gamma_{i,j} > 0$ will experience smaller demand $x^*_i,j$ as users become less price-elastic ($\alpha_{i,j}$ increases) if $p_a (p_c (1 + s_j) - d_{i,j}) < p_c c_{i,j}$.

Intuitively, as users become less price-elastic ($\alpha_{i,j}$ increases), they do not increase their demands as much in response to CPs’ sponsoring data to lower prices. CPs thus do not benefit as much from sponsorship and sponsor less data.

In Figure 4c, we see that $\gamma_{i,j}$ indeed decreases to zero as $\alpha_{i,j}$ increases. As $\gamma^*_{i,j}$ decreases, user demand $x^*_i,j$ also eventually decreases in Figures 4a and 4b. Proposition 4 shows that if a user’s cost awareness is low enough ($c_{i,j}$ is high), the decrease in $\gamma^*_{i,j}$ will cause user demand to decrease at any $\alpha_{i,j}$ with $\gamma^*_{i,j} > 0$, as in Figure 4b.

We next consider the amount of ads sponsored, $s_j$. Without sponsorship, user utility decreases as $s_j$ increases, since users must pay for the ads’ data and do not experience much utility from ads. However, Figure 5 shows that with sponsorship, user utility can increase with $s_j$ if $\gamma^*_{i,j} > 0$ ($s_j \geq 0.09$), though promotion CPs always observe a lower user utility as $s_j$ increases. We now see that this increase is due to revenue CPs sponsoring more data as $s_j$ increases:

\textbf{Proposition 5:} Consider a revenue CP $j$ earning its revenue from ads. User $i$’s utility $V_{i,j}(x^*_i,j, \pi^*_i,j, p_a, \gamma^*_{i,j})$ from CP $j$ increases with ads shown ($s_j$) if $ar_{i,j} > p_c$ and $\gamma^*_{i,j} > 0$.

Revenue CPs’ marginal revenue $ar_{i,j} s_j$ increases with $s_j$. Thus, if the expected revenue per volume of ads, $ar_{i,j}$, is sufficiently high, CPs sponsor more data as $s_j$ increases. The resulting decrease in users’ effective prices $\pi^*_i,j$ is enough to offset their lower utilities from ads. The $a, r_{i,j}$, and $p_c$ values used in Figure 5 satisfy Prop. 5’s condition, indicating that users will gain more utility from CPs that show more ads if those CPs also sponsor data. For instance, if the New York Times shows more ads than the Washington Post, it can attract more usage by sponsoring more data.

B. Distributions of Demand and Utility

We next examine the global implications of changes in demand and utility due to content sponsorship. In particular,
TABLE II: Jain’s fairness index for distributions of (demands, user utilities, CP utilities) before and after sponsorship over all user-CP pairs.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Before (Revenue CP)</th>
<th>After (Revenue CP)</th>
<th>Before (Promotion CP)</th>
<th>After (Promotion CP)</th>
</tr>
</thead>
<tbody>
<tr>
<td>α_{ij} (Figure 4a)</td>
<td>(0.674, 0.214, 0.674)</td>
<td>(0.912, 0.219, 0.719)</td>
<td>(0.65, 0.21, 0.256)</td>
<td>(0.922, 0.228, 0.267)</td>
</tr>
<tr>
<td>α_{ij} (Figure 4b)</td>
<td>(0.694, 0.218, 0.694)</td>
<td>(0.961, 0.224, 0.757)</td>
<td>(0.709, 0.221, 0.276)</td>
<td>(0.998, 0.237, 0.289)</td>
</tr>
<tr>
<td>c_{ij} (Figure 6a)</td>
<td>(0.503, 0.503, 0.503)</td>
<td>(0.503, 0.503, 0.503)</td>
<td>(0.57, 0.57, 0.768)</td>
<td>(0.856, 0.676, 0.868)</td>
</tr>
<tr>
<td>d_{ij} (Figure 6b)</td>
<td>(1, 1, 0.769)</td>
<td>(0.643, 0.859, 0.673)</td>
<td>(1, 1, 0.769)</td>
<td>(0.769, 0.927, 0.705)</td>
</tr>
</tbody>
</table>

As users become less price-elastic (α_{ij} increases), their relative demands decrease since they increase their demands less in response to pricing changes. Thus, using Lemma 1, the fairness of user demands x^∗_{i,j} with and without sponsorship depends on whether x^∗_{i,j} increases or decreases as α_{ij} increases (from Figure 4a, x^∗_{i,j} can be non-monotonic). In Figures 4a and 4b’s simulations, we find that the fairness of user demands and utilities increases with sponsorship (Table II). While this is not always the case for users with different price elasticities (Prop. 6), sponsorship does favor more cost-aware users, allowing them to disproportionately increase their utility and demand.

Proposition 7 (Fairness across user cost awareness): Consider a set of users who vary only in their cost awarenesses c_{i,j} and a CP j with the same d_{i,j} for all users. If CP j is a promotion CP, the distributions of user demands and utilities across different cost awarenesses become more fair (F(\{x^∗_{i,j}\}) and F(\{V_{i,j}\})) increase with sponsorship. Users’ relative demands and benefits increase as c_{i,j} increases.

If CP j is a revenue CP, relative demand and benefit is independent of cost awareness, so fairness does not change.

User demand increases as users become less cost-aware, with and without sponsorship. Thus, we might expect less cost-aware users to benefit more from sponsorship; indeed, with revenue CPs, relative benefit is independent of cost awareness. However, promotion CPs experience diminishing marginal utility from greater user demand, and we see from (8) that they sponsor less content for less cost-aware (high c_{i,j}) users. Promotion CPs thus increase the effective prices π^∗_{i,j} as c_{i,j} increases, dampening user demand enough to ensure that relative demand decreases with c_{i,j} (increases with c_{i,j}^{-1}).

Figure 6a illustrates Prop. 7’s result; user utility not only increases as c_{i,j} increases, but is never less than the utility before sponsorship. Table II shows that the demand and user utility distributions become more fair for the promotion CP; fairness does not change for the revenue CP.

If we instead consider homogeneous users and vary CP cost awareness, we find the opposite effect: the distributions of demand and utility always become less fair.

Proposition 8 (Fairness across CP cost awareness): Consider a set of homogeneous users and a set of either...
TABLE III: Relative demands and benefits for users and CPs with optimal sponsorship.

<table>
<thead>
<tr>
<th></th>
<th>Relative demand</th>
<th>Relative user benefit</th>
<th>Relative CP benefit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Revenue CP</td>
<td>$(\frac{1}{1-\alpha_{i,j}} - \frac{d_{i,j}}{p_u(1-\alpha_{i,j})(1+x_j)})^{\frac{1}{\pi_{i,j}}} - 1$</td>
<td>$(\frac{1}{1-\alpha_{i,j}} - \frac{d_{i,j}}{p_u(1-\alpha_{i,j})(1+x_j)})^{\frac{1}{\pi_{i,j}}} - 1$</td>
<td>$\frac{\alpha_{i,j}}{1-\alpha_{i,j}} \left(\frac{p_u(1+x_j)}{d_{i,j}} - 1\right) \left(1 - \frac{\alpha_{i,j}}{1-\alpha_{i,j}} \left(\frac{d_{i,j}}{p_u(1-\alpha_{i,j})(1+x_j)}\right)\right)^{\frac{1}{\pi_{i,j}}} - 1$</td>
</tr>
<tr>
<td>Promotion CP</td>
<td>$(1 - \alpha_{i,j} + \frac{d_{i,j}p_u(1+r_{i,j}s_j)x_{i,j}}{c_{i,j}p_c(1+r_{i,j}s_j)})^{\frac{1}{\alpha_{i,j}}} - 1$</td>
<td>$(1 - \alpha_{i,j} + \frac{d_{i,j}p_u(1+r_{i,j}s_j)x_{i,j}}{c_{i,j}p_c(1+r_{i,j}s_j)})^{\frac{1}{\alpha_{i,j}}} - 1$</td>
<td>$(1 - \alpha_{i,j} + \frac{d_{i,j}p_u(1+r_{i,j}s_j)x_{i,j}}{c_{i,j}p_c(1+r_{i,j}s_j)})^{\frac{1}{\alpha_{i,j}}} - 1$</td>
</tr>
</tbody>
</table>

Sponsored content favors less cost-aware CPs, as some have feared [6]. Less cost-aware CPs receive higher utilities even without sponsorship, due to their greater valuation of user demand. As we would expect, they can also sponsor more data, disproportionately increasing their utility. Figure 6b illustrates Prop. 8 by showing CP utilities as $d_{i,j}$ varies. Sponsorship increases the unevenness of these utilities, making their distribution more unfair (Table II).

