

Offering Supplementary Wireless Technologies: Adoption Behavior and Offloading Benefits

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Abstract—To alleviate the congestion caused by rapid growth in demand for mobile data, ISPs have begun encouraging users to offload some of their traffic onto a supplementary, better quality network technology, e.g., offloading from 3G or 4G to WiFi and femtocells. With the growing popularity of such offerings, a deeper understanding of the underlying economic principles and their impact on technology adoption is necessary. To this end, we develop a model for user adoption of a base wireless technology and a bundle of the base plus a supplementary technology. In our model, individual users make their adoption decisions based on several factors, including the technologies’ intrinsic qualities, throughput degradation due to congestion externalities from other subscribers, and the flat access rates that an ISP charges. We study the adoption dynamics and show that they converge to a unique equilibrium for a given set of exogenously determined system parameters. In particular, we characterize the occurrence of interesting adoption behaviors, including a possible decrease in the adoption of the supplementary technology as its coverage increases. Similar behaviors occur at an ISP’s profit-maximizing prices and the optimal coverage area for the supplementary technology. To account for the potential benefits from offloading in practice, we collect 3G and WiFi usage and location data from twenty mobile users. We then use this data to numerically investigate the profit-maximizing adoption levels when an ISP accounts for its cost of deploying the supplemental technology and savings from offloading traffic onto this technology.

I. INTRODUCTION

Internet service providers (ISPs) have recently begun to experience the effects of a projected 78% annual growth rate in mobile data demand over the next five years [1]. Their existing networks are increasingly unable to accommodate this growth, leading ISPs to search for ways to curb network congestion without hurting their profit margins [2]. For instance, Verizon now offers femtocells in order to supplement its 4G network capacity [3], while AT&T has deployed WiFi hotspots in New York to manage persistent 3G congestion [4]. Though AT&T’s WiFi is currently free, as mobile demand keeps growing, ISPs may soon begin to charge consumers for access to these supplemental networks [5]. Some ISPs, in fact, have already begun to do so: for instance, T-Mobile and Virgin Mobile respectively charge their subscribers an extra \$20 and \$15 a month for WiFi access [6], [7]. Given these developments, ISPs will soon require economic models that help them understand *how to price* access to such supplementary technologies and the implications of those pricing decisions.

We develop an analytical framework to understand user adoption decisions between a base technology and a bundled

offering of a base plus supplemental technology; users may adopt the base technology, no technology, or the bundle of both technologies. We assume that the users have heterogeneous valuations of each technology’s quality and account for the negative externalities of congestion effects as a technology’s adoption increases. We use this framework to identify the impact of various factors, such as, coverage and pricing on the equilibrium adoption levels and an ISP’s profit.

Our work is inspired by two research areas: the study of user technology adoption and that of network offloading. Though both areas have separately received considerable attention from economics and networking researchers, our contribution lies in incorporating user adoption models to study tradeoffs between deployment costs and offloading benefits for a supplementary technology. We give analytic conditions under which non-intuitive adoption behaviors occur and collect usage data to substantiate our study of an ISP’s profit and savings from offloading. In particular, our model shows interesting outcomes in the following scenarios (Section IV-A):

- An ISP wishes to expand its femtocell network to offload more 4G data traffic, but cannot change its pricing structure due to exogenous factors, e.g., the presence of a major competitor. We show that while increasing femtocell coverage can increase the volume of offloaded traffic, it may also decrease femtocell adoption: increased congestion induces some users to drop the bundled femtocell service and only subscribe to 4G. We then specify conditions under which this decrease occurs even when the ISP offers revenue-maximizing prices.
- An ISP tries to induce heavy users to leave its 3G network by increasing the access price. However, by doing so, an ISP may actually increase user adoption of 3G: increasing the price of 3G and the 3G + WiFi bundle by the same amount can lead some users to drop their WiFi subscriptions and adopt only 3G.

Given these adoption behaviors, we then consider an ISP’s optimal operating point—at what access prices and coverage area does an ISP maximize its profit, and what adoption levels do these correspond to? In determining the profit, we consider the ISP revenue, the cost of deploying the supplemental technology, and the ISP’s savings from offloading. Using empirical data to estimate the offloading benefits, we consider the following scenarios (Section V-B):

- Suppose that an ISP seeks to optimize its profit with respect to femtocell coverage and access prices for its bundled 4G and femtocell offerings. We show that adoption of both 4G and the 4G + femtocell bundle may increase when the marginal savings in the amount of traffic offloaded increases.
- If an ISP maximizes its profit in densely populated areas, femtocell adoption may increase with deployment costs.

We give an overview of related work on user adoption and offloading models in Section II. In Section III, we introduce our model and characterize the equilibrium adoption levels. Section IV uses these equilibria to derive analytic conditions under which the equilibrium adoption can behave non-intuitively. We then investigate the equilibrium adoption levels when the ISP maximizes its profit for a range of different cost structures (Section V), and conclude the paper in Section VI. All the proofs, omitted here due to space constraints, can be found in Appendix A.

II. RELATED WORK

Our work relates to two topics in the literature: technology adoption dynamics and the economics of offloading traffic to supplemental networks (e.g., 3G/4G to WiFi/femtocells).

A. Technology Adoption

Many works in economics have studied technology adoption in various contexts [8]. Katz and Shapiro, for instance, consider competing network technologies with positive externalities in a homogeneous user population [9], while Cabral [10] presents a diffusion model for a single technology adoption by users with heterogeneous network valuations. Economides and Viard [11] provide a static analysis for the adoption of two complementary technologies with positive externalities and heterogeneity in user evaluations. Sen, et. al. [12] study the dynamics of competition between two generic network technologies with positive network externalities, focusing on the role of converters in affecting the equilibrium outcomes.

While our work follows these in modeling user heterogeneity, it differs in that (a) our model's dominant externality is the negative one of network congestion, and (b) we consider a non-competitive scenario in which the supplementary technology is bundled with the base technology to relieve its congestion. We investigate the impact of profit-maximizing pricing and coverage decisions on equilibrium adoption outcomes.

Another work related to ours is by Ren, Park, and van der Schaar [13], who consider the market entry and spectrum sharing decisions of femtocell providers. In contrast, we do not consider an entrant-incumbent competition scenario; we model the adoption of a supplementary technology offered by a monopolist provider and the resulting tradeoffs between the deployment costs and the savings from offloading. Our work is thus focused on traffic offloading, unlike [13].

B. Traffic Offloading

Shetty, Parekh and Walrand analyze user adoption of split- and common-spectrum 4G and femtocell networks and use

their results to study ISP revenue maximization [14]. They consider the utility of heterogeneous users under both spectrum sharing schemes and account for congestion effects with detailed throughput models. However, [14] does not consider ISP costs or savings from offloading and relies on numerical studies due to the complexity of their throughput models.

Other works have also studied traffic offloading, but without developing an analytical model of user adoption decisions. For instance, [15] considers the problem of offloading 3G traffic to WiFi networks, focusing on the implications for ISP revenue. User adoption is here modeled using given demand functions, which depend on the prices of 3G and WiFi. Offloading onto femtocell networks is studied in [16], which considers ISP revenue and social welfare under flat and usage-based pricing of both open and closed femtocell networks. Our work contributes to these efforts by providing a fairly generic analytical framework, complemented with data collected from real users, to study the role of economic and technological decisions on the possible outcomes of the adoption process.

III. TECHNOLOGY ADOPTION MODEL

In this section, we introduce an analytic framework to model the dynamics of user adoption based on the user's utility of subscribing to the base and supplemental technologies, denoted as Technologies 1 and 2, respectively.¹ We consider a monopolist ISP, i.e., one ISP that does not compete for users with other service providers. Users may choose to adopt only the base technology (Technology 1), adopt a bundle of the base and supplemental technologies (Technologies (1+2)), or to not adopt either technology. This choice is governed by the value that each of the above options provides to the user, as described in Section III-A. Users' choices evolve over time in response to changes in the technologies' adoption and congestion levels; we analytically formulate these dynamics and characterize the steady-state equilibrium adoption levels. In Section III-B, we show that exactly one asymptotically stable equilibrium exists for any given set of exogenous system parameters.

A. Utility Functions

A user's value or utility from subscribing to a particular wireless technology depends on several factors, such as the intrinsic quality of the technology (e.g. the user's monetary valuation of the maximum throughput), the negative externality of congestion (i.e., reduced throughput), and the access price charged by the service provider. Following [12] and [14], we account for these factors in defining the utility functions associated with each technology adoption option. For the two options, base and the base plus supplementary plans, the respective utility functions are given by (1) and (2); the utility of non-adoption is assumed to be zero.

$$U_1(t) = \theta q_1 + T_1(x_1 + (1 - \eta)x_{1+2}) - p_1 \quad (1)$$

$$U_{1+2}(t) = (1 - \eta)(\theta q_1 + T_1(x_1 + (1 - \eta)x_{1+2})) + \eta(\theta q_2 + T_2(\eta x_{1+2})) - p_2 - p_1. \quad (2)$$

¹We choose this notation to emphasize that our model is *generic*, and may be applied to any market with base and supplementary technologies.

The above utility functions have three separate value components, as we discuss here. The intrinsic qualities (e.g., monetary values of the maximum achievable throughput or delay in the absence of congestion) of Technologies 1 and 2 are denoted by q_i , $i = 1, 2$, and we assume that the supplemental technology has a higher intrinsic quality than the base ($q_2 > q_1$). For example, femtocells and WiFi typically deliver much higher maximum throughput than the base 4G or 3G networks. While the actual throughput, delay, etc. at a given instant of time can vary depending on such factors as distance from a cell tower, we assume that users will make adoption decisions based on the *overall* quality that they experience. The valuation of this intrinsic quality is weighted by a random variable $\theta \in [0, 1]$ to account for heterogeneity in users' valuation of each technology's quality. Users who stream a lot of video, for instance, might have a high θ value, while those who mainly surf the web will have a low θ value.²

We assume that the supplemental technology, Technology 2, has a limited coverage area (e.g., users are not always within range of a hotspot or a femtocell) that determines the "coverage factor" η , such that a fraction η of traffic from adopters of the technology bundle travels over Technology 2's network. We assume that users are homogeneous in their usage volumes and that they are distributed uniformly throughout the coverage area; then the amount of traffic offloaded to Technology 2 is proportional to the fraction of users adopting the technology bundle, multiplied by η . We let $x_1(t)$ denote the fraction of users adopting only Technology 1 at time t and $x_{1+2}(t)$ the fraction of users adopting both technologies, and note that $x_1(t)$, $x_{1+2}(t)$, and $x_1(t) + x_{1+2}(t) \in [0, 1]$. Thus, the amount of traffic on Technology 1 is $x_1(t) + (1 - \eta)x_{1+2}(t)$, while the amount offloaded to Technology 2 is $\eta x_{1+2}(t)$.

We use decreasing functions $T_1(x_1(t) + (1 - \eta)x_{1+2}(t))$ and $T_2(\eta x_{1+2}(t))$ to represent the throughput degradation as a function of the traffic volume for Technologies 1 and 2 respectively, normalized to monetary units (e.g., the decrease in the technologies' monetary value).³ The wireless service provider prices the access for the two options at p_1 for the base technology and $p_1 + p_2$ for the base plus supplemental technology bundle (i.e., p_2 is the extra price that a user pays for the bundled option). For notational convenience, the time argument of $x_1(t)$ and $x_{1+2}(t)$ will be assumed from here on to be implicit in the utility functions (1) and (2).

