Abstract

The objective of this study is to develop a majorization-based tool to compare financial networks with a focus on the implications of liability concentration. Specifically, we quantify liability concentration by applying the majorization order to the liability matrix that captures the interconnectedness of banks in a financial network. We develop notions of balancing and unbalancing networks to bring out the qualitatively different implications of liability concentration on the system’s loss profile. We illustrate how to identify networks that are balancing or unbalancing, and make connections to interbank structures identified by empirical research, such as perfect and imperfect tiering schemes. An empirical analysis of the network formed by the banking sectors of eight representative European countries suggests that the system is either unbalancing or close to it, persistently over time. This empirical finding, along with the majorization results, supports regulatory policies aiming at limiting the size of gross exposures to individual counterparties.

Keywords: systemic risk, financial network, interbank liabilities, majorization.

1 Introduction

The financial industry is fraught with cases of bank failures due to large exposures to certain counterparties. One such example is Johnson Matthey Bankers in the United Kingdom in 1984. Bank assets more than doubled between 1980 and 1984, and loans became concentrated only to a few borrowers, including Mahmoud Sipra, Rajendra Sethia and ESAL Commodities, and Abdul Shamji. The quality of some of these loans turned out to be worse than expected, and Bank of England had to intervene to prevent a financial crisis. Another example is the Korean banking system during the crisis in the late 1990s, when the country’s bank assets were concentrated on five largest banks. These cases have prompted regulatory authorities in recent years to impose limits on banks’ exposures. For example, the Core Principles for Effective Banking Supervision (Core Principle 19) sets prudent limits on large exposures to a single borrower.

The goal of our study is to develop an analytical framework to assess how concentration of liabilities affects the exposures towards individual entities, and in turn induces losses in a financial system. Our starting point is the basic model of Eisenberg and Noe (2001), enhanced with
bankruptcy costs as in Glasserman and Young (2014), which captures the interlinking exposures among financial institutions. The model yields the loss contributed by each node (bank) in the network, computed as the difference between its total liability and payment. We then use the majorization order (Arnold et al. (2011)) among vectors to express preferences between losses, i.e. one loss profile (vector) is preferred to another only if it is majorized by the latter. This allows us to capture the desired preference via a broad class of functions known as Schur convex functions (Arnold et al. (2011)), which preserve the majorization order. Such a class includes functions such as summation and max; thus, a loss profile is preferred to another if it results in a smaller total loss or a smaller worst-case loss.

More importantly, we use matrix majorization to compare relative liability matrices in terms of concentration of liabilities. When the relative liability matrix of network $a$ is majorized by the corresponding matrix of network $b$, it means that in network $a$ the interbank liabilities are more evenly distributed across the nodes in the network, as compared with $b$, where the liabilities are more concentrated. (Refer to Figure 1, where the relative liability matrix of the network on the left is majorized by the one on the right, in both upper and lower panels.)

In addition, we develop two new notions of financial networks, referred to as balancing and unbalancing. Note, the two notions are not orthogonal to each other, notwithstanding what their names may suggest. Both notions are defined in terms of the primitive data from the financial network; in particular, they both concern the pre-clearing equity position of the banks (nodes), with respect to their liabilities. The balancing notion is defined in terms of the base-liability configuration, i.e., assuming all nodes will pay their full liabilities; in which case, it stipulates that a node with a smaller liability is associated with a larger equity. This, of course, needs not be the case post-clearing, as some nodes might default. The unbalancing notion, instead, is defined in terms of a (minimally) reduced level of liability, which guarantees that all nodes will make full payment at clearing, and stipulates that a node with a smaller liability has a smaller equity. Thus, in an unbalancing system, a node with a smaller liability is more likely to default, and so does a node with a larger liability in a balancing system. (Both are the consequence of a smaller equity before clearing.)

Liability concentration has a knock-on effect when compounded with balancing and unbalancing systems. A high concentration of liability means that a node with a large (small) liability will receive more (less) payments relative to the case of low concentration. For example, when a balancing system is coupled with a lower concentration (of liability), a node with a larger liability tends to receive less payments and thus incurs a larger loss. Moreover, a low concentration means that the payment flows among the nodes are more uniform; hence, the loss at one node is more likely to propagate to its neighboring nodes, and their neighboring nodes, and so forth. Thus, the low concentration implies (a potentially) more serious systemic consequence in the balancing case. In an unbalancing system, on the other hand, a higher liability concentration will yield the similar systemic consequence. In this case, a node with a smaller liability will tend to receive less payments, resulting in a larger loss. Since liabilities are more concentrated, the loss induced by a defaulted bank will have to be absorbed by a smaller number of its creditors, which may induce more defaults; hence, the contagion effect.