C. User and CP Benefits

Having compared user and CP utility among different users and CPs, in this section we compare users’ and CPs’ utilities to each other. We first show that if demand increases with content sponsorship as in Prop. 4, user utility also increases:

Lemma 2: User $i$’s utility $V_{i,j}$ from CP $j$ increases with sponsorship if and only if the CP sponsors content ($\gamma_{i,j} > 0$).

Note that Lemma 2 does not require the amount sponsored to be chosen optimally; it is enough for CPs to sponsor any amount of data. We use this result to show that users, CPs, and ISPs can all benefit from content sponsorship:

Proposition 9 (Overall utility increase): Suppose the ISP’s chosen price $p_u^*$ is not greater than users’ data price $p_u$ before sponsorship. Then users and CPs do not decrease their utilities and the ISP does not decrease its profit with sponsorship.

If users’ data price $p_u$ does not increase, their effective prices $\pi_{i,j}^*$ will decrease with sponsorship; user demands then increase, which benefits users, CPs, and ISPs. CPs and users, however, do not benefit equally. We consider the ratio of CP to user utility when $\gamma_{i,j} > 0$ (i.e., some data is sponsored) and show that it decreases with sponsorship, meaning that CPs experience lower increases in utility compared to users.

Proposition 10 (User and CP utilities): Table IV gives the ratio of CP to user utility for each user-CP pair before and after sponsorship. This ratio is always lower after sponsorship.

Examining Table IV, we see that surprisingly, CPs benefit more relative to users (i.e., the CP-user utility ratio increases) as their data price $p_c$ increases: CPs then sponsor less data, bringing the utility ratio closer to that before sponsorship.
TABLE IV: CP-to-user utility ratios before and after sponsorship.

<table>
<thead>
<tr>
<th></th>
<th>Revenue CP</th>
<th>Promotion CP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Before sponsorship</td>
<td></td>
<td></td>
</tr>
<tr>
<td>With optimal sponsorship</td>
<td>$\frac{p_{i,j}(1+\alpha_{i,j}s_j)}{p_{u}(1+\gamma_{u,j}s_j)}$</td>
<td>$\frac{d_{i,j}(1+\alpha_{i,j}s_j)}{\gamma_{i,j}(1+\alpha_{i,j}s_j)}$</td>
</tr>
</tbody>
</table>

$c_{i,j}$ values; while the $\alpha_{i,j}$ values cluster towards the very high and very low, a more comprehensive user base would likely exhibit greater behavioral variation.

2) Simulation Results: We assume that ISPs charge their optimal data prices $p_{u}^*$ = $17$/GB and $p_{c}^*$ = $6.19$/GB, which are close to ISPs’ currently offered prices [21]; the maximum traffic is $X = 30$ GB. Revenue CPs take $d_{i,j} = ar_{i,j}s_j = 5.4$ as calculated in Section IV-A.

Figure 9a shows the distribution of the optimal amounts sponsored $\gamma_{i,j}^*$ for each user-CP pair. While $\gamma_{i,j}^*$ = 0 for 28% of the user-CP pairs, a few CPs sponsor $>100\%$ of users’ content (i.e., they also sponsor some ads, which add 15% to the content volume). For instance, some enterprises may sponsor all traffic on a corporate email app. Despite the $\alpha_{i,j}$ estimates clustering around very low and very high values, except for the concentration at 0 the $\gamma_{i,j}$ values are approximately evenly distributed between 0 and 1.15, likely due to different cost awarenesses $c_{i,j}^{-1}$ and $d_{i,j}^{-1}$.

Figure 9b shows the ratios of CP to user utility for each user-CP pair before and after sponsorship; as in Prop. 10, the ratios decrease with sponsorship, indicating that users benefit more than CPs. The decrease is most apparent for larger CP-to-user ratios, likely because these correspond to users with very low utility. Low user and CP utilities increase the most with sponsorship, as shown in Figure 9c, though both CP and user utilities generally increase. In fact, the fairness of the CP and user utility distributions over all user-CP pairs increases with sponsorship. Jain’s index for user utilities increases from 0.158 to 0.172, as is consistent with Prop. 7 for users facing one CP, and Jain’s index for CP utilities increases from 0.126 to 0.134. While sponsorship makes CP utilities more unfair with homogeneous users (Prop. 8), in reality CPs face different user demands, allowing the distribution of CP utility to become (slightly) more fair in our simulation.

Not only do user and CP utilities become more fair with sponsorship, but user demand also increases (Figure 10a). However, different CPs experience different changes in the distribution of these demands. Figure 10b shows that some CPs experience more even and some more uneven demand with sponsorship. Similarly, each individual user faces all six CPs. While the CPs have the same cost awarenesses $d_{i,j}^{-1}$ and fractions of ads sponsored $s_j$ for each user, users’ price elasticities $\alpha_{i,j}^{-1}$ and cost awarenesses $c_{i,j}^{-1}$ do vary for each CP. Thus, the distribution of demands for different CPs changes from user to user, as shown in Figure 10c.

B. Which Content to Sponsor?

While Props. 1 and 2 give CPs the optimal amount of content to sponsor, they do not help decide which content to sponsor. Since most users choose the content they view (e.g., which videos to watch), sponsorship can influence demand for different content. If the CP sponsors more popular content, more users will benefit, but CPs may wish to promote less popular content by sponsoring it.

We suppose that a CP $j$ has $K_j$ types of content (e.g., videos), each with a probability $v_{i,j}^k$ ($p_{i,j}^k$) $\in [0,1]$ of being viewed by type $i$ users, where $p_{i,j}^k$ is the fraction of content type $k$ that is sponsored. The CP chooses the $p_{i,j}^k$ so as to maximize an objective function $G \{ (v_{i,j}^k (p_{i,j}^k)) \}$. For instance, to spread demand across different types of content the CP can equalize the $v_{i,j}^k$ ($p_{i,j}^k$) (i.e., make $G$ a fairness function [24]). To target more popular content, it can maximize the sum of all viewing probabilities: $G = \sum_k v_{i,j}^k (p_{i,j}^k)$.

The CP maximizes $G$ subject to two constraints. First, the total fraction of content sponsored should equal $\gamma_{i,j}^*$, as given by (7) or (8): $K \sum_k p_{i,j}^k z_k = \gamma_{i,j}^* \sum_k z_k$, where $z_k$ is the volume of type $k$ content. Second, users’ overall demand should be $x_{i,j} (\pi_{i,j}^*)$, i.e., $\sum_k v_{i,j}^k (p_{i,j}^k) z_k = x_{i,j} (\pi_{i,j}^*)$. If the CP cannot sponsor fractions of each content type (e.g., a content type is a single video), we constrain $p_{i,j}^k \in \{0,1\}$. We then have a knapsack problem with multiple constraints [25].