Given these functions, we can find the threshold value of θ , $\theta_{(1,0)}$, for which users will prefer to adopt Technology 1 (i.e., $U_1 > 0$). Similarly, we can also find the value of $\theta_{(1+2,1)}$ for which Technology 1 users will prefer the bundle of both Technologies (1+2) (i.e., $U_{1+2} > U_1 > 0$). We note that each θ threshold is a (time-dependent) function of x_1 and x_{1+2} .

The threshold $\theta_{(1,0)}$ for preferring Technology 1 occurs

when $U_1 = 0$, i.e.,

$$\theta_{(1,0)} = \frac{p_1 - T_1(x_1 + (1 - \eta)x_{1+2})}{q_1}. \quad (3)$$

The threshold $\theta_{(1+2,1)}$ for adopting Technology 2 in addition to Technology 1 occurs when $U_{1+2} = U_1 \geq 0$, i.e.,

$$\theta_{(1+2,1)} \geq \frac{T_2(\eta x_{1+2}) - T_1(x_1 + (1 - \eta)x_{1+2}) - \frac{p_2}{\eta}}{q_1 - q_2}. \quad (4)$$

Finally, we solve for the threshold $\theta_{(1+2,0)}$ above which users will prefer to adopt both technologies, rather than have no connectivity. This occurs when $U_{1+2} = 0$, or

$$\theta_{(1+2,0)} = \frac{-(1 - \eta)T_1(x_1 + (1 - \eta)x_{1+2}) - \eta T_2(\eta x_{1+2})}{(1 - \eta)q_1 + \eta q_2} + \frac{p_2 + p_1}{(1 - \eta)q_1 + \eta q_2}. \quad (5)$$

In the remainder of this paper, we take the throughput degradation functions T_1 and T_2 to be linear, i.e., $T_i(x) = -\gamma_i x$, $i = 1, 2$, where γ_1 and γ_2 are (positive) approximation constants.⁴ With the linear T_1 and T_2 , (3-5) become

$$\theta_{(1,0)} = \frac{p_1 + \gamma_1(x_1 + (1 - \eta)x_{1+2})}{q_1}. \quad (6)$$

$$\theta_{(1+2,1)} = \frac{-\eta\gamma_2 x_{1+2} + \gamma_1(x_1 + (1 - \eta)x_{1+2}) - \frac{p_2}{\eta}}{q_1 - q_2}. \quad (7)$$

$$\theta_{(1+2,0)} = \frac{(1 - \eta)\gamma_1(x_1 + (1 - \eta)x_{1+2}) + \eta^2\gamma_2 x_{1+2}}{(1 - \eta)q_1 + \eta q_2} + \frac{p_2 + p_1}{(1 - \eta)q_1 + \eta q_2}. \quad (8)$$

For given adoption levels x_1 and x_{1+2} , the ordering of these threshold values (6-8) determines whether a user of type θ is willing to adopt a particular technology. Thus, we can determine the fraction of users $H_1(x_1(t), x_{1+2}(t))$ and $H_{1+2}(x_1(t), x_{1+2}(t))$ willing to adopt Technology 1 and Technologies (1+2) respectively. In doing so, we recall that $\theta \in [0, 1]$; for instance, if $\theta_{(1,0)} < 0$, *all* users receive positive utility from adopting Technology 1. We thus let $[\cdot]_{[0,1]}$ denote the *projection* onto the $[0, 1]$ interval.⁵

We first consider the case $\theta_{(1,0)} < \theta_{(1+2,0)}$, i.e., the threshold for preferring the base technology to no adoption is smaller than that of preferring both technologies to no adoption. We show that $\theta_{(1+2,1)} > \theta_{(1+2,0)}$; thus, if a user receives positive utility from Technology 1 and increases it by adopting Technology 2 as well ($\theta_{(1,0)} < \theta_{(1+2,1)} < \theta$), she cannot receive negative utility from adopting both technologies ($\theta_{(1,0)} < \theta_{(1+2,1)} < \theta < \theta_{(1+2,0)}$):

Proposition 1: If $\theta_{(1,0)} < \theta_{(1+2,0)}$, then $\theta_{(1+2,0)} < \theta_{(1+2,1)}$. If $\theta_{(1+2,0)} < \theta_{(1,0)}$, then $\theta_{(1+2,1)} < \theta_{(1+2,0)}$.

Thus, if $\theta_{(1,0)} < \theta_{(1+2,0)}$, the fraction of users H_1 willing to adopt Technology 1 equals the fraction for whom

²The exact values of the q_i parameters depend on the particular technology being considered, while the distribution of θ values can be estimated from established techniques in marketing research, e.g., conjoint analysis [17].

³We note that we limit our model to scenarios in which Technology 2's throughput is unaffected by the users on Technology 1, e.g., non-interfering wireless technologies, as in a split-spectrum 4G and femtocell deployment.

⁴This assumption is often used in the literature on network technology adoption [13]. In Appendix B, we derive analytical bounds on the approximation error for typical throughput functions.

⁵That is, $[y]_{[0,1]} = y$ if $y \in [0, 1]$, 0 if $y < 0$, and 1 if $y > 1$.

TABLE I
EXPRESSIONS FOR H_1 AND H_{1+2} IN DIFFERENT REGIONS OF (x_1, x_{1+2}) .

	Conditions on θ	H_1	H_{1+2}
a	$\theta_{(1+2,0)} < \theta_{(1+2,1)} < 0$ $\theta_{(1+2,1)} < \theta_{(1+2,0)} < 0$	0	1
b	$\theta_{(1,0)} < 0 < \theta_{(1+2,1)} < 1$	$\theta_{(1+2,1)}$	$1 - \theta_{(1+2,1)}$
c	$0 < \theta_{(1,0)} < \theta_{(1+2,1)} < 1$	$\theta_{(1+2,1)} - \theta_{(1,0)}$	$1 - \theta_{(1+2,1)}$
d	$0 < \theta_{(1,0)} < 1 < \theta_{(1+2,1)}$	$1 - \theta_{(1,0)}$	0
e	$0 < \theta_{(1+2,0)} < 1 < \theta_{(1,0)}$ $0 < \theta_{(1+2,0)} < \theta_{(1,0)} < 1$	0	$1 - \theta_{(1+2,0)}$
f	$\theta_{(1,0)} < 0 < 1 < \theta_{(1+2,1)}$	1	0
g	$1 < \theta_{(1,0)} < \theta_{(1+2,0)}$ $1 < \theta_{(1+2,0)} < \theta_{(1,0)}$	0	0

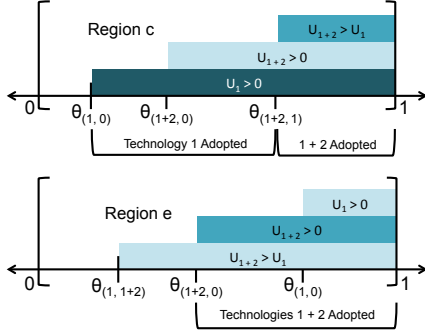


Fig. 1. Visualization of θ and H values for regions c and e in Table I.

$\theta_{(1+2,1)} > \theta > \theta_{(1,0)}$, and the fraction of users H_{1+2} willing to adopt Technologies (1 + 2) equals the fraction for which $\theta > \theta_{(1+2,1)}$. For simplicity, we assume that users' valuations θ are uniformly distributed in the interval $[0, 1]$; then

$$\begin{aligned} H_1(x_1, x_{1+2}) &= [\theta_{(1+2,1)}]_{[0,1]} - [\theta_{(1,0)}]_{[0,1]}, \\ H_{1+2}(x_1, x_{1+2}) &= 1 - [\theta_{(1+2,1)}]_{[0,1]}. \end{aligned} \quad (9)$$

If the thresholds are reversed, i.e., $\theta_{(1+2,0)} < \theta_{(1,0)}$, then we may use Prop. 1 to derive

$$H_1(x_1, x_{1+2}) = 0, \quad H_{1+2}(x_1, x_{1+2}) = 1 - [\theta_{(1+2,0)}]_{[0,1]}. \quad (10)$$

Following standard economic models, we assume a negligible cost of switching between the adoption choices [12], [13].

By dividing the dynamical space into seven regions, we can explicitly write out (9 - 10) as in Table I. Figure 1 visually represents the adoption expressions in two regions, and Fig. 2 uses the θ threshold values (6 - 8) to map them to the adoption levels x_1 and x_{1+2} .⁶ The user dynamics are then

$$\begin{aligned} \dot{x}_1(t) &= \rho [H_1(x_1(t), x_{1+2}(t)) - x_1(t)] \\ \dot{x}_2(t) &= \rho [H_{1+2}(x_1(t), x_{1+2}(t)) - x_{1+2}(t)], \end{aligned} \quad (11)$$

where $\rho \in (0, 1]$ denotes the rate of adoption. At any time t , the fraction of users adopting each technology equals the fraction willing to adopt, less those who have already done so. Given these dynamics, we now derive the possible equilibrium points in each region, i.e., the values of x_1 and x_{1+2} for which $H_1(x_1, x_{1+2}) = x_1$ and $H_{1+2}(x_1, x_{1+2}) = x_{1+2}$ for the H expressions in Table I. Tables II and III summarize the expressions for possible equilibria in each region.

⁶A qualitatively similar figure with the same adjacent regions will be obtained even for nonlinear T_1 and T_2 .

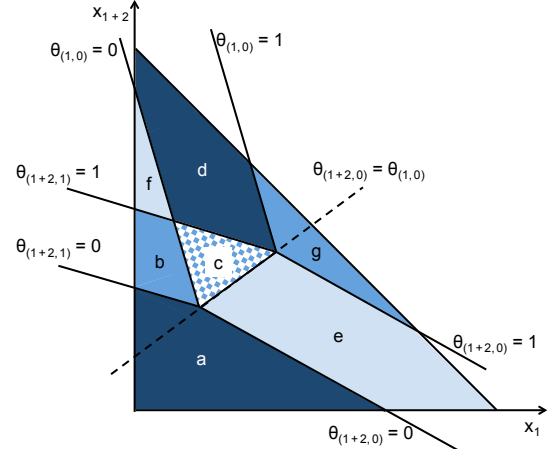


Fig. 2. Visualization of Table I's regions in terms of the adoption levels.

B. Convergence and Stability

We now examine the stability of the equilibrium points in Tables II and III.⁷

Proposition 2: Assuming that an equilibrium point exists, it is asymptotically stable.

While we assume that throughput degradation (T_1 and T_2) is linear in the previous section, our stability analysis depends only on the Jacobian of the dynamics (11) at given adoption levels (x_1, x_{1+2}) . Since our only assumption on the slopes γ_1 and γ_2 of the throughput degradation T_1 and T_2 is positivity, the Jacobian expressions are not affected by nonlinear forms of the T_i . Thus, if T_1 and T_2 are continuously differentiable and strictly decreasing, Prop. 2's conclusion still holds.

In other words, for any set of exogenous parameters values and initial adoption levels, the adoption dynamics must converge to some stable equilibrium:

Proposition 3: With the adoption dynamics (11), no periodic orbit can exist: for any initial values $x_1(0)$ and $x_{1+2}(0)$, $x_1(t)$ and $x_{1+2}(t)$ converge to an equilibrium point.

Moreover, only one such equilibrium point exists:

Theorem 1: For given values of the system parameters q_1 , q_2 , η , γ_1 , γ_2 , p_1 , and p_2 , the adoption levels $x_1(t)$ and $x_{1+2}(t)$ converge to a unique, asymptotically stable equilibrium that does not depend on the initial values $x_1(0)$ and $x_{1+2}(0)$.

In the remainder of the paper, we use \bar{x}_1 and \bar{x}_{1+2} to denote the unique equilibrium adoption levels.