VI. CONCLUSION

In this work, we derive the optimal sponsored data behaviors for users, CPs, and ISPs and then consider their implications for heterogeneous CPs and users. In particular, we find that sponsorship can reverse some of our intuition on how user demand changes with price sensitivity and the amount of ads shown by CPs. These changes affect the distributions of CP and user utilities: while sponsored data exacerbates CP inequality, it can even out user demand and utility, and it benefits users proportionally more than CPs. We illustrate our results with data from an ISP pricing trial.

Though sponsored data offers a seemingly simple new choice to CPs, its effect will be felt throughout the mobile data market: as we show in this work, CPs’ newfound market power over mobile data can significantly change user demand.
and CP and user utilities. Our work is necessarily an idealized approximation to user, CP, and ISP behavior, but nevertheless provides a guide towards understanding these implications.

REFERENCES

A. Proof of Proposition 1

We first note that since (5–6) is additively separable with respect to the $\gamma_{i,j}$ variables, we can simply choose each $\gamma_{i,j}$ so as to maximize $W_{i,j}(\gamma_{i,j})$. Moreover, $W_{i,j}$ is maximized at one of three possible values: $\gamma_{i,j} = 0$, $\gamma_{i,j} = 1 + s_j$, or a critical point of $W_{i,j}(\gamma_{i,j})$. Since we assume that $d_{i,j} < p_c(1 + s_j)$, taking $\gamma_{i,j} = 1 + s_j$ yields a negative profit for the CP. Thus, either the optimal value of $\gamma_{i,j}$, which we denote as $\gamma^*_{i,j}$, is either 0 or a critical point of $W_{i,j}$.

We find a unique critical point of $W_{i,j}(\gamma_{i,j})$ by taking the first derivative of (5) and setting it equal to zero:

$$
p_u \left( \frac{d_{i,j} - p_c \gamma_{i,j}}{1 + r_{i,j} s_j} \right) \left( p_u(1 - \gamma_{i,j} + s_j) \right)^{-\frac{1}{\alpha_{i,j}}} - \alpha_{i,j} \left( \frac{p_c}{1 + r_{i,j} s_j} \right) \left( p_u(1 - \gamma_{i,j} + s_j) \right)^{-\frac{1}{\alpha_{i,j}}} = 0.
$$

We then solve (11) for $\gamma^*_{i,j}$:

$$
\gamma^*_{i,j} = \frac{d_{i,j}}{p_c(1 - \alpha_{i,j})} - \frac{\alpha_{i,j}(1 + s_j)}{1 - \alpha_{i,j}}.
$$

Note that (12) satisfies the constraint $\gamma^*_{i,j} \leq 1 + s_j$ if and only if $d_{i,j} \leq 1 + s_j$, which is true by assumption.

We now show that taking $\gamma^*_{i,j} = 0$ yields a higher CP profit than (12) if and only if (12) is negative, i.e., $d_{i,j} < \alpha_{i,j} p_c(1 + s_j)$. We find the ratio of CP utility with $\gamma^*_{i,j}$ as in (12) to that with $\gamma^*_{i,j} = 0$ to be

$$
\sigma_{i,j} = \frac{\alpha_{i,j}}{\mu_{i,j} - 1} \left( \frac{1}{1 - \alpha_{i,j}} - 1 \right) \left( 1 - \frac{1}{\mu_{i,j} - 1} \right)^{-\frac{1}{\alpha_{i,j}}}.
$$

where for notational convenience we define $\mu_{i,j} = p_c(1 + s_j)/(d_{i,j}) > 1$. Setting $\sigma_{i,j} \geq 1$, we find the equivalent condition

$$
(\mu_{i,j} - 1)^{\frac{1}{\alpha_{i,j}}} \geq \frac{p_{c,\gamma^*_{i,j}}}{\alpha_{i,j}} (1 - \alpha_{i,j})^{1 - \frac{1}{\alpha_{i,j}}}.
$$

Raising both sides to the power $\alpha_{i,j}/(\alpha_{i,j} - 1)$, we find the equation

$$
\frac{\alpha_{i,j}}{\alpha_{i,j} - 1} (1 - \alpha_{i,j})^{\frac{1}{\alpha_{i,j} - 1}} - \mu_{i,j} + 1 \geq 0.
$$

At $\mu_{i,j} = \alpha_{i,j}^{-1}$, we see by inspection that this inequality is satisfied with equality. Taking the derivative with respect to $\mu_{i,j}$, we find the expression

$$
\alpha_{i,j} \left( \frac{\alpha_{i,j}}{\alpha_{i,j} - 1} \right)^{\frac{1}{\alpha_{i,j} - 1}} - \mu_{i,j} + 1 = 0,
$$

which is positive if and only if $\mu_{i,j} \geq \alpha_{i,j}^{-1}$. Thus, $\sigma_{i,j} \geq 1$ and $\gamma_{i,j} = 0$ yields lower profit than (12) for all $\mu_{i,j} \geq \alpha_{i,j}^{-1}$, or equivalently, $\alpha_{i,j} p_c(1 + s_j) \geq d_{i,j}$. 

B. Proof of Proposition 2

As in Prop. 1, we first observe that it suffices to consider the $\gamma_{i,j}$ separately, and that the optimal value of each $\gamma_{i,j}$ is either 0, 1 + $s_j$, or a critical point of $W_{i,j}(\gamma_{i,j})$. We now observe that

$$
\lim_{\gamma_{i,j} \to 1 + s_j} W_{i,j}(\gamma_{i,j}) = \frac{d_{i,j}}{1 - \beta_{i,j}} \left( \frac{p_u(1 - \gamma_{i,j} + s_j)}{c_{i,j}} \right)^{\frac{\beta_{i,j} - 1}{\alpha_{i,j}}} - \frac{p_c \gamma_{i,j}}{c_{i,j}} \left( \frac{p_u(1 - \gamma_{i,j} + s_j)}{c_{i,j}} \right)^{\frac{1}{\alpha_{i,j}}}
$$

Thus, we first find the unique critical point of $W_{i,j}$ and then derive conditions under which this critical point yields greater CP utility than taking $\gamma_{i,j} = 0$. We thus take the derivative and find

$$
\frac{dW_{i,j}}{d\gamma_{i,j}} \propto \frac{d_{i,j}}{\gamma_{i,j}} p_u (1 + r_{i,j} s_j)^{\alpha_{i,j}} - \frac{p_c \gamma_{i,j}}{\alpha_{i,j}} (1 + r_{i,j} s_j) - \frac{p_c \gamma_{i,j}}{\alpha_{i,j}} (1 + r_{i,j} s_j) \left( \frac{p_u(1 - \gamma_{i,j} + s_j)}{c_{i,j}} \right)^{-1}.
$$

After multiplying through by common factors, we find the equation

$$
p_u (1 + s_j) \left( \frac{d_{i,j}}{p_c(1 - \gamma_{i,j})} - \frac{\alpha_{i,j}(1 + s_j)}{1 - \alpha_{i,j}} \right) = \gamma_{i,j} \left( \frac{p_u p_c}{\alpha_{i,j} c_{i,j}} + \frac{p_u}{p_c} \left( \frac{d_{i,j}}{p_c} (1 + r_{i,j} s_j)^{\alpha_{i,j}} - \frac{p_{c,\gamma^*_{i,j}}}{\alpha_{i,j} c_{i,j}} \right) \right),
$$

which simplifies to

$$
\gamma^*_{i,j} = \frac{d_{i,j}}{p_c(1 - \alpha_{i,j})} - \frac{\alpha_{i,j}(1 + s_j)}{1 - \alpha_{i,j}}.
$$

It is easy to see that $\gamma^*_{i,j} < 1 + s_j$.