IV. ADOPTION BEHAVIORS

In this section, we investigate the dependence of the equilibrium adoption on both prices and the coverage factor. Section IV-A highlights non-intuitive adoption behaviors, such as the possibility of a decrease in the adoption of both the bundled technologies and the total adoption level when the coverage factor of the supplementary technology increases. In Section

⁷Though the overall fractions of users x_1 and x_{1+2} will not change at equilibrium, individual users may change their adoption decisions.

TABLE II
EQUILIBRIUM POINTS $(\bar{x}_1, \bar{x}_{1+2})$ OF THE DIFFERENT REGIONS IN TABLE I.

	$(\bar{x}_1, \bar{x}_{1+2})$	Region Constraints
a	(0, 1)	$p_1 + p_2 < -(1-\eta)^2\gamma_1 - \eta^2\gamma_2$ $p_2 < \eta((1-\eta)\gamma_1 - \eta\gamma_2)$
b	$\left(\frac{\eta((1-\eta)\gamma_1 - \eta\gamma_2) - p_2}{\eta(q_1 - q_2) - \eta^2(\gamma_1 + \gamma_2)}, \frac{p_2 - \eta\gamma_1 + \eta(q_1 - q_2)}{\eta(q_1 - q_2) - \eta^2(\gamma_1 + \gamma_2)}\right)$	$(\eta(\gamma_1 + \gamma_2) - q_1 + q_2)p_1 + \gamma_1 p_2 < -\eta\gamma_1\gamma_2 + (1-\eta)\gamma_1(q_1 - q_2)$ $\eta((1-\eta)\gamma_1 - \eta\gamma_2) < p_2 < \eta(\gamma_1 + q_2 - q_1)$
c	See Table III.	See Table III.
d	$\left(\frac{q_1 - p_1}{q_1 + \gamma_1}, 0\right)$	$-\gamma_1 < p_1 < q_1, p_2 + \frac{\eta\gamma_1 p_1}{q_1 + \gamma_1} > \eta\left(q_2 - q_1 + \frac{\gamma_1 q_1}{q_1 + \gamma_1}\right)$
e	$\left(0, \frac{(1-\eta)q_1 + \eta q_2 - p_1 - p_2}{(1-\eta)q_1 + \eta q_2 + (1-\eta)^2\gamma_1 + \eta^2\gamma_2}\right)$	$-(1-\eta)^2\gamma_1 - \eta^2\gamma_2 < p_1 + p_2 < (1-\eta)q_1 + \eta q_2$ $\eta(q_2 - q_1 - (1-\eta)\gamma_1 + \eta\gamma_2)p_1 - (q_1 + (1-\eta)\gamma_1)p_2 > \eta^2 q_1 \gamma_2 - \eta(1-\eta)\gamma_1 q_2$
f	(1, 0)	$p_1 < -\gamma_1, p_2 > \eta(q_2 + \gamma_1 - q_1)$
g	(0, 0)	$p_1 > q_1$ $p_2 + p_1 > (1-\eta)q_1 + \eta q_2$

TABLE III
EQUILIBRIUM POINTS $(\bar{x}_1, \bar{x}_{1+2})$ OF REGION C IN TABLE I.

\bar{x}_1	$\frac{-\eta\gamma_2 q_1 + (1-\eta)\gamma_1 q_2 + p_1(\eta\gamma_2 - (1-\eta)\gamma_1 + q_2 - q_1) + p_2(-(1-\eta)\gamma_1 - q_1)/\eta}{-\gamma_1 q_2 - \eta\gamma_1\gamma_2 + q_1((1-\eta)\gamma_1 - \eta\gamma_2 + q_1 - q_2)}$
\bar{x}_{1+2}	$\frac{-\gamma_1 q_2 + q_1(q_1 - q_2) + p_1\gamma_1 + p_2(\gamma_1 + q_1)/\eta}{-\gamma_1 q_2 - \eta\gamma_1\gamma_2 + q_1((1-\eta)\gamma_1 - \eta\gamma_2 + q_1 - q_2)}$
Constraints	$p_1(\eta\gamma_2 - (1-\eta)\gamma_1 + q_2 - q_1) + p_2(-(1-\eta)\gamma_1 - q_1)/\eta < \eta\gamma_2 q_1 - (1-\eta)\gamma_1 q_2$ $p_1\gamma_1 + p_2(\gamma_1 + q_1)/\eta < \gamma_1 q_2 - q_1(q_1 - q_2)$ $p_1(\eta\gamma_2 + \eta\gamma_1 + q_2 - q_1) + p_2\gamma_1 > -\eta\gamma_1\gamma_2 + (1-\eta)\gamma_1(q_1 - q_2)$

IV-B, we consider the ISP's revenue maximization problem, and find that this behavior persists under the optimal prices.⁸

A. Observations

We first consider the adoption behavior for a range of coverage factors (η) , e.g., Section I's example of an ISP that increases its femtocell coverage to offload more traffic from 4G, but cannot change its access prices due to the presence of a competitor. Figure 3a shows the equilibrium adoption levels for a set of exogenous system parameters. At large (> 0.7) values of η , adoption \bar{x}_{1+2} of the bundled technologies decreases with η , even though the coverage area increases. As η increases, a larger portion of traffic $\eta\bar{x}_{1+2}$ is offloaded onto Technology 2, and the resulting increase in congestion can lower adoption of Technologies (1 + 2). Then depending on the adoption \bar{x}_1 of Technology 1, the total adoption $\bar{x}_1 + \bar{x}_{1+2}$ may increase or decrease as \bar{x}_{1+2} decreases. In Fig. 3a, \bar{x}_1 is positive and increasing, and the total adoption also increases. Figure 3b shows an example in which \bar{x}_{1+2} decreases with η , but $\bar{x}_1 = 0$. Thus, $\bar{x}_1 + \bar{x}_{1+2} = \bar{x}_{1+2}$ and *the total adoption may decrease as the coverage increases*. In fact, \bar{x}_1 is crucial to the behavior of $\bar{x}_1 + \bar{x}_{1+2}$:

Proposition 4: Suppose that no users adopt Technology 1 ($\bar{x}_1 = 0$), and that some, but not all, adopt both technologies ($\bar{x}_{1+2} \in (0, 1)$). Then \bar{x}_{1+2} decreases with η if

$$(1-\eta)^2\gamma_1 q_1 + (1-\eta^2)\gamma_1 q_2 + \eta(\eta-2)\gamma_2 q_1 - \eta^2\gamma_2 q_2 + (p_1 + p_2)(q_2 - q_1 - 2(1-\eta)\gamma_1 + 2\eta\gamma_2) < 0. \quad (12)$$

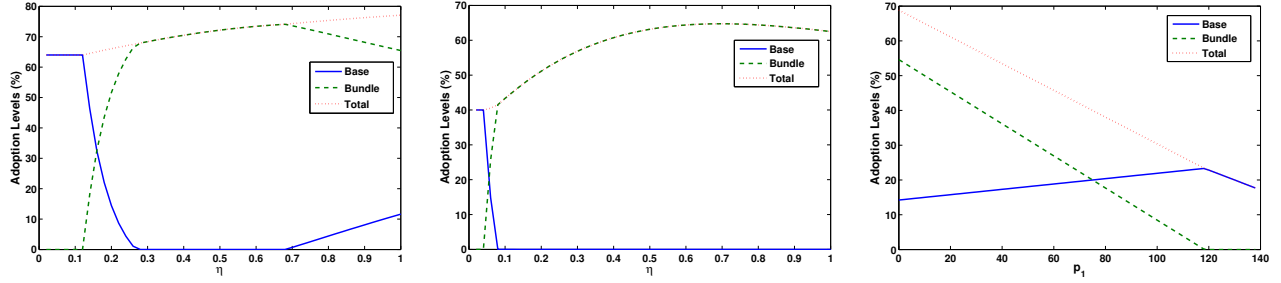
⁸In Appendix C, we show that similar behaviors occur when the user heterogeneity variable θ is non-uniformly distributed.

If some users adopt Technology 1, some adopt both technologies, and some neither ($\bar{x}_1 > 0, \bar{x}_{1+2} > 0$ and $\bar{x}_1 + \bar{x}_{1+2} < 1$), then total adoption $\bar{x}_1 + \bar{x}_{1+2}$ increases with η .

We note that the mathematical condition (12) is decreasing in $\eta\gamma_2$; Technology 2's throughput degradation coefficient γ_2 , multiplied by the coverage factor η , must be sufficiently high for \bar{x}_{1+2} to decrease. On the other hand, the presence of the positive $(p_1 + p_2)(q_2 - q_1)$ term indicates that if the marginal difference in the intrinsic quality between the two technologies ($q_2 - q_1$) is large, users may adopt the bundled technologies even if Technology 2 is very congested.

Another interesting feature of Fig. 3b is the abrupt switch from all users adopting the base technology to all users adopting the bundled technologies when $\eta < 0.1$. In this example, the access price p_2 of Technology 2 is relatively low, as is its throughput degradation coefficient γ_2 when compared to the intrinsic quality q_2 . Thus, as η increases slightly, the utility of adopting Technologies (1 + 2) increases quickly: the user need not pay much more for Technology 2, which provides higher quality service with relatively little throughput degradation. Thus, many users adopt the supplemental technology in addition to the base one. As η grows further to 0.08, the utility of adopting Technologies (1 + 2) becomes larger than that of adopting only the base technology, save for those users who adopt neither technology due to low valuation levels θ .

Finally, we consider adoption behaviors for fixed Technology 2 access price p_2 and coverage factor η . For instance, as proposed in the introduction, the ISP may increase the access price p_1 of its base technology in an attempt to induce heavy users to leave their network. We find that in some cases,



(a) Adoption as η varies, $\bar{x}_1 > 0$ for $\eta > 0.7$. (b) Adoption as η varies, $\bar{x}_1 = 0$ for $\eta > 0.1$. (c) Adoption as p_1 increases.

Fig. 3. As the supplemental technology's coverage area η increases, (a) \bar{x}_{1+2} decreases for large η , while (b) total adoption may also decrease. As (c) the base technology's access price p_1 increases, the base technology's adoption increases. Parameter values are (a) $q_1 = 200$, $q_2 = 250$, $\gamma_1 = 50$, $\gamma_2 = 20$, $p_1 = 40$, $p_2 = 10$; (b) $q_1 = 100$, $q_2 = 300$, $\gamma_1 = 50$, $\gamma_2 = 100$, $p_1 = 40$, $p_2 = 10$; and (c) $q_1 = 200$, $q_2 = 225$, $\gamma_1 = 150$, $\gamma_2 = 50$, $p_2 = 30$, $\eta = 0.5$.

increasing p_1 actually increases Technology 1's adoption \bar{x}_1 . Figure 3c shows an example; we note that though \bar{x}_1 increases, the total adoption $\bar{x}_1 + \bar{x}_{1+2}$ decreases. We can in fact fully characterize the conditions under which this behavior occurs:

Proposition 5: Suppose that some users adopt Technologies 1 and (1 + 2), while some adopt neither technology ($\bar{x}_1 > 0$, $\bar{x}_{1+2} > 0$, $\bar{x}_1 + \bar{x}_{1+2} < 1$). Then the base technology's adoption \bar{x}_1 increases with the access price p_1 if

$$q_2 - q_1 < (1 - \eta)\gamma_1 - \eta\gamma_2. \quad (13)$$

Qualitatively, (13) indicates that Technology 1's adoption \bar{x}_1 increases with p_1 if the quality differential ($q_2 - q_1$) from adoption of Technology 2 is outweighed by the marginal savings in throughput degradation from adopting only Technology 1 ($(1 - \eta)\gamma_1 - \eta\gamma_2$). As p_1 increases, users adopt Technology 1, rather than the bundled Technologies (1 + 2).

B. Revenue Maximization

We now examine the behavior of Technologies (1 + 2)'s equilibrium adoption \bar{x}_{1+2} as the coverage factor η varies and the ISP chooses prices so as to maximize its revenue $p_1(\bar{x}_1 + \bar{x}_{1+2}) + p_2\bar{x}_{1+2}$. We first use Tables II and III's expressions for the equilibrium \bar{x}_1 and \bar{x}_{1+2} to find the revenue-maximizing prices p_1^* and p_2^* at each possible equilibrium, as shown in Table IV. To emphasize their dependence on price, in the remainder of this section we use the notation $\bar{x}_1(p_1^*, p_2^*)$ and $\bar{x}_{1+2}(p_1^*, p_2^*)$ to denote the equilibrium adoption levels given the optimal prices p_1^* and p_2^* .

We see from Table IV that the ISP earns non-positive revenue if it maximizes its revenue at equilibria in regions a, f, and g. Intuitively, in these regions, all users adopt at least one technology at the equilibrium ($\bar{x}_1 + \bar{x}_{1+2} = 1$ in Table II). Yet if users' technology valuations θq_i are sufficiently close to zero due to a small θ value, their utility functions (1) and (2) will be negative unless the prices are negative. Thus, the ISP must offer *negative* prices in order to guarantee full adoption in these regions. Using Table IV, we prove that revenue is greatest under partial adoption of both technologies:

Proposition 6: If η is free to vary, its revenue-maximizing value is $\eta = 1$, i.e., full coverage of the supplemental

technology. For any fixed η , if

$$\eta \frac{\gamma_2}{\gamma_1} \geq (1 - \eta) \frac{q_2}{q_1}, \quad (14)$$

the revenue-maximizing equilibrium adoption levels lie in region c of Table I: some users adopt Technology 1, some adopt Technologies (1 + 2), and some adopt neither technology ($\bar{x}_1(p_1^*, p_2^*) > 0$, $\bar{x}_{1+2}(p_1^*, p_2^*) > 0$, $\bar{x}_1(p_1^*, p_2^*) + \bar{x}_{1+2}(p_1^*, p_2^*) < 1$). If (14) does not hold, then no users adopt Technology 1, but some adopt Technologies (1 + 2) ($\bar{x}_1(p_1^*, p_2^*) = 0$, $\bar{x}_{1+2}(p_1^*, p_2^*) > 0$).

The condition in Prop. 6 can be interpreted as stating that when the quality q_1 of Technology 1 is sufficiently high and the marginal throughput degradation γ_1 sufficiently low relative to Technology 2, then for a large coverage factor η , some users will adopt Technology 1 at the optimal prices. However, under these conditions the adoption $\bar{x}_{1+2}(p_1^*, p_2^*)$ of Technologies (1 + 2) will decrease, as shown in Fig. 4's example.⁹ As in Fig. 3a, in Fig. 4 the volume of traffic $\eta\bar{x}_{1+2}(p_1^*, p_2^*)$ offloaded onto Technology 2 increases with η , leading some users to adopt only Technology 1. However, since $p_2^* = \eta(q_2 - q_1)/2$ increases with η (cf. region c in Table IV), ISP revenue increases with η . Overall adoption $\bar{x}_1(p_1^*, p_2^*) + \bar{x}_{1+2}(p_1^*, p_2^*)$ also increases; the increase in $\bar{x}_1(p_1^*, p_2^*)$ offsets the decrease in $\bar{x}_{1+2}(p_1^*, p_2^*)$. Formally,

Proposition 7: If (14) holds, then adoption $\bar{x}_{1+2}(p_1^*, p_2^*)$ of Technologies (1 + 2) decreases and total adoption $\bar{x}_1(p_1^*, p_2^*) + \bar{x}_{1+2}(p_1^*, p_2^*)$ increases with η .

V. OPERATING COSTS AND PROFIT

In addition to considering adoption under revenue maximization, as in Section IV-B, ISPs must take into account the savings from offloading traffic and the cost of deploying the supplemental technology. In this section, we first use empirical usage data to estimate the amount of traffic that can be offloaded onto the supplemental technology's network at times of peak usage on the base technology. We then use cost parameters appropriate for a WiFi deployment to investigate user adoption under ISP profit maximization.

⁹We note that the overall adoption level in Fig. 4 is low when compared with those of Fig. 3. With different parameters (e.g. q_i and γ_i values), the overall adoption levels may change; we use the ones here to reflect current smartphone penetration rates in the U.S. [18].

TABLE IV
REVENUE-MAXIMIZING PRICES ASSUMING EQUILIBRIUM ADOPTION LEVELS IN REGIONS A-G (CF. TABLES I-III).

	p_1^*	p_2^*	Revenue
a	$< -(1-\eta)\gamma_1$	$-(1-\eta)^2\gamma_1 - \eta^2\gamma_2 - p_1$	$-(1-\eta)^2\gamma_1 - \eta^2\gamma_2$
b*	$\frac{-\eta\gamma_1\gamma_2 + (1-\frac{\eta}{2})(q_1-q_2)}{\eta(\gamma_1+\gamma_2) - q_1 + q_2}$	$\frac{\eta(q_2-q_1)}{2}$	$\frac{\frac{\eta}{4}(q_1-q_2)^2 + (1-\eta)\gamma_1(q_1-q_2) - \eta\gamma_1\gamma_2}{\eta(\gamma_1+\gamma_2) + q_2 - q_1}$
c†	$\frac{q_1}{2}$	$\frac{\eta(q_2-q_1)}{2}$	$\frac{q_1^2\eta\gamma_2 + q_2^2\eta\gamma_1 + q_1^2(q_2-q_1) + \eta q_1(q_1-q_2)^2}{4(\gamma_1 q_2 + \eta\gamma_1\gamma_2 + q_1(q_2-q_1) + \eta\gamma_2 - (1-\eta)\gamma_1)}$
d	$\frac{q_1}{2}$	$\geq \eta\left(q_2 - q_1 + \frac{\gamma_1 q_1}{2(\gamma_1 + \gamma_1)}\right)$	$\frac{q_1^2}{4(\gamma_1 + \gamma_1)}$
e	$\frac{(1-\eta)q_1 + \eta q_2}{2} - p_2$	$\leq p_1\left(\frac{-\eta q_2 + \eta^2\gamma_2}{q_1 + (1-\eta)\gamma_1} - \eta\right) - \frac{\eta^2 q_1\gamma_2 - \eta(1-\eta)\gamma_1 q_2}{q_1 + (1-\eta)\gamma_1}$	$\frac{((1-\eta)q_1 + \eta q_2)^2}{4((1-\eta)q_1 + \eta q_2 + (1-\eta)^2\gamma_1 + \eta^2\gamma_2)}$
f	$-\gamma_1$	$\geq \eta(q_2 + \gamma_1 - q_1)$	$-\gamma_1$
g	$> q_1$	$\geq \eta(q_2 - q_1)$	0
If $2(1-\eta)\gamma_1 - 2\eta\gamma_2 > q_2 - q_1$, the revenue-maximizing prices for region b are instead: $p_1^ = \frac{(1-\eta)\gamma_1(-\gamma_1 + q_1 - q_2)}{\eta(\gamma_1 + \gamma_2) + q_2 - q_1}$, $p_2^* = (1-\eta)\gamma_1 - \eta\gamma_2$, revenue = $\frac{-\eta\gamma_1\gamma_2 + ((1-\eta)^2\gamma_1 + \eta^2\gamma_2)(q_1 - q_2) - \eta((1-\eta)\gamma_1 - \eta\gamma_2)^2}{\eta(\gamma_1 + \gamma_2) + q_2 - q_1}$			
†If $\eta\gamma_2 q_1 \leq (1-\eta)\gamma_1 q_2$, then at the optimal prices $\bar{x}_1 = 0$ and the equilibrium lies in region e.			

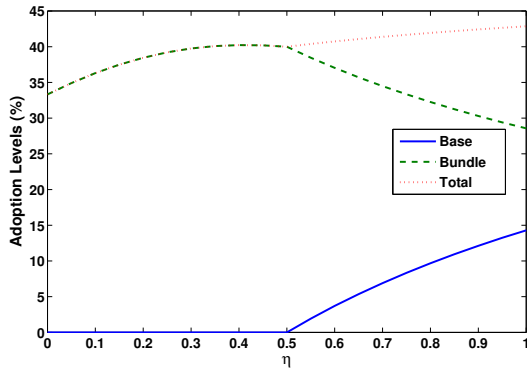


Fig. 4. Adoption levels for $\eta \in [0, 1]$ and revenue-maximizing prices ($q_1 = 50$, $q_2 = 100$, $\gamma_1 = 50$, $\gamma_2 = 100$). As η increases, total adoption $\bar{x}_1(p_1^*, p_2^*) + \bar{x}_{1+2}(p_1^*, p_2^*)$ increases, driven by the increase in $\bar{x}_1(p_1^*, p_2^*)$.

A. Trial Data

We gather 3G and WiFi usage data from 20 Android smartphones over six days. Since ISP cost is driven by peak-hour traffic, we focus on usage when the 3G network is most heavily utilized [19]. Our goal is twofold: first, to estimate the fraction of 3G traffic that occurs at this peak time; and second, to estimate the probability of WiFi access at this time, given the overall WiFi access probability. We can then estimate the amount of traffic that will be offloaded to WiFi at the peak time, given the WiFi adoption and coverage factor.

We implemented a simple data monitoring app and released it to users in the United States. Figure 5 shows app screenshots; in each hour, we recorded the volume of 3G and WiFi usage and WiFi base station IDs. We find that the probability of WiFi access in the hour of highest 3G usage is 82% of the overall probability of WiFi access. On average, 55% of 3G traffic occurs in these peak hours, corroborating existing findings that 3G data usage exhibits severe peaks during the day [19].

B. Optimizing Profit

In addition to its revenue, ISP profit includes its savings from offloading, less the cost of deploying a supplemental

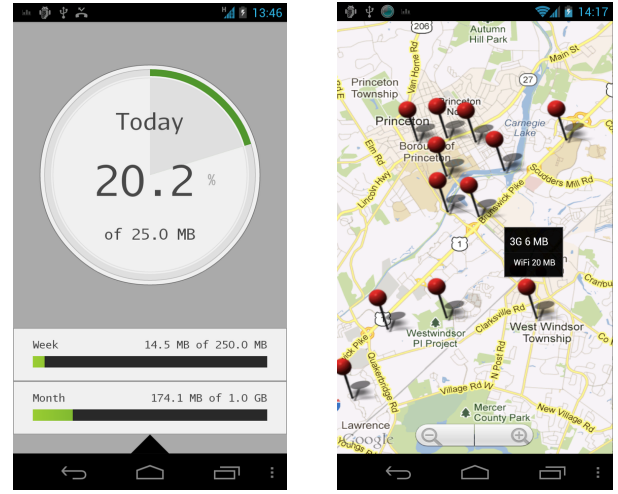


Fig. 5. Screenshots of our usage monitoring app on the Android platform.

technology.¹⁰ Since these parameters depend on the market conditions, we consider three scenarios: a small city; a large, sparsely populated city (e.g., in California); and a large, more densely populated city (e.g., New York or Philadelphia). We refer to the latter two cities as the “sparse” and “dense” cities.