We now find the ratio of CP utility with (13) to that with $\gamma_{i,j} = 0$, showing that taking $\gamma^*_{i,j}$ as in (13) yields higher utility when (13) is positive. We find the ratio

$$
\sigma_{i,j} = \left( 1 - (1 - \alpha_{i,j}) (1 + r_{i,j} s_j)^{\alpha_{i,j} - 1} + c_{i,j} p_c \gamma_{i,j} (1 - \alpha_{i,j}) \right) \left( \frac{d_{i,j} p_u}{c_{i,j} p_c} (1 + r_{i,j} s_j)^{\alpha_{i,j} - 1} \right) \left( 1 - (1 + \alpha_{i,j} (\gamma_{i,j} - 1)) \right)^{\frac{1}{\alpha_{i,j} - 1}}.
$$

For notational convenience, we define $\nu_{i,j} = d_{i,j} p_u (1 + r_{i,j} s_j)^{\alpha_{i,j}} / (\alpha_{i,j} c_{i,j} p_c)$ and rewrite $\sigma_{i,j} \geq 1$ as

$$
(1 + (1 - \alpha_{i,j}) (1 + r_{i,j} s_j)^{\alpha_{i,j} - 1} (1 - 1) ) \left( 1 + \alpha_{i,j} (\gamma_{i,j} - 1) \right)^{\frac{1}{\alpha_{i,j} - 1}} \geq 1.
$$

Simplifying and multiplying through by $\nu_{i,j}$, we find the condition

$$
(\nu_{i,j} - 1 - \alpha_{i,j}) (1 + r_{i,j} s_j)^{\alpha_{i,j} - 1} (\nu_{i,j} - 1) \left( \frac{1}{\alpha_{i,j} - 1} \right) \geq \nu_{i,j}.
$$
We now show that
\[ \nu_{i,j} - \left( 1 - \alpha_{i,j} \right) \left( 1 + r_{i,j} s_j \right)^{\alpha_{i,j} - 1} (\nu_{i,j} - 1) > 1 - \alpha_{i,j} (\nu_{i,j} - 1), \]
which is equivalent to
\[ (\nu_{i,j} - 1) (1 - \alpha_{i,j}) \left( 1 - \left( 1 + r_{i,j} s_j \right)^{\alpha_{i,j} - 1} \right) > 0. \]

Since \( \nu_{i,j} > 1 \) when (13) is positive, this inequality holds and \( \sigma_{i,j} \geq 1 \) becomes
\[ 0 \geq \nu_{i,j}^{\alpha_{i,j}} - \alpha_{i,j} (\nu_{i,j} - 1) - 1. \]

At \( \nu_{i,j} = 1 \), this inequality holds by inspection; we see that the derivative of the right-hand side with respect to \( \nu_{i,j} \), \( \alpha_{i,j}^{\nu_{i,j}^{\alpha_{i,j}} - 1} - \alpha_{i,j} \), is negative if \( \nu \geq 1 \). Thus, \( \sigma_{i,j} \geq 1 \) for all \( \nu_{i,j} \geq 1 \), i.e., (13) positive.

**C. Proof of Proposition 3**

We first note that since each CP optimally chooses \( \gamma_{i,j} \), \( dW_{i,j}(\gamma_{i,j})/d\gamma_{i,j} \) must equal zero if \( \gamma_{i,j} \in (0, 1 + s_j) \), where we compute
\[
\frac{dW_{i,j}}{d\gamma_{i,j}} = \frac{d_{i,j} p_u}{\alpha_{i,j} c_{i,j} (1 + r_{i,j} s_j)^{1 - \beta_{i,j}}} \left( \frac{p_u (1 - \gamma_{i,j} + s_j)}{c_{i,j}} \right)^{\beta_{i,j} - 1} \gamma_{i,j}^{\alpha_{i,j} - 1} \\
- \frac{p_c}{1 + r_{i,j} s_j} \left( \frac{p_u (1 - \gamma_{i,j} + s_j)}{c_{i,j}} \right)^{\alpha_{i,j}} \\
- \frac{p_c p_u}{\alpha_{i,j} c_{i,j} (1 + r_{i,j} s_j)} \left( \frac{p_u (1 - \gamma_{i,j} + s_j)}{c_{i,j}} \right)^{\alpha_{i,j} - 1}. \]

Thus, by taking \( \gamma_{i,j}^* \) as a function of \( p_c \) and taking the total derivative of (14) with respect to \( p_c \), we find that
\[
\frac{d\gamma_{i,j}^*}{dp_c} = -\alpha_{i,j} p_c (1 + s_j) - \gamma_{i,j}^* p_c (1 - \alpha_{i,j}) - \gamma_{i,j}^* p_c (1 - \alpha_{i,j}) < 0. \]

We now find that
\[
\frac{dx_{i,j}^*}{dp_c} = \frac{p_u}{\alpha_{i,j} c_{i,j}} \left( \frac{p_u (1 - \gamma_{i,j}^* + s_j)}{c_{i,j}} \right)^{\alpha_{i,j} - 1} \gamma_{i,j}^{\alpha_{i,j} - 1} \frac{d\gamma_{i,j}^*}{dp_c} < 0. \]

We now define \( R_{i,j} = (\pi_{i,j}^* + p_c \gamma_{i,j}^*) x_{i,j}^* \) and compute
\[
\frac{dR_{i,j}}{dp_u} = \left( (1 - \alpha_{i,j}) c_{i,j} x_{i,j}^* - \alpha_{i,j} x_{i,j}^* + p_c \gamma_{i,j}^* \right) \frac{d\gamma_{i,j}^*}{dp_c} \\
+ \alpha_{i,j} x_{i,j}^* + p_c x_{i,j}^* \frac{d\gamma_{i,j}^*}{dp_c} \]
\[
+ \frac{p_u p_c \gamma_{i,j}^*}{c} \left( \frac{p_u (1 - \gamma_{i,j}^* + s_j)}{c_{i,j}} \right)^{\alpha_{i,j}} \frac{d\gamma_{i,j}^*}{dp_c} \\
+ \frac{p_u p_c \gamma_{i,j}^*}{c} (1 + r_{i,j} s_j) x_{i,j}^* \frac{d\gamma_{i,j}^*}{dp_c} \\
- \alpha_{i,j} x_{i,j}^* p_c (1 + s_j) \\
- p_c (1 - \alpha_{i,j}) (\alpha_{i,j} (1 + s_j) + \gamma_{i,j}^* (1 - \alpha_{i,j})) \]
\[
(15) \]
We now consider revenue CPs and compute

\[ x_{i,j}^*(\pi_{i,j}^*) = \frac{1}{1 + r_{i,j}s_j} \left( \frac{p_a p_e (1 + s_j) - p_a d_{i,j}}{c_{i,j} p_e (1 - \alpha_{i,j})} \right)^{\frac{1}{\alpha_{i,j}}}. \]  

(17)

Taking the derivative of (17) with respect to \( \alpha_{i,j} \), we obtain

\[ \frac{\partial x_{i,j}^*}{\partial \alpha_{i,j}} \propto \frac{1}{\alpha_{i,j}^{\frac{1}{\alpha_{i,j}}}} \ln \left( \frac{p_a (p_e (1 + s_j) - d_{i,j})}{p_e c_{i,j} (1 - \alpha_{i,j})} \right) - \frac{1}{\alpha_{i,j} (1 - \alpha_{i,j})}. \]

This derivative is negative if

\[ p_a p_e (1 + s_j) - p_a d_{i,j} < c_{i,j} p_e (1 - \alpha_{i,j}) \exp \left( -\frac{\alpha_{i,j}}{1 - \alpha_{i,j}} \right). \]

Since \((1 - \alpha_{i,j}) \exp(\alpha_{i,j} / (1 - \alpha_{i,j}))\) is minimized at \( \alpha = 0 \), \( x_{i,j}^* \) is always decreasing in \( \alpha_{i,j} \) for \( (0, 1) \) if

\[ p_a p_e (1 + s_j) - d_{i,j} < p_e c_{i,j}. \]