We model the cost savings introduced by user offloading as a linear function of the amount offloaded during the peak hour, i.e., the marginal cost of peak traffic, multiplied by the amount offloaded [19]. We take this marginal cost to be 1.0 ¢/MB in the small city, 1.9 ¢/MB in the sparse city, and 2.9 ¢/MB in the dense city; these values are based on AT&T’s and Verizon’s data plan overage charges in the U.S. From our trial data, we find that each user consumes on average 1200MB in each month, with 660MB occurring at peak hours of the day. As described in the previous section, the probability of peak-hour WiFi access is 82% of the overall access probability; we

¹⁰We assume that the deployment cost of the base technology is independent of the adoption levels, e.g., an already-deployed 3G network.

take this overall probability to be the coverage factor η . Thus, each user offloads $(0.82\eta)(660\text{MB}) = 541\eta$ MB at the peak hours over one month. Multiplying by the fraction of users adopting both technologies and the c/MB marginal savings from offloading, we write the ISP's monetary savings from offloading as $c_{\text{WF}}\eta\bar{x}_{1+2}$, with $c_{\text{WF}} = 5.4, 10.6, \text{ or } 15.8$ for the small, sparse, and dense cities respectively.

We next consider the cost of deploying the supplemental technology. We assume that the ISP's access point (AP) deployment in each type of city is such that the throughput degradation is the same function of the fraction of users on Technology 2's network (i.e., equal γ_2 values). In more densely populated cities, the ISP may utilize a denser AP deployment in order to accommodate the larger number of users in the sparse and dense cities. These additional APs do not increase the geographical coverage area, but rather accommodate more users within the same area.

We model the cost of deployment as a linear function of η : $c_{\text{AP}}\eta$. We thus maximize the ISP's total profit at equilibrium

$$p_1(\bar{x}_1 + \bar{x}_{1+2}) + p_2\bar{x}_{1+2} + c_{\text{WF}}\eta\bar{x}_{1+2} - c_{\text{AP}}\eta \quad (15)$$

with respect to the optimization variables η , p_1 , and p_2 . In the remainder of the section, we use η^* , p_1^* , and p_2^* to denote the optimal values of η , p_1 , and p_2 , respectively; the corresponding adoption levels are denoted by \bar{x}_1^* and \bar{x}_{1+2}^* .

To find c_{AP} , we use parameters appropriate for a WiFi deployment. We assume that each additional AP increases the coverage factor η by a fixed amount $\Delta\eta$ and costs the ISP a fixed amount C_{AP} per month. From [20], we estimate C_{AP} as a monthly operational cost of \$20, plus capital investment of \$1200 spread over 12 months, so that $C_{\text{AP}} = \$120$. For simplicity, we interpret the WiFi access probability η as the physical area covered by APs, e.g., uniform user mobility. The cost of covering an area η with APs is then $C_{\text{AP}}[\eta/\Delta\eta] \approx (C_{\text{AP}}/\Delta\eta)\eta$. Normalizing by the user population, we find that

$$\begin{aligned} c_{\text{AP}}\eta &= \frac{C_{\text{AP}} (\text{Market area})}{(\text{AP coverage area}) (\text{Market population})} \eta \\ &= \frac{\$120}{(\text{AP coverage area}) (\text{Population density})} \eta. \end{aligned}$$

We use population densities of 2000, 5000, and 12000 people per square mile for the small, sparse, and dense cities respectively. The AP coverage area is assumed to be 0.01 square miles, (a 130 meter radius), for the small city, 0.005 square miles for the sparse city, and 0.002 square miles for the dense city. We then find $c_{\text{AP}} = 6.2, 4.9, \text{ and } 11.5$ for the small, sparse, and dense cities respectively.

In Figure 6, we show the adoption levels \bar{x}_1^* and \bar{x}_{1+2}^* at the ISP's optimal operating point for a range of c_{WF} and c_{AP} values, obtained by varying the marginal savings from offloading and cost of one AP's deployment. Though some characteristics persist for all scenarios—for instance, adoption \bar{x}_{1+2}^* of Technologies (1 + 2) increases with the marginal offloading savings c_{WF} —there are some noticeable differences. For the small and sparse cities, as c_{WF} increases at small values, \bar{x}_{1+2}^* does not noticeably increase, but adoption \bar{x}_1^*

of Technology 1 does. As c_{WF} increases at larger values, more Technology 1 users also adopt Technology 2, i.e., \bar{x}_{1+2}^* increases and \bar{x}_1^* decreases for all three cities.

As the marginal cost of deployment c_{AP} increases, the coverage factor η^* decreases, as does the adoption \bar{x}_1^* of Technology 1. For the small city, this decrease in η^* induces behavior similar to that of Fig. 4: as η^* decreases, adoption \bar{x}_{1+2}^* of Technologies (1 + 2) first increases, then decreases. A large η^* implies that the traffic offloaded $\eta^*\bar{x}_{1+2}^*$ is also large, and the resulting congestion induces some users to adopt only Technology 1 and leave Technology 2. The same effect is observed for the sparse and dense cities, without the final decrease in \bar{x}_{1+2}^* for large c_{AP} (small η^*). Thus, in cities with denser populations, a decrease in coverage due to higher costs may in fact increase adoption of Technology 2.

Finally, we examine adoption behavior at the profit-maximizing prices p_1^* and p_2^* for fixed coverage factor η and cost parameters c_{AP} and c_{WF} . From Fig. 7, we see that as η increases, adoption $\bar{x}_{1+2}(p_1^*, p_2^*)$ of Technologies (1 + 2) first increases and then decreases. For smaller values of η , the ISP induces users to adopt Technology 2 and offload traffic onto this network. As η grows, however, the ISP allows $\bar{x}_{1+2}(p_1^*, p_2^*)$ to decrease. Since a large coverage factor allows the ISP to charge a high access price p_2^* for Technology 2, ISP revenue increases, offsetting the decrease in savings from offloading less traffic. We note, however, that this threshold η value is largest for the dense city at about 80%, and smallest for the small city at about 64%. This observation is consistent with the dense city's larger marginal savings from offloading.

VI. CONCLUSION

In this paper, we develop a model of user adoption for base and supplemental wireless network technologies that accounts for both heterogeneity in users' technology valuations, congestion effects, and pricing decisions. We show that user adoption converges to a unique, stable equilibrium point, and derive analytical conditions under which non-intuitive adoption behaviors occur. We then show that these may persist when ISPs maximize either their revenue or profit. To derive a realistic profit model, we use empirical usage data to characterize an ISP's savings from offloading traffic to the supplemental network. We find that the population density of the ISP's market can significantly affect the equilibrium adoption behavior.

Though we use empirical data to realistically study ISP savings from offloading traffic onto a supplementary network, our parameters can only approximate true market structures. Similarly, our user adoption model makes approximating assumptions, one of which is that users' technology valuations are uniformly distributed. While many of the reported qualitative adoption behaviors are also observed for non-uniform distributions (cf. Appendix C), more numerical investigations are needed. We expect that further exploration and validation of this proposed analytical framework will yield additional insights into the benefits of congestion alleviation through offloading in wireless networks.

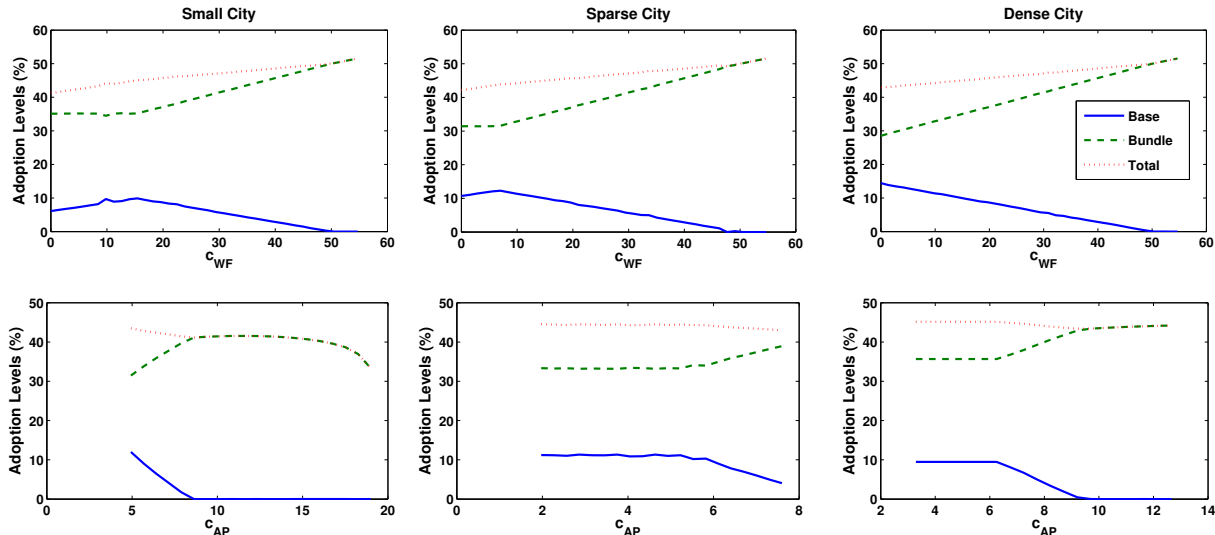


Fig. 6. Adoption levels for the profit-maximizing prices and coverage factor ($q_1 = 50$, $q_2 = 100$, $\gamma_1 = 25$, $\gamma_2 = 50$) in different scenarios. Nominal cost parameters are $(c_{WF}, c_{AP}) = (5.4, 6.2)$, $(10.6, 4.9)$ and $(15.9, 11.5)$ for the small, sparse, and dense cities respectively. Qualitatively, the dynamics as c_{AP} and c_{WF} vary are seen to depend on the user population density.

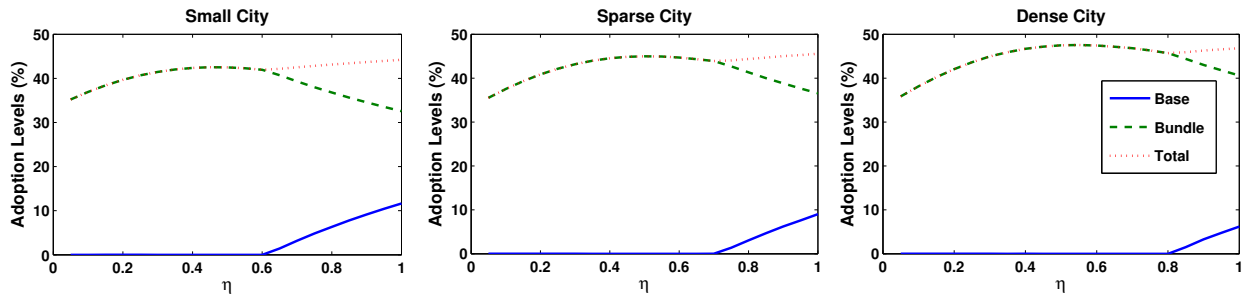


Fig. 7. Adoption levels for the profit-maximizing prices and fixed coverage factor ($q_1 = 50$, $q_2 = 100$, $\gamma_1 = 25$, $\gamma_2 = 50$) in different scenarios. Nominal cost parameters are $(c_{WF}, c_{AP}) = (5.4, 6.2)$, $(10.6, 4.9)$ and $(15.9, 11.5)$ for the small, sparse, and dense cities respectively. As η increases past a threshold value, adoption $\bar{x}_{1+2}(p_1^*, p_2^*)$ of Technologies (1 + 2) decreases despite the potential to offload more traffic as $\bar{x}_{1+2}(p_1^*, p_2^*)$ increases.