Finally, we consider promotion CPs when \( \gamma_{i,j}^* > 0 \) and compute

\[ x_{i,j}^*(\pi_{i,j}^*) = \left( \frac{p_a p_e (1 + s_j) (1 + r_{i,j}s_j)_{\alpha_{i,j}}^{\alpha_{i,j}}}{(1 - \alpha_{i,j}) c_{i,j} p_e + d_{i,j} p_a (1 + r_{i,j}s_j)_{\alpha_{i,j}}^{\alpha_{i,j}}} \right)^{\frac{1}{\alpha_{i,j}}}. \]

(18)

We then find that

\[ \frac{\partial x_{i,j}^*}{\partial \alpha_{i,j}} \propto \frac{\alpha_{i,j}}{\alpha_{i,j}^{\frac{1}{\alpha_{i,j}}}} \ln \left( 1 + r_{i,j}s_j \right) - c_{i,j} p_e (1 + s_j) - c_{i,j} p_e (1 + r_{i,j}s_j)_{\alpha_{i,j}}^{\alpha_{i,j}} \]

\[ + \ln \left( \frac{p_a p_e (1 + s_j)}{1 - \alpha_{i,j}} c_{i,j} p_e + d_{i,j} p_a (1 + r_{i,j}s_j)_{\alpha_{i,j}}^{\alpha_{i,j}} \right). \]

The second term is always negative by assumption, so it suffices to derive conditions under which the first term is always negative. We have the criterion \( p_a p_e (1 + s_j) < (1 - \alpha_{i,j}) c_{i,j} p_e + d_{i,j} p_a (1 + r_{i,j}s_j)_{\alpha_{i,j}}^{\alpha_{i,j}} \), so it is sufficient for

\[ p_a (p_e (1 + s_j) - d_{i,j} (1 + r_{i,j}s_j)_{\alpha_{i,j}}^{\alpha_{i,j}}) < c_{i,j} p_e, \]

the left-hand side of which is maximized at \( \alpha_{i,j} = 0 \).

E. Proof of Proposition 5

We first find that user \( i \)'s utility for a revenue CP \( j \), \( V_{i,j}^* \), is

\[ c_{i,j} \left( \alpha_{i,j} + r_{i,j}s_j \right) \frac{p_a p_e (1 + s_j) - p_a a_{r_{i,j}s_j} \frac{1 - \alpha_{i,j}}{\alpha_{i,j}}}{(1 - \alpha_{i,j})(1 + r_{i,j}s_j)} \]

(19)

when \( \gamma_{i,j}^* > 0 \). We can then see that the second term,

\[ \frac{p_a p_e (1 + s_j) - p_a a_{r_{i,j}s_j} \frac{1 - \alpha_{i,j}}{\alpha_{i,j}}}{c_{i,j} p_e (1 - \alpha_{i,j})}, \]

is increasing in \( s_j \) if \( a_{r_{i,j}s_j} > p_e \). It then suffices to show that

\[ \frac{\partial}{\partial s_j} \left( \frac{c_{i,j} \left( \alpha_{i,j} + r_{i,j}s_j \right)}{(1 - \alpha_{i,j})(1 + r_{i,j}s_j)} \right) = \frac{1 - \alpha_{i,j}}{(1 + r_{i,j}s_j)^2} > 0. \]

Each term of \( V_{i,j}^* \) is then increasing in \( s_j \).

F. Proof of Lemma 1

Since \( F \) is Schur-concave and homogeneous, it is sufficient to show that \( \vec{x} / | \vec{x}| \) can be obtained from \( \vec{y} / | \vec{y}| \) via a finite set of Robin-Hood operations. We wish to find a threshold index \( k^* \) such that for \( l < k^* \), \( \frac{\vec{y}_l}{| \vec{y}|} \leq \frac{\vec{x}_l}{| \vec{x}|} \), and for \( l \geq k^* \), \( \frac{\vec{y}_l}{| \vec{y}|} \geq \frac{\vec{x}_l}{| \vec{x}|} \). If such a \( k^* \) exists, we can easily obtain \( \vec{x} / | \vec{x}| \) from \( \vec{y} / | \vec{y}| \) by noting that for \( l \geq k^* \), we need to reduce \( \vec{y}_l / | \vec{y}| \), and for \( l < k^* \), we must increase \( \vec{y}_l / | \vec{y}| \). We do so by starting from \( \vec{y}_K \) (the largest element) and reducing it by the required amount, distributing this amount to \( \vec{y}_l \) with \( l < k^* \).

Since \( | \vec{y}/| \vec{y}| = | \vec{x}/| \vec{x} | = 1 \), this procedure will recover \( \vec{x} / | \vec{x}| \).

We thus need to show that such a threshold \( k^* \) exists from our assumption that \( \vec{y}_k / | \vec{y}| \leq \vec{y}_{k+1} / | \vec{y}| \) for all \( k \). It suffices to show that \( \vec{y}_1 / | \vec{y}| \leq \vec{y}_n / | \vec{y}| \leq \vec{y}_n / | \vec{y}| \) for all \( k \).

We do so by induction on \( K \): clearly, the assertion is true for \( K = 1 \). Assuming it is true for \( K = n \), we suppose that \( \vec{y}_n / | \vec{y}| \leq \vec{y}_n / | \vec{y}| \)

\[ \sum_{k \leq n} x_k^* + x_{n+1} \leq \frac{x_{n+1}}{\sum_{k \leq n} x_k^* + x_{n+1}} \]

which we find by multiplying out the terms and using the fact that \( \sum_{k \leq n} y_k / \sum_{k \leq n} x_k \leq \vec{y}_n / | \vec{y}| \leq \vec{y}_n / | \vec{y}| \).

Similarly,

\[ \sum_{k \leq n} \vec{x}_k / \sum_{k \leq n} \vec{x}_k \geq \vec{x}_1 / \vec{x}_1. \]

G. Proof of Proposition 6

1) Revenue CPs: We first show that users with lower \( \alpha_{i,j} \) receive a proportionally greater increase in demand with content sponsorship. Taking the derivative of \( x_{i,j}^* (\pi_{i,j}^*) / (x_{i,j}^* (p_a (1 + s_j))) \) with respect to \( \alpha_{i,j} \), we find that it is proportional to

\[ \frac{1}{\alpha_{i,j} - \alpha_{i,j}} + \frac{1}{\alpha_{i,j}^2} \ln \left( \left( \frac{1}{1 - \alpha_{i,j}} \right) \left( \frac{1 - d_{i,j}}{p_e (1 + s_j)} \right) \right), \]

which is always negative if \( \gamma_{i,j}^* > 0 \), and 0 at \( d_{i,j} = \alpha_{i,j} p_e (1 + s_j) \).

For \( d_{i,j} < \alpha_{i,j} p_e (1 + s_j) \), \( \gamma_{i,j}^* = 0 \), so relative demand equals 1; thus, relative demand can never increase as \( \alpha_{i,j} \) increases.

We now show that if \( p_a (1 + s_j) < c_{i,j} \), then users with lower \( \alpha_{i,j} \) have higher demands if no content is sponsored. A user’s demand for CP \( j \) without content sponsorship is \((1 / (1 + r_{i,j}s_j)) (p_a (1 + s_j) / c_{i,j})^{1/\alpha_{i,j}} \), which is decreasing with \( \alpha_{i,j} \) if and only if \( p_a (1 + s_j) < c_{i,j} \).

Given this result, we now let \( \vec{x} \) denote a \( K \)-element vector containing the demands before sponsorship and \( \vec{x}^* \) that after sponsorship, both sorted in increasing order (i.e., decreasing \( \alpha_{i,j} \)). We have shown that \( \vec{x}^*/| \vec{x}| \leq \vec{x}_{k+1}^*/| \vec{x}_{k+1}| \) for all \( k \), so the result follows from Lemma 1.