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APPENDIX A
PROOFS

A. Proposition 1

First we suppose that $\theta_{1,0} < \theta_{1+2,0}$. Then using (4) and (5), the inequality $\theta_{1+2,0} < \theta_{1,1+2}$ is equivalent to

$$p_1(q_1 - q_2) > q_1 T_2(\eta x_{1+2}) - \eta^{-1} p_2 q_1 - q_2 T_1(x_1 + (1 - \eta)x_{1+2}). \quad (16)$$

But expanding the inequality $\theta_{1,0} < \theta_{1+2,0}$ using (3) and (4), we have the exact same inequality. On the other hand, if $\theta_{1+2,0} < \theta_{1,0}$, then we switch the sign of (16) to show that $\theta_{1+2,0} < \theta_{1,0}$ implies $\theta_{1,1+2} < \theta_{1+2,0}$. ■

B. Proposition 2

In regions a, f and g, it is simple to show that the equilibria are stable: letting $x = (x_1, x_{1+2})$, we denote (11) as $\dot{x} = f(x)$ so that

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

at each equilibrium in regions a, f and g. Since this matrix is clearly negative definite, the equilibria are asymptotically stable.

In region b, the Jacobian of the dynamics (11) is

$$\begin{bmatrix} \frac{\gamma_1}{q_1 - q_2} - 1 & \frac{-\eta\gamma_2 + (1-\eta)\gamma_1}{q_1 - q_2} \\ \frac{-\gamma_1}{q_1 - q_2} & \frac{\eta\gamma_2 - (1-\eta)\gamma_1}{q_1 - q_2} - 1 \end{bmatrix}$$

The eigenvalues of this matrix are $\lambda = \eta \frac{\gamma_1 + \gamma_2}{q_1 - q_2} - 1$, $\lambda = -1$. Since we assume $q_1 < q_2$, both eigenvalues are real and negative and the equilibrium in region b is stable. Similarly, in region c the Jacobian has the form

$$\begin{bmatrix} \frac{\gamma_1}{q_1 - q_2} - \frac{\gamma_1}{q_1} - 1 & \frac{-\eta\gamma_2 + (1-\eta)\gamma_1}{q_1 - q_2} - \frac{(1-\eta)\gamma_1}{q_1} \\ \frac{-\gamma_1}{q_1 - q_2} & \frac{\eta\gamma_2 - (1-\eta)\gamma_1}{q_1 - q_2} - 1 \end{bmatrix},$$

which has eigenvalues λ satisfying

$$2(\lambda + 1) = \frac{\eta(\gamma_2 + \gamma_1)}{q_1 - q_2} - \frac{\gamma_1}{q_1} \pm \sqrt{\left(\frac{\eta(\gamma_2 + \gamma_1)}{q_1 - q_2} - \frac{\gamma_1}{q_1}\right)^2 + \frac{4\eta\gamma_1\gamma_2}{q_1(q_1 - q_2)}},$$

which has solutions $\lambda = \eta \frac{\gamma_2 + \gamma_1}{q_1 - q_2}$, $-\frac{\gamma_1}{q_1}$. As both of these are real and negative, we see that the equilibrium in region c is asymptotically stable.

Finally, in regions d and e we have the triangular Jacobian matrices

$$\begin{bmatrix} \frac{-\gamma_1}{q_1} - 1 & \frac{-(1-\eta)\gamma_1}{q_1} \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ \frac{-(1-\eta)\gamma_1}{(1-\eta)q_1 + \eta q_2} & \frac{-(1-\eta)^2\gamma_1 - \eta^2\gamma_2}{(1-\eta)q_1 + \eta q_2} - 1 \end{bmatrix}.$$

Since all diagonal entries in these matrices are negative, the equilibria in both regions are asymptotically stable. ■

C. Proposition 3

The result follows from Bendixson's criterion. We can use the Jacobian expressions in the proof of Prop. 2 to show that the divergence $\partial f_1/\partial x_1 + \partial f_2/\partial x_{1+2}$ of the dynamical equations (11) is negative in each region. Then Bendixson's criterion tells us that no periodic orbits can exist.¹¹ Since no periodic orbits exist, each trajectory must converge to an equilibrium point. ■

D. Theorem 1

The stability of the equilibrium follows from Prop. 2, while existence of at least one such equilibrium follows from the boundedness of the x_i and the non-existence of a periodic orbit (Prop. 3). Thus, it remains to show that at most one equilibrium point can exist.

Suppose that two stable equilibrium points exist, and consider a bounded neighborhood S of $\{x_1 \geq 0, x_{1+2} \geq 0, x_1 + x_{1+2} \leq 1\}$. We suppose that the dynamical equations (11) are continuously extended to all of S for the purposes of the proof, and that no new equilibria are created. If we consider all trajectories lying in S , the regions of attraction for both equilibria are open, connected, and invariant sets, whose boundaries are formed by trajectories [21]. However, since no periodic orbits exist, the boundaries must be the boundary of S itself. Then since S is connected, there exists at least one point in S that is in neither region of attraction. But this is a contradiction, as the trajectory starting from this point must approach a compact limit set.

We note that in the case of linear throughput functions, an analysis of the existence criteria in Tables II and III yields the same uniqueness result. We present some details below. In this discussion, we use the notation

$$C_b = \eta(\gamma_1 + \gamma_2) + q_2 - q_1 \\ C_e = \eta(q_2 - q_1) - \eta(1 - \eta)\gamma_1 + \eta^2\gamma_2$$

We first consider region a, and show that when paired with any of regions b - f, the regional equilibrium constraints in Tables II and III satisfy the corollary:

- **Region b:** Prices that satisfy both sets of constraints must satisfy $p_2 = \eta((1 - \eta)\gamma_1 - \eta\gamma_2)$, so that the equilibrium point in region b $(0, 1)$, which is the equilibrium in region a.
- **Region c:** We can rearrange the third inequality of region c to find that

$$(p_1 + p_2)\gamma_1 + p_1(\eta\gamma_2 - (1 - \eta)\gamma_1 + q_2 - q_1) \geq -\eta\gamma_1\gamma_2 + (1 - \eta)\gamma_1(q_1 - q_2),$$

¹¹Bendixson's criterion relies on Green's theorem for the vector field f , which is typically assumed to be continuously differentiable. In our case, f is merely piecewise-linear and continuous. However, since we are working in \mathbb{R}^2 and f is continuously differentiable almost everywhere, the proof of Green's theorem is still valid.

and then substitute region a's bound on $p_1 + p_2$ to find that

$$\begin{aligned} p_1 (\eta\gamma_2 - (1 - \eta)\gamma_1 + q_2 - q_1) &\geq \\ (1 - \eta)^2\gamma_1^2 - \eta(1 - \eta)\gamma_1\gamma_2 + (1 - \eta)\gamma_1 (q_1 - q_2). \end{aligned}$$

Substituting into the first inequality of region c, we find that

$$\begin{aligned} -p_2 \left(\frac{(1 - \eta)\gamma_1 + q_1}{\eta} \right) &\leq \\ (q_1 + (1 - \eta)\gamma_1) (\eta\gamma_2 - (1 - \eta)\gamma_1). \end{aligned}$$

Then this inequality must be an equality by the conditions for region a, and we have $p_2 = -\eta^2\gamma_2 + \eta(1 - \eta)\gamma_1$, with $p_1 + p_2 = -(1 - \eta)^2\gamma_1 - \eta^2\gamma_2$. Then the equilibrium point for region c becomes $(0, 1)$, the same as for region a.

- **Region d:** We add the first constraint from region a with the negative of the second constraint from region d to find that

$$\begin{aligned} \eta \left(q_2 - q_1 + \frac{\gamma_1 (q_1 - p_1)}{q_1 + \gamma_1} \right) & \\ < -p_1 - (1 - \eta)^2\gamma_1 - \eta^2\gamma_2, \end{aligned}$$

so that

$$\begin{aligned} p_1 (q_1 + (1 - \eta)\gamma_1) &< -\gamma_1 q_1 \\ (q_1 + \gamma_1) (\eta (q_1 - q_2) - (1 - \eta)^2\gamma_1 - \eta^2\gamma_2). \end{aligned}$$

On the other hand, if we add the second constraint from region a to the negative of the second constraint from region d, we find that

$$\begin{aligned} p_1\gamma_1 &> \gamma_1 q_1 + \\ (q_2 - q_1 + \eta\gamma_2 - (1 - \eta)\gamma_1) (q_1 + \gamma_1). \end{aligned}$$

Combining these two constraints on p_1 , we find that

$$\begin{aligned} \frac{q_2 - q_1 + \eta\gamma_2 - (1 - \eta)\gamma_1 + \frac{\gamma_1 q_1}{q_1 + \gamma_1}}{\gamma_1} &< \\ \frac{\eta (q_1 - q_2) - (1 - \eta)^2\gamma_1 - \eta^2\gamma_2 - \frac{\gamma_1 q_1}{q_1 + \gamma_1}}{q_1 + (1 - \eta)\gamma_1}. \end{aligned}$$

Dropping some terms, we find the necessary condition

$$\begin{aligned} \frac{q_1 (q_1 + (1 - \eta)\gamma_1)}{q_1 + \gamma_1} - (1 - \eta) (q_1 + (1 - \eta)\gamma_1) &< \\ \eta (q_1 - q_2) - (1 - \eta)^2\gamma_1 - \eta^2\gamma_2 - \frac{\gamma_1 q_1}{q_1 + \gamma_1}. \end{aligned}$$

Rearranging, we find that

$$\frac{(1 - \eta)\gamma_1 q_1}{q_1 + \gamma_1} < -\eta q_2 - \eta^2\gamma_2 < 0,$$

which is a contradiction.

- **Region e:** Prices that satisfy both sets of constraints must satisfy $p_1 + p_2 = -(1 - \eta)^2\gamma_1 - \eta^2\gamma_2$. But then the equilibrium point becomes $(0, 1)$ for both regions.
- **Region f:** Prices satisfying region a's constraints must have $p_2 - \eta\gamma_1 \leq -\eta^2(\gamma_2 + \gamma_1)$, while those satisfying

region f's constraints must have $p_2 - \eta\gamma_1 \geq \eta(q_2 - q_1) > 0$. Thus, we have a contradiction.

- **Region g:** Prices satisfying region a's constraints must have $p_1 + p_2 \leq -(1 - \eta)^2\gamma_1 - \eta^2\gamma_2 < 0$, while those satisfying region g's constraints must have $p_1 + p_2 > (1 - \eta)q_1 + \eta q_2 > 0$. Thus, we have a contradiction.

We now consider region b, paired with regions c - g.

- **Region c:** We first note that the first constraint of region b and the third constraint of region c must hold with equality, so that

$$\begin{aligned} p_2 &= -\eta\gamma_2 + (1 - \eta) (q_1 - q_2) - \\ &\frac{\eta\gamma_2 + \eta\gamma_1 + q_2 - q_1}{\gamma_1} p_1, \end{aligned}$$

and $\theta_{1,0} = 0$ at the equilibrium point of region c, and $x_1 + x_{1+2} = 1$ at the equilibrium points of both regions. But then these points are both in region b, and must be the same.