To show the second part of the proposition, we use the fact that, relative to the case with sponsorship, users with higher \( \alpha_{i,j} \) experience proportionally higher increases in demand without content sponsorship. Thus, by Lemma 1, it suffices to show that users with higher \( \alpha_{i,j} \) also experience higher levels of demand with content sponsorship under the conditions of
\[ \gamma^*_{i,j} \text{ always increasing in } c_{i,j}, \text{ i.e., } d/(p_c(1+s_j)). \]

We now derive the condition
\[ p_u(p_c(1+s_j) - d_{i,j}) > c_{i,j}p_c \left( \frac{p_c(1+s_j) - d_{i,j}}{p_c(1+s_j)} \right) \exp \left( \frac{d_{i,j}}{p_c(1+s_j) - d_{i,j}} \right), \]
which yields the condition in the proposition upon canceling common factors. We now show that when \( \gamma^*_{i,j} = 0 \), \( x^*_{i,j} \) is also increasing in \( \alpha_{i,j} \); then by continuity of \( x^*_{i,j} \) as a function of \( \alpha_{i,j} \), \( x^*_{i,j} \) is always increasing in \( \alpha_{i,j} \). For \( \gamma^*_{i,j} = 0 \), we have \( x^*_{i,j} = (p_u(1+s_j)/p_c)^{-1/\alpha_{i,j}}/(1+r_{i,j}s_j) \), which is increasing with \( \alpha_{i,j} \) if \( p_u(1+s_j) > c_{i,j} \). Since \( \exp (d_{i,j}/(p_u(1+s_j) - d_{i,j})) > 1 \), this is always the case.

2) Promotion CPs: We first take the derivative of
\[ \frac{\partial}{\partial \alpha_{i,j}} \left( \frac{\ln (x^*_{i,j}(\pi^*_{i,j})/x^*_{i,j}(p_u(1+s_j)))}{\alpha_{i,j}} \right) \text{ when } \gamma^*_{i,j} > 0 \]
to find the expression
\[ \frac{-1}{\alpha_{i,j}} \ln \left( 1 - \alpha_{i,j} + \frac{d_{i,j}p_u(1+r_{i,j}s_j)^{\alpha_{i,j}}}{c_{i,j}p_c} \right) + \frac{d_{i,j}p_u(1+r_{i,j}s_j)^{\alpha_{i,j}}}{c_{i,j}p_c} \ln \left( 1 + \frac{r_{i,j}s_j}{1 - \alpha_{i,j}} \right) \frac{c_{i,j}p_c}{d_{i,j}p_u(1+r_{i,j}s_j)^{\alpha_{i,j}}}, \]
(21)
The first term is negative if
\[ 1 - \alpha_{i,j} + \frac{d_{i,j}p_u(1+r_{i,j}s_j)^{\alpha_{i,j}}}{c_{i,j}p_c} > 1, \]
which is always the case since \( d_{i,j}p_u(1+r_{i,j}s_j)^{\alpha_{i,j}} > c_{i,j}p_c \) if \( \gamma^*_{i,j} > 0 \). The second term is negative if
\[ \frac{d_{i,j}p_u(1+r_{i,j}s_j)^{\alpha_{i,j}}}{c_{i,j}p_c} \ln \left( 1 + \frac{r_{i,j}s_j}{1 - \alpha_{i,j}} \right) < 1, \]
which is true by assumption. Thus, relative demand always decreases as \( \alpha_{i,j} \) increases if \( \gamma^*_{i,j} > 0 \). At \( d_{i,j}p_u(1+r_{i,j}s_j)^{\alpha_{i,j}} = c_{i,j}p_c \), \( \gamma^*_{i,j} = 0 \) and (21) is still negative. If \( d_{i,j}p_u(1+r_{i,j}s_j)^{\alpha_{i,j}} < c_{i,j}p_c \), then \( \gamma^*_{i,j} = 0 \) and relative demand is constant at 1. Thus, relative demand is always non-increasing as \( \alpha_{i,j} \) increases.

To show that fairness decreases for \( S_1 \), note that if \( p_u(1+s_j) < c_{i,j} \), then demand without sponsorship \( \left((p_u(1+s_j)/c_{i,j})^{1/(1-\alpha_{i,j})}/(1+r_{i,j}s_j)\right) \) is decreasing with \( \alpha_{i,j} \). Since relative demand also decreases with \( \alpha_{i,j} \), we see that fairness decreases with sponsorship using the same argument as in Prop. 6.

To show the second part of the proposition, it suffices to show that when \( \alpha_{i,j}p_c(1+s_j) > \exp(\alpha_{i,j})d_{i,j}(1+r_{i,j}s_j)^{\alpha_{i,j}} \), and \( p_u(1+s_j) > c_{i,j} \), demand with sponsorship increases with \( \alpha_{i,j} \). From the proof of Prop. 4, we find that optimal demand increases with \( \alpha_{i,j} \) when \( \gamma^*_{i,j} > 0 \) if and only if
\[ \frac{1}{\alpha_{i,j}^2} \ln \left( \frac{p_u(p_c(1+s_j))}{(1-\alpha_{i,j})c_{i,j}p_c + d_{i,j}p_u(1+r_{i,j}s_j)^{\alpha_{i,j}}} \right) + \frac{d_{i,j}p_u(1+r_{i,j}s_j)^{\alpha_{i,j}}}{c_{i,j}p_c} \ln \left( 1 + \frac{r_{i,j}s_j}{1 - \alpha_{i,j}} \right) - c_{i,j}p_c \]
\[ > 0. \]
Using the fact that \( c_{i,j}p_c \gamma^*_{i,j} \) is strictly increasing in \( c_{i,j}p_u(1+r_{i,j}s_j)^{\alpha_{i,j}} \), we find the sufficient condition
\[ \frac{1}{\alpha_{i,j}^2} \ln \left( \frac{p_u(p_c(1+s_j))}{(1-\alpha_{i,j})c_{i,j}p_c + d_{i,j}p_u(1+r_{i,j}s_j)^{\alpha_{i,j}}} \right) > \frac{1}{\alpha_{i,j}} \]
Exponentiating and substituting \( (1/(\alpha_{i,j} - 1)d_{i,j}p_u(1+r_{i,j}s_j)^{\alpha_{i,j}}) \) for \( (1-\alpha_{i,j})c_{i,j}p_c \), we find the condition
\[ \frac{d_{i,j}p_u(1+s_j)}{c_{i,j}(1+r_{i,j}s_j)^{\alpha_{i,j}}} \geq \exp \left( \alpha_{i,j} \right), \]
which is exactly that given in the proposition. We now show that when \( \gamma^*_{i,j} = 0 \), demand with sponsorship is still increasing with \( \alpha_{i,j} \). Equivalently, if \( d_{i,j}p_u(1+r_{i,j}s_j)^{\alpha_{i,j}} < c_{i,j}p_c \), then user demand is increasing with \( \alpha_{i,j} \) if \( p_u(1+s_j) > c_{i,j} \).
We now observe that since \( \gamma_{i,j} \) is independent of \( c_{i,j} \) for revenue CPs (Prop. 1), either \( \gamma_{i,j} = 0 \) for all users or \( \gamma_{i,j} > 0 \) for all users. Since relative demand and relative user benefit is independent of \( c_{i,j} \) for revenue CPs if \( \gamma_{i,j} > 0 \) (Table III), content sponsorship merely multiplies users’ demands and utilities by a constant. Since our fairness function \( F \) is homogeneous, fairness does not change. ■

I. Proof of Proposition 8

1) Relative Demands: We first note that, from Table III, relative demand for revenue and promotion CPs increases with \( d_{i,j} \) if \( d_{i,j} \geq \alpha_{i,j} p_{c}(1 + s_{j}) \) (for revenue CPs) or \( d_{i,j} p_{u}(1 + r_{i,j})^{\alpha_{i,j}} \geq \alpha_{i,j} c_{i,j} p_{c} \) (for promotion CPs). If \( d_{i,j} \leq \alpha_{i,j} p_{c}(1 + s_{j}) \) (revenue CPs) or \( d_{i,j} p_{u}(1 + r_{i,j})^{\alpha_{i,j}} < \alpha_{i,j} c_{i,j} p_{c} \) (promotion CPs), \( \gamma_{i,j}^* = 0 \) and relative demand is a constant 1. Thus, relative demand is nondecreasing in \( d_{i,j} \).

Following the method used in the proof of Prop. 6, it now suffices to show that user demand without sponsorship also increases in \( d_{i,j} \). Since user demand is independent of \( d_{i,j} \) this follows immediately.

2) Relative Benefits: We first consider revenue CPs. CP utility before sponsorship is then \( d_{i,j} x_{i,j}^{*} (p_{u}(1 + s_{j})) \), which is clearly increasing in \( d_{i,j} \).

We find that the CP’s relative benefit when \( d_{i,j} \geq \alpha_{i,j} p_{c}(1 + s_{j}) \) (then \( \gamma_{i,j}^* > 0 \) unless \( d_{i,j} = \alpha_{i,j} p_{c}(1 + s_{j}) \), from Corollary 1) is

\[
\frac{\alpha_{i,j}}{1 - \alpha_{i,j}} \left( \frac{p_{c}(1 + s_{j})}{d_{i,j}} - 1 \right) \left( 1 - \frac{d_{i,j}}{p_{c}(1 + s_{j})} \right)
\]

whose derivative with respect to \( d_{i,j} \) is proportional to

\[
\frac{-p_{c}(1 + s_{j})}{(d_{i,j})^{2}(1 - \alpha_{i,j})} \left( 1 - \frac{d_{i,j}}{p_{c}(1 + s_{j})} \right)
\]

\[
+ \left( 1 - \frac{d_{i,j}}{p_{c}(1 + s_{j})} \right) \frac{1}{\alpha_{i,j} p_{c}(1 + s_{j})(1 - \alpha_{i,j})},
\]

which is positive if

\[
(1 + \alpha_{i,j}) d_{i,j} p_{c}(1 + s_{j}) > p_{c}^{2}(1 + s_{j})^{2} \alpha_{i,j} + (d_{i,j})^{2}.
\]

Gathering terms and factoring, we find the condition \( (d_{i,j} - \alpha_{i,j} p_{c}(1 + s_{j}))(p_{c}(1 + s_{j}) - d_{i,j}) < 0 \), which holds by assumption. Again, if \( \gamma_{i,j} = 0 \), then relative benefit is 1, and independent of \( d_{i,j} \). Thus, both CP utility before sponsorship and relative benefit are nondecreasing in \( d_{i,j} \), which by the same argument in Prop. 4 proves the desired fairness result.

We now consider promotion CPs. CP utility before sponsorship is \( d_{i,j} x_{i,j}^{*} (p_{u}(1 + s_{j}))^{1-\alpha_{i,j}} / (1-\alpha_{i,j}) \), is increasing in \( d_{i,j} \); thus, it suffices to show that CPs’ relative benefit increases with \( d_{i,j} \). We take the derivative of the relative benefit (Table III if \( d_{i,j} p_{u}(1 + r_{i,j})^{\alpha_{i,j}} \geq \alpha_{i,j} c_{i,j} p_{c} \) with respect to \( d_{i,j} \) to find that it is proportional to

\[
\frac{-c_{i,j} p_{u} \alpha_{i,j}}{d_{i,j}^{2} p_{u}(1 + r_{i,j} s_{j})} \left( 1 - \alpha_{i,j} + \frac{d_{i,j} p_{u}(1 + r_{i,j} s_{j})^{\alpha_{i,j}}}{c_{i,j} p_{c}} \right)
\]

\[
+ \frac{p_{u}(1 + r_{i,j} s_{j})^{\alpha_{i,j}}}{c_{i,j} p_{c} \alpha_{i,j}} \left( 1 - (1 - \alpha_{i,j})(1 + r_{i,j} s_{j})^{\alpha_{i,j}-1} \right)
\]

\[
+ \frac{c_{i,j} p_{u} \alpha_{i,j}(1 - \alpha_{i,j})}{d_{i,j}^{2} p_{u}(1 + r_{i,j} s_{j})}.
\]

Upon multiplying out this expression, we find the equivalent condition

\[
\frac{c_{i,j}^{2} \gamma_{i,j}^{*2} (1 - \alpha_{i,j})^{2} + d_{i,j} p_{u}(1 + r_{i,j} s_{j})^{\alpha_{i,j}}}{c_{i,j} p_{c} \alpha_{i,j}} \left( 1 + (1 - \alpha_{i,j})(1 + r_{i,j} s_{j})^{\alpha_{i,j}-1} - \frac{c_{i,j} p_{u} \alpha_{i,j}(1 - \alpha_{i,j})}{d_{i,j}^{2} p_{u}(1 + r_{i,j} s_{j})} \right).
\]

We now observe that \( (1 + r_{i,j} s_{j})^{1+\alpha_{i,j}} > (1 + r_{i,j} s_{j})^{2\alpha_{i,j}} < 0 \) to find the sufficient condition

\[
\frac{c_{i,j}^{2} \gamma_{i,j}^{*2} (1 - \alpha_{i,j})^{2} + d_{i,j} p_{u}(1 + r_{i,j} s_{j})^{\alpha_{i,j}}}{c_{i,j} p_{c} \alpha_{i,j}} \left( 1 + (1 - \alpha_{i,j})(1 + r_{i,j} s_{j})^{\alpha_{i,j}-1} - \frac{c_{i,j} p_{u} \alpha_{i,j}(1 - \alpha_{i,j})}{d_{i,j}^{2} p_{u}(1 + r_{i,j} s_{j})} \right) < 0.
\]

Factoring the left-hand side of the equation, we obtain

\[
\frac{\left( 1 + \frac{1}{c_{i,j}^{2} \gamma_{i,j}^{*2} (1 - \alpha_{i,j})^{2} + d_{i,j} p_{u}(1 + r_{i,j} s_{j})^{\alpha_{i,j}}}{c_{i,j} p_{c} \alpha_{i,j}} \left( 1 + (1 - \alpha_{i,j})(1 + r_{i,j} s_{j})^{\alpha_{i,j}-1} - \frac{c_{i,j} p_{u} \alpha_{i,j}(1 - \alpha_{i,j})}{d_{i,j}^{2} p_{u}(1 + r_{i,j} s_{j})} \right) \right)}{(d_{i,j} p_{u}(1 + r_{i,j} s_{j})^{\alpha_{i,j}} - \alpha_{i,j} c_{i,j} p_{c})}
\]

\[
\times (d_{i,j} p_{u}(1 + r_{i,j} s_{j})^{\alpha_{i,j}} - \alpha_{i,j} c_{i,j} p_{c}) < 0.
\]

The first term in this product is always negative, and the second is always positive when \( \gamma_{i,j} > 0 \) from (8). Thus, if \( d_{i,j} p_{u}(1 + r_{i,j} s_{j})^{\alpha_{i,j}} \geq \alpha_{i,j} c_{i,j} p_{c} \) relative benefit is always decreasing in \( d_{i,j} \); if \( d_{i,j} p_{u}(1 + r_{i,j} s_{j})^{\alpha_{i,j}} < \alpha_{i,j} c_{i,j} p_{c} \), relative benefit is constant, so the relative benefit is always non-decreasing with \( d_{i,j} \). ■