- **Region d:** The second constraint of region d yields

$$p_2 \geq \eta \left(q_2 - q_1 + \frac{\gamma_1 (q_1 - p_1)}{q_1 + \gamma_1} \right),$$

while the first constraint of region b yields

$$\begin{aligned} p_2 &\leq -\eta\gamma_2 + (1 - \eta) (q_1 - q_2) - \\ &\frac{\eta (\gamma_1 + \gamma_2) - q_1 + q_2}{\gamma_1} p_1. \end{aligned}$$

We thus have the necessary condition

$$\begin{aligned} p_1 \left(\frac{-\eta\gamma_2^2}{q_1 + \gamma_1} + \eta (\gamma_1 + \gamma_2) - q_1 + q_2 \right) & \\ \leq -\eta\gamma_2 + q_1 - q_2 - \frac{\eta\gamma_1 q_1}{q_1 + \gamma_1}, \end{aligned}$$

which simplifies to $p_1 \leq -\gamma_1$. Combining this result with the inequality $p_1 \geq -\gamma_1$ from region d, we find that $p_1 = -\gamma_1$ and $p_2 = \eta(q_2 - q_1 + \gamma_1)$. But under these conditions, regions b and d have the same equilibrium point $(1, 0)$.

- **Region e:** From the first constraint in region b, we have

$$\begin{aligned} p_2 &\leq -\eta\gamma_2 + (1 - \eta) (q_1 - q_2) - \\ &\frac{\eta (\gamma_1 + \gamma_2) - q_1 + q_2}{\gamma_1} p_1, \end{aligned}$$

which combined with the constraint $p_2 \geq \eta(1 - \eta)\gamma_1 - \eta^2\gamma_2$ from region b gives

$$\begin{aligned} p_1 &\leq \frac{\gamma_1}{\eta (\gamma_1 + \gamma_2) + q_2 - q_1} \times \\ &(-\eta(1 - \eta) (\gamma_1 + \gamma_2) + (1 - \eta) (q_1 - q_2)) \\ &= -(1 - \eta)\gamma_1. \end{aligned}$$

We can then take the second inequality for region e and find that

$$p_2 \leq \frac{C_e p_1 - \eta^2 q_1 \gamma_2 + \eta(1 - \eta)\gamma_1 q_2}{q_1 + (1 - \eta)\gamma_1},$$

which may be combined with region e's first inequality $p_2 \geq -(1-\eta)^2\gamma_1 - \eta^2\gamma_2 - p_1$ to yield

$$p_1 \left(\frac{-q_1 - (1-\eta)^2\gamma_1 + \eta(q_1 - q_2) - \eta^2\gamma_2}{q_1 + (1-\eta)\gamma_1} \right) \\ \leq (1-\eta)^2\gamma_1 + \eta^2\gamma_2 + \frac{\eta(1-\eta)\gamma_1 q_2 - \eta^2 q_1 \gamma_2}{q_1 + (1-\eta)\gamma_1}.$$

We then substitute $p_1 \leq -(1-\eta)\gamma_1$ to find the necessary condition $0 \leq 0$. But then $p_1 = -(1-\eta)\gamma_1$, and $p_2 = -(1-\eta)^2\gamma_1 - \eta^2\gamma_2 + (1-\eta)\gamma_1$. Then the equilibrium expressions in regions b and e are the same.

- **Region f:** We have $p_2 = \eta(q_2 + \gamma_1 - q_1)$, so that the equilibrium in region b becomes $(1, 0)$: the same as that in region f.
- **Region g:** We see from the constraints on region g that $p_2 > \eta(q_2 - q_1)$, so that $C_b p_1 < -\gamma_1 \gamma_2 + \gamma_1(q_1 - q_2)$. But $C_b > 0$ and $p_1 > q_1 > 0$ from the constraints on region g, so we have a contradiction.

We next consider region c, paired with regions d - g.

- **Region d:** We note that the second constraints for regions c and d imply that

$$p_1 \gamma_1 + \frac{p_2}{\eta} (q_1 + \gamma_1) \\ = \gamma_1 q_2 - q_1 (q_1 - q_2),$$

so that the equilibrium point for both regions becomes the same.

- **Region e:** The first constraint for region c and the second constraint for region e imply that both hold with equality. One can then check that the equilibrium points for both regions are the same.
- **Region f:** We first multiply the second constraint for region c by η and add it to the third constraint, yielding

$$p_1 (\eta \gamma_2 + q_2 - q_1) - p_2 q_1 \geq \\ - \eta \gamma_1 (\gamma_2 + q_2) + ((1-\eta)\gamma_1 + \eta q_1) (q_1 - q_2),$$

which since $p_1 \leq -\gamma_1$ implies the necessary condition

$$p_2 \leq \eta \gamma_1 - \eta (q_1 - q_2).$$

Combining this with region f's second constraint, this inequality holds with equality, and $p_1 = -\gamma_1$. We thus have the equilibrium point $(1, 0)$ for both regions.

- **Region g:** We find from the second inequality of region c that

$$p_2 \left(\frac{q_1 + (1-\eta)\gamma_1}{\eta} \right) \leq \\ \gamma_1 q_2 - q_1 (q_1 - q_2) - \gamma_1 ((1-\eta)q_1 + \eta q_2),$$

where we have substituted region g's second inequality $p_2 + p_1 \geq (1-\eta)q_1 + \eta q_2$. From region g's constraints, we then find that $p_2 \geq \eta(q_2 - q_1)$. Combining this with the above inequality, we have the necessary condition

$$\eta(1-\eta)\gamma_1 (q_2 - q_1) - q_1 (q_1 - q_2) \geq \\ \eta(q_2 - q_1) (q_1 + (1-\eta)\gamma_1),$$

which yields $0 \geq 0$. Then this inequality must hold with equality, so that $p_2 = \eta(q_2 - q_1)$ and $p_1 = q_1$. Then regions c and g both contain the equilibrium $(0, 0)$.

We now consider region d, paired with regions e - g.

- **Region e:** The second constraint of region d yields the inequality

$$p_2 \geq \eta \left(q_2 - q_1 + \frac{\gamma_1 (q_1 - p_1)}{q_1 + \gamma_1} \right),$$

while the second constraint of region e yields

$$p_2 \leq \frac{C_e p_1 - \eta^2 q_1 \gamma_2 + \eta(1-\eta)\gamma_1 q_2}{q_1 + (1-\eta)\gamma_1}.$$

Thus, we combine these inequalities and simplify to find the necessary condition

$$p_1 \left(\frac{C_e}{q_1 + (1-\eta)\gamma_1} + \frac{\eta \gamma_1}{q_1 + \gamma_1} \right) \geq \\ \eta (q_2 - q_1) + \frac{\eta \gamma_1 q_1}{q_1 + \gamma_1} + \\ \frac{\eta^2 q_1 \gamma_2 - \eta(1-\eta)\gamma_1 q_2}{q_1 + (1-\eta)\gamma_1}.$$

The first constraint in region d yields $p_1 \leq q_1$, which when combined with the above inequality gives the necessary condition

$$q_1 (q_1 + \gamma_1) (\eta^2 \gamma_2 - \eta(1-\eta)\gamma_1) \geq \\ \eta(1-\eta)\gamma_1 (q_2 - q_1 + \eta^2 q_1 \gamma_2) (q_1 + \gamma_1) \\ + (q_1 + \gamma_1) (\eta^2 q_1 \gamma_2 - \eta(1-\eta)\gamma_1 q_2),$$

which simplifies to the inequality $0 \leq 0$. Thus, the previous inequalities must all hold with equality, so that $p_1 = q_1$ and $p_2 = \eta(q_2 - q_1)$. But then the equilibrium points in regions d and e are both $(0, 0)$.

- **Region f:** We must have $p_1 = -\gamma_1$, so that the equilibrium points in both regions are the same: $(1, 0)$.
- **Region g:** We must have $p_1 = q_1$, so that the equilibrium points in both regions are $(0, 0)$.

We next consider region e, paired with regions f and g.

- **Region f:** We can multiply the first constraint for region e by $q_1 + (1-\eta)\gamma_1$ and add it to the second constraint to find

$$p_1 (\eta q_2 + (1-\eta)q_1 + (1-\eta)^2\gamma_1 + \eta^2\gamma_2) > \\ \eta^2 q_1 \gamma_2 - \eta(1-\eta)\gamma_1 q_2 - \\ (q_1 + (1-\eta)\gamma_1) ((1-\eta)^2\gamma_1 + \eta^2\gamma_2).$$

Combining this with the constraint $p_1 > -\gamma_1$ from region f, we have

$$-(1-\eta)\gamma_1 q_1 > \eta^2 q_1 \gamma_2 + \eta^2 \gamma_1 q_2 \\ + (\eta \gamma_1 - q_1) ((1-\eta)^2\gamma_1 + \eta^2 \gamma_1),$$

which simplifies to a contradiction:

$$0 > \eta \gamma_1 ((1-\eta)^2\gamma_1 + \eta^2 \gamma_2 + (1-\eta)q_1 + \eta q_2).$$

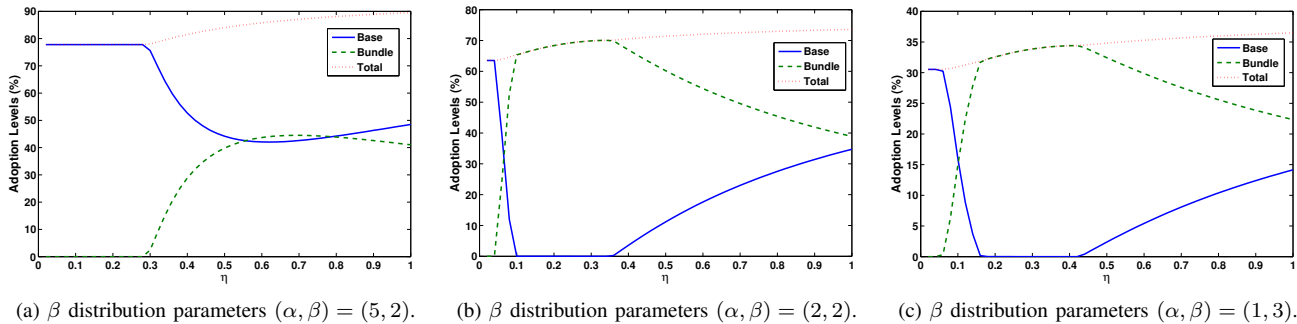


Fig. 8. Equilibrium adoption levels as the coverage factor η varies for different β distributions of the user heterogeneity variable θ . System parameters are (a) $q_1 = 150, q_2 = 250, \gamma_1 = 50, \gamma_2 = 150, p_1 = 50, p_2 = 40$; (b) and (c) $q_1 = 200, q_2 = 280, \gamma_1 = 50, \gamma_2 = 150, p_1 = 50, p_2 = 5$.

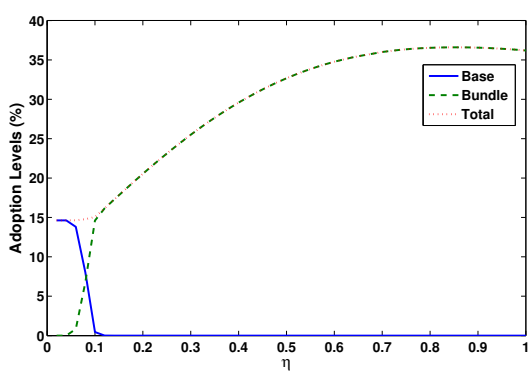


Fig. 9. Equilibrium adoption levels as the coverage factor η increases, $\bar{x}_1 = 0$ for large η . User heterogeneity θ follows a β distribution with parameters $(\alpha, \beta) = (1, 3)$ (cf. Fig. 10). System parameters are $q_1 = 100, q_2 = 300, \gamma_1 = 50, \gamma_2 = 100, p_1 = 40, p_2 = 10$.