J. Proof of Lemma 2

Suppose that CP \( j \) sponsors \( \gamma_{i,j} \) amount of content for user \( i \). Then users’ utility is

\[
\left( \frac{1}{c_{i,j}^{2} \gamma_{i,j}^{*2} (1 + r_{i,j} s_{j} + \alpha_{i,j})}{(1 - \alpha_{i,j})(1 + r_{i,j} s_{j})} \right) \left( p_{u}(1 - \gamma_{i,j} + s_{j}) \right)^{1-\alpha_{i,j}}. \tag{22}
\]

Since \( \alpha_{i,j} < 1 \), we see that (22) is increasing with \( \gamma_{i,j} \). Thus, the user will always benefit if a CP decides to sponsor some data. ■

K. Proof of Proposition 9

We first show that ISPs benefit relative to before sponsorship. Suppose that the ISP chooses \( p_{u}^{*} = p_{u}^{c} \) to be the same as \( p_{u} \) before sponsorship. Then the ISP price per unit of content is independent of the \( \gamma_{i,j}^{*} \). Moreover, since user demand can only increase with sponsorship (Prop. 4), ISP revenue can only increase. If the constraint on total demand \( \sum_{i,j} x_{i,j}^{*} \leq X \) does not hold with \( p_{u}^{*} = p_{u}^{c} \) equal to the price before sponsorship, then the ISP can increase \( p_{u} \) until it holds. Its marginal price \( p_{u}(1 + s_{j} + \gamma_{i,j}^{*} (p_{c} - p_{u})) \) must then increase relative to that before sponsorship, while its demand is the maximum possible...
and therefore at least that before sponsorship; its total revenue is then at least that before sponsorship. Without the constraint \( p_u^* = p_u^c \), ISPs have more flexibility to choose their prices, so their revenue can only increase further.

We now consider users and CPs. If \( p_u^* \) decreases relative to that before sponsorship, users’ effective data prices \( p_u(1 - \gamma_{i,j}^* + s_j) \) must decrease (for any \( \gamma_{i,j}^* > 0 \)). Lemma 2 then shows the result for end users. Finally, we show that CPs benefit since they optimally choose \( \gamma_{i,j}^* \). Thus, their utilities must be at least as large as that obtained by choosing \( \gamma_{i,j} = 0 \), i.e., no sponsorship. If CPs choose not to sponsor data, their cost of sponsorship is zero, as it is before sponsorship. However, the data component \( \mathcal{U}_{i,j} \) of their utility function (3) increases since user demand increases due to lower \( p_u \), i.e., no sponsorship. If CPs choose not to sponsor data, their cost of sponsorship is zero, as it is before sponsorship.

\[ (1 + s_j)p_c \left( \frac{p_u p_u(1 + s_j)}{(1 - \alpha_{i,j})p_u(1 + r_{i,j}s_j)^{\gamma_{i,j}} \alpha_{i,j}} \right)^{\frac{1}{\gamma_{i,j}}} \]

\[ \left( \frac{p_u p_u(1 + r_{i,j}s_j) - (1 - \alpha_{i,j})}{(1 + r_{i,j}s_j) + d_{i,j}p_u + (1 + r_{i,j}s_j)^{\gamma_{i,j}}} \right) \]

\[ \left( \frac{\alpha_{i,j} - (1 - \alpha_{i,j})}{p_u p_u(1 + s_j) - p_u d_{i,j}} \right)^{\frac{1}{\gamma_{i,j}}} \]

\[ \left( d_{i,j}(1 + r_{i,j}s_j)^{\gamma_{i,j}} \right) \]

where \( \gamma_{i,j} > 0 \). With promotion CPs, we have

\[ d_{i,j}p_u(1 + r_{i,j}s_j - (1 - \alpha_{i,j})) (1 + r_{i,j}s_j)^{\alpha_{i,j}} \]

\[ \alpha_{i,j}(1 - \alpha_{i,j})c_{i,j}p_c \]

\[ \leq d_{i,j}(1 + r_{i,j}s_j) \]
Table V: Jain’s fairness index before and after sponsorship over all CP-user pairs.

Index values are reported for distributions of (demands, user utilities, CP utilities).

<table>
<thead>
<tr>
<th>Variable</th>
<th>Before (Revenue CP)</th>
<th>After (Revenue CP)</th>
<th>Before (Promotion CP)</th>
<th>After (Promotion CP)</th>
<th>After (Promotion CP, $\beta \neq \alpha$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_{i,j}$ (Figure 11a)</td>
<td>(0.674, 0.214, 0.674)</td>
<td>(0.912, 0.219, 0.719)</td>
<td>(0.65, 0.21, 0.256)</td>
<td>(0.922, 0.228, 0.267)</td>
<td>(0.971, 0.217, 0.82)</td>
</tr>
<tr>
<td>$c_{i,j}$ (Figures 12a, 12b)</td>
<td>(0.503, 0.503, 0.503)</td>
<td>(0.503, 0.503, 0.503)</td>
<td>(0.57, 0.57, 0.768)</td>
<td>(0.856, 0.676, 0.868)</td>
<td>(0.649, 0.6, 0.698)</td>
</tr>
<tr>
<td>$d_{i,j}$ (Figures 12c, 12d)</td>
<td>(1, 1, 0.769)</td>
<td>(0.643, 0.859, 0.673)</td>
<td>(1, 1, 0.769)</td>
<td>(0.769, 0.927, 0.705)</td>
<td>(0.793, 0.943, 0.709)</td>
</tr>
</tbody>
</table>

Fig. 11: With sponsorship, (a) demand decreases as price elasticity decreases for users of a revenue ($c_{i,j} = 5.8, d_{i,j} = 5.4$) and promotion ($c_{i,j} = 5.3, d_{i,j} = 4.8$) CP; (b) user and (c) CP utility increases with ads for a user of revenue CPs ($\alpha_{i,j} = 0.3, c_{i,j} = 4$) and decreases for a user of promotion CPs ($\alpha_{i,j} = 0.4, c_{i,j} = 4$); (d) demand increases as CP price elasticity decreases for a user of revenue CPs ($\alpha_{i,j} = 0.3, c_{i,j} = 4$) or a user of promotion CPs ($\alpha_{i,j} = 0.4, c_{i,j} = 4, d_{i,j} = 2$).

Fig. 12: (a) User utility and (b) demand increases as user cost awareness decreases; similarly, (c) demand and (d) CP utility increases as CP cost awareness decreases. We use (a,b) one revenue ($\alpha_{i,j} = 0.4$) and one promotion ($\alpha_{i,j} = 0.5, d_{i,j} = 3$) CP; (c,d) one user of revenue ($\alpha_{i,j} = 0.4, c_{i,j} = 4, r_{i,j} \in [0.002, 0.03]$) and one of promotion CPs ($\alpha_{i,j} = 0.5, c_{i,j} = 4, d_{i,j} = 2$).

Fig. 13: Distribution of CP-to-user utility ratios over all CP-user pairs for all simulations in Figures 11a–11c and 12.

demand and utility become more fair, though to a lesser degree than for $\beta_{i,j} = \alpha_{i,j}$ (Prop. 7). As CP cost awareness varies, the distributions of demand and user utility both become more unfair for $\beta_{i,j} = \alpha_{i,j}$ and for $\beta_{i,j} = 0.3$. Before sponsorship, the fairness of CP utilities was 0.793 for $\beta_{i,j} = 0.3$, so we see that the distribution of CP utility becomes more unfair as well for both values of $\beta_{i,j}$ (Prop. 8).

Finally, in Figure 13, we show the ratios of CP to user utility before and after sponsorship for all simulations varying $\alpha_{i,j}$, $s_{i,j}$, $c_{i,j}$, and $d_{i,j}$ in Figures 11a–11c and 12. We see that for all types of CPs (revenue, promotion with $\beta_{i,j} = \alpha_{i,j}$, and promotion with $\beta_{i,j} \neq \alpha_{i,j}$), the utility ratio decreases with sponsorship (Prop. 10).