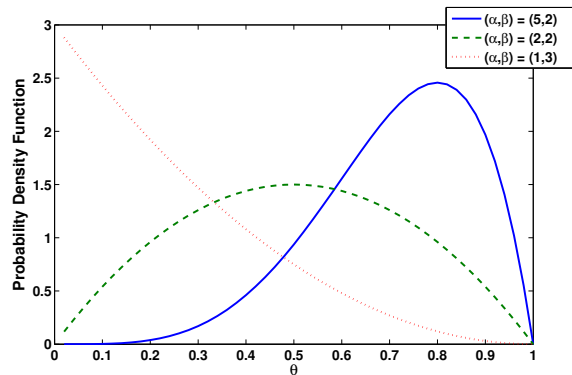


Fig. 10. Different β distributions used for the user heterogeneity variable θ in Fig. 8.

- **Region g:** We have $p_1 + p_2 = (1 - \eta)q_1 + \eta q_2$, so that the equilibrium in both regions is $(0, 0)$.

Finally, we consider regions f and g: any equilibrium point satisfying both these constraints must satisfy $q_1 \geq p_1 > 0 > -\gamma_1 \geq p_1$, which is a contradiction. ■

E. Proposition 4

The condition follows upon noting that the assumptions $x_1 = 0, x_1 + x_{1+2} < 1$, and $x_{1+2} > 0$ imply that the adoption dynamics lie in region e. Differentiating Table II's equilibrium expression for x_{1+2} in this region, we find the condition (12). If $x_1 > 0$, the adoption dynamics lie in region c. We may differentiate the equilibrium expressions for $x_1 + x_{1+2}$ and x_{1+2} in this region and use the constraints in Table II to show that these quantities are increasing in η . ■

F. Proposition 5

Under the given conditions for x_1 and x_{1+2} , the dynamics lie in region c of Table I. Thus, we differentiate the equilibrium expression for x_1 in region c in Table III to obtain the desired result. ■

G. Proposition 6

We first show that the revenue in region b cannot exceed that in region c or e. Next, for $\eta\gamma_2q_1 \geq (1 - \eta)\gamma_1q_2$, we show

that the revenue in region e cannot exceed that in region c. Under these conditions, the derivative with respect to η of the maximum revenue in region e is nonnegative. We then take the derivative of the revenue in region c with respect to η , and show that it is nonnegative for $\eta \in [0, 1]$. Since revenue in region c equals that in region d at $\eta = 0$, the proposition follows.

Region b: We first consider the case $2(1 - \eta)\gamma_1 - 2\eta\gamma_2 > q_2 - q_1$. By inspection, in this case region b has negative revenue. Since regions c, d and e have positive revenue, this cannot be optimal.

We now suppose that $2(1 - \eta)\gamma_1 - 2\eta\gamma_2 \leq q_2 - q_1$ and $\eta\gamma_2q_1 < (1 - \eta)\gamma_1q_2$. We show that the maximum revenue in region b is less than the maximum revenue in region e. It suffices, from Table IV, to show that

$$\begin{aligned} & \left(\eta(q_2 - q_1)^2 + 4(1 - \eta)\gamma_1(q_1 - q_2) \right) \\ & \times \left((1 - \eta)q_1 + \eta q_2 + (1 - \eta)^2\gamma_1 + \eta^2\gamma_2 \right) \\ & \leq \left((1 - \eta)q_1 + \eta q_2 \right)^2 (\eta(\gamma_1 + \gamma_2) + q_2 - q_1). \end{aligned}$$

Simplifying and neglecting some terms, we find the sufficient

condition

$$\begin{aligned} & \eta\gamma_1q_1^2 + \eta\gamma_2q_1^2 + q_1^2(q_2 - q_1) + 2\eta^2\gamma_1q_1(q_2 - q_1) \\ & + 2\eta^2\gamma_2q_2(q_2 - q_1) + \eta q_1(q_2 - q_1)^2 + \eta^2\gamma_1(q_2 - q_1)^2 \\ & + 4\eta(1 - \eta)(4 - 1 + \eta)\gamma_1(q_2 - q_1)^2 \geq 0, \end{aligned}$$

which is clearly true, since each term in the sum is nonnegative.

Next, we suppose that $\eta\gamma_2q_1 \geq (1 - \eta)\gamma_1q_2$ and show that the revenue at region c's revenue-maximizing point is larger than the maximum revenue in region b. From Table IV, it suffices to show that

$$\begin{aligned} \frac{\frac{\eta}{4}(q_1 - q_2)^2}{\eta(\gamma_1 + \gamma_2) + q_2 - q_1} & \leq \frac{q_1^2\eta\gamma_2 + q_2^2\eta\gamma_1}{4A} \\ & + \frac{q_1^2(q_2 - q_1)}{4A} \\ & + \frac{\eta q_1(q_1 - q_2)^2}{4A} \end{aligned}$$

for any given η , where

$$\begin{aligned} A & = \gamma_1q_2 + \eta\gamma_1\gamma_2 \\ & + q_1(q_2 - q_1 + \eta\gamma_2 - (1 - \eta)\gamma_1). \end{aligned}$$

Multiplying out the fractions, we find the sufficient condition

$$\begin{aligned} & (\eta\gamma_1q_2 + \eta^2\gamma_1\gamma_2)(q_1 - q_2)^2 \\ & - \eta(1 - \eta)q_1\gamma_1(q_1 - q_2)^2 \\ & \leq (\eta\gamma_1 + \eta\gamma_2 + q_2 - q_1)(\eta\gamma_2q_1^2 + \eta\gamma_1q_2^2). \end{aligned}$$

We now expand the left side of this inequality to find that it is

$$\begin{aligned} & \leq -\eta\gamma_1q_1q_2^2 + \eta\gamma_1q_2^3 \\ & + \eta^2\gamma_1\gamma_2q_1^2 - 2\eta^2\gamma_1\gamma_2q_1q_2 + \eta^2\gamma_1\gamma_2q_2^2 \\ & = \eta\gamma_1q_2^2(q_2 - q_1) + \eta^2\gamma_1\gamma_2q_1^2 \\ & - 2\eta^2\gamma_1\gamma_2q_1q_2 + \eta^2\gamma_1\gamma_2q_2^2. \end{aligned}$$

We now obtain

$$\begin{aligned} -2\eta^2\gamma_1\gamma_2q_1q_2 & \leq \eta^2\gamma_1^2q_2^2 + \eta^2\gamma_2^2q_1^2 \\ & + \eta\gamma_1q_2^2(q_2 - q_1). \end{aligned}$$

Since the left-hand side is clearly negative, while the right-hand side is positive, we have the desired result.

Region e: We multiply the optimal revenue expressions in Table IV and simplify terms to find the necessary and sufficient condition

$$\eta((1 - \eta)\gamma_1q_2 - \eta\gamma_2q_1)^2 \geq 0,$$

which clearly holds for all η .

If $\eta\gamma_2q_1 < (1 - \eta)\gamma_1q_2$, then the revenue-maximizing point lies in region e. We thus take the derivative of revenue in region e with respect to η to find that it equals

$$C\eta \left[(q_2 - q_1)^2 - 2\gamma_1q_2 - 2\gamma_2q_1 \right] + 2\gamma_1q_2 + q_1(q_2 - q_1),$$

where

$$C = \frac{((1 - \eta)q_1 + \eta q_2)}{4((1 - \eta)q_1 + \eta q_2 + (1 - \eta)^2\gamma_1 + \eta^2\gamma_2)^2}.$$

Since $\eta\gamma_2q_1 < (1 - \eta)\gamma_1q_2$, we see that this quantity is nonnegative, and thus that the revenue increases as η increases. The revenue-maximizing point is thus the largest value of η for which $\eta\gamma_2q_1 < (1 - \eta)\gamma_1q_2$.

Region d: We first show that the revenue in region c is increasing in η . By inspection, the revenue expressions in regions c, d, and e are equal at $\eta = 0$, which will complete the proof. We calculate that

$$\begin{aligned} \frac{dR_c}{d\eta} & = \frac{\gamma_1^2q_2^2(q_2 - q_1) + 2\gamma_1q_1q_2(q_2 - q_1)^2}{4X^2} \\ & + \frac{q_1^2(q_2 - q_1)^2 + (1 - \eta)\gamma_1q_1^3(q_2 - q_1)}{4X^2}, \end{aligned}$$

where

$$\begin{aligned} X & = \gamma_1q_2 + \eta\gamma_1\gamma_2 \\ & + q_1(q_2 - q_1 + \eta\gamma_2 - (1 - \eta)\gamma_1). \end{aligned}$$

Thus, $dR_c/d\eta \geq 0$ for all values of $\eta \in [0, 1]$. ■

H. Proposition 7

If $\eta\gamma_2q_1 \geq (1 - \eta)\gamma_1q_2$, the ISP's revenue is maximized when the dynamics lie in region c. Thus, we can use Table IV's expressions for the optimal prices in region c to find the corresponding adoption levels in Table II. Differentiating with respect to η yields the proposition. ■

APPENDIX B THROUGHPUT LINEARIZATION

In Section III, we use *linear* models to represent the decrease in utility due to throughput degradation. We justify this assumption here by analyzing the accuracy of a linear approximation to previously proposed throughput measures.

Prior works on technology adoption [14] take congestion levels into account with a Markov chain analysis, assuming Poisson arrivals of rate λx and exponentially distributed session length with mean μ^{-1} . The expected throughput is then

$$-R_0(1 - \nu x) \frac{\log(1 - \nu x)}{\nu x}, \quad (17)$$

where R_0 is the average time of service without interference or queueing, and $\nu = \lambda/\mu$ is assumed to be less than 1. Thus, the *throughput degradation* equals (17), less the maximum throughput. We now use Taylor's remainder theorem to bound the error of a linear approximating this quantity. The second derivative of (17) is

$$\begin{aligned} R_0 \left(\frac{2}{x^2} + \frac{\nu}{x} + \frac{\nu^2}{1 - \nu x} + \frac{2 \log(1 - \nu x)}{\nu x^3} \right) & = \\ R_0 \nu^2 \sum_{n=0}^{\infty} \frac{n+1}{n+3} (\nu x)^n, & \end{aligned}$$

where the infinite sum uses the Taylor series for $\log(1 - \nu x)$ and the geometric series for $(1 - \nu x)^{-1}$. Thus, approximating (17) at $x = 0.5$, we can bound the error by

$$\begin{aligned} & R_0 \nu^2 \max_{x \in (0,1)} \frac{(x - 0.5)^2}{2} \sum_{n=0}^{\infty} \frac{n+1}{n+3} (\nu x)^n \\ & \leq \frac{R_0 \nu^2}{8} \max_{x \in (0,1)} \sum_{n=0}^{\infty} \frac{n+1}{n+3} (\nu x)^n. \end{aligned}$$

Approximating the sum as the geometric series of νx , we let $x = 1$ to obtain an upper bound of $R_0 \nu^2 / (8 - 8\nu)$. Numerically, this bound is in fact conservative; for instance, taking $R_0 = 1$ and $\nu = 0.5$ produces a maximum error of 0.013, as opposed to an analytical bound of 0.0625.

APPENDIX C

NON-UNIFORMLY DISTRIBUTED VALUATIONS

In Fig. 8, we show some adoption behaviors for fixed system parameters when the user heterogeneity variable θ , introduced in Section III-A, is not uniformly distributed. We consider three different distributions of θ (probability density functions shown in Fig. 10) and investigate the equilibrium adoption levels as the coverage factor η varies. As in Fig. 3a in Section IV-A, we observe that as the coverage increases, adoption \bar{x}_{1+2} decreases, while total adoption $\bar{x}_1 + \bar{x}_{1+2}$ increases. In Fig. 9, we present an example in which $\bar{x}_1 = 0$ for $\eta > 0.12$; then as \bar{x}_{1+2} decreases, so does the total adoption.