Our results are related to Amini et al. (2010), who show that a network with higher concentration of exposures is less resilient to shocks. In their framework, the loss incurred by the creditors of each node does not depend on the interbank liability structure. Rather it is given by the exposure of the creditor to the defaulted node multiplied by an exogenously specified loss given default rate. Moreover, their analysis is performed asymptotically as the number of institutions grows to infinity.
We use consolidated banking sector data of the eight largest European countries, consisting of balance sheet data and interbank exposures, to investigate the state of the network. Our analysis reveals that it is either unbalancing or close to it (see Table 3 for details) persistently over different time periods. These empirical findings, along with the above discussed theoretical results, support regulatory policies of the Basel Committee (BCBS (2014)) aiming at limiting the size of gross exposures to individual counterparties. Moreover, our results add to the understanding of preventive policies in bringing out their consequences and implications on the network. The regulator will monitor the interbank system and limit gross exposures toward banks that have small capital. In an unbalancing system, banks with smaller outstanding liabilities will incur larger losses. Hence, reducing gross exposures to them means to push banks with larger liabilities to lend less to those with smaller liabilities, making the matrix of interbank liabilities less concentrated and driving the network closer to a balancing state. When this transition happens, banks with larger outstanding liabilities will suffer larger losses. The regulator would then need to incentivize them to lend more to banks with smaller liabilities. This will reduce the net exposure of banks with smaller liabilities to those with large liabilities, and thereby make liabilities more concentrated.

A brief overview of related literature is in order. Most studies on interbank networks have focused on understanding the impact of shocks, originating in a specific part of the network, on the overall financial system. Allen and Gale (2001) employ an equilibrium approach to model the propagation of financial distress in a credit network. Gai and Kapadia (2010) model how contagion spreads in a random network, and analyze the knock-on effects of distress. Battiston et al. (2012a) and Battiston et al. (2012b) characterize feedback effects arising from changing financial conditions of the network nodes. Rogers and Veraart (2012) improve the realism of the Eisenberg and Noe (2001) model by including liquidation costs at default. Elsinger et al. (2006) distinguish between fundamental and contagious defaults in the Eisenberg and Noe (2001) framework, and analyze feedback and domino effects via an empirical analysis. Glasserman and Young (2014) show that, under a wide range of shock distributions, the contagion effects via network spillovers are usually small for realistic interbank networks. Furfine (2003) provides an empirical analysis quantifying contagion risk resulting from interbank federal funds exposures data. Haldane and May (2011) draw analogies with ecosystems and analyze how growth in interbank claims leads to instability. Other studies have explored the relation between the topological structure of the network and the magnitude of defaults it experiences. Gai et al. (2011) analyze the degree to which networks with a smaller number of key strongly interconnected players is affected by target shocks. Elliott et al. (2013) discuss the dependence of the probability of default cascades on integration and diversification. Acemoglu et al. (2014) develop a theoretical framework to explain the robust-yet-fragile tendency of financial networks. On the empirical side, Cont et al. (2013) and Angelini et al. (1996) analyze, respectively, Brazilian and Italian interbank systems, and show how contagion through the payment system can originate systemic crisis.

The rest of the paper is organized as follows. We start with preliminaries on both the Eisenberg-Noe model and the majorization order in §2, and formalize the notion of a loss profile and loss preference in a financial network. We then spell out in §3 the technical details in modeling a) the concentration of liabilities using matrix majorization, and b) the notion of balancing versus unbalancing networks and their implications on loss preference. §4 presents concrete examples to illustrate and enhance the notions of balancing and unbalancing systems. In particular, we make connections to the studies on German and Italian banks respectively presented in Craig and Von Peter (2014) and in Fricke and Lux (2013), where a tiering (or, core-periphery) structure has been identified as the primary configuration of interbank liabilities. We apply the balancing/unbalancing notions to the tiering structure and bring out the distinction between perfect and imperfect tiering.
schemes. §5 provides an empirical analysis of the network induced by the eight largest European banking sectors and develop policy implications. Concluding remarks are summarized in §6. Proofs of technical results are delegated to an appendix.

2 Loss Preferences

We describe the majorization method used to express loss preferences in Section 2.1. We recall the Eisenberg-Noe framework enhanced with bankruptcy costs in Section 2.2. We describe the objective of the study in Section 2.3.

2.1 Loss Comparison Using Majorization

We start by providing basic notations and definitions related to majorization and refer to Arnold et al. (2011) for a complete treatment of the subject.

Majorization is a preorder on vectors of real numbers, which measures the dispersion among the elements in a vector. For any vector \( x \in \mathbb{R}^n \), we use \( x[1], \ldots, x[n] \) to denote the ordered entries of \( x \) from largest to smallest (\( x[1] \) being the largest and \( x[n] \) the smallest). Moreover, we use \( x(1), \ldots, x(n) \) to denote the ordered entries of \( x \) from smallest to largest (\( x(1) \) being the smallest and \( x(n) \) the largest).

Definition 1. \( x \) is majorized by \( y \), denoted by \( x < y \), if

\[
\sum_{i=1}^{k} x[i] \leq \sum_{i=1}^{k} y[i] \quad \text{for} \quad k = 1, \ldots, n-1, \quad \sum_{i=1}^{n} x[i] = \sum_{i=1}^{n} y[i].
\]

or equivalently,

\[
\sum_{i=1}^{k} x(i) \geq \sum_{i=1}^{k} y(i) \quad \text{for} \quad k = 1, \ldots, n-1, \quad \sum_{i=1}^{n} x(i) = \sum_{i=1}^{n} y(i).
\]

\( x < y \) indicates that the vector \( x \) is more evenly distributed than \( y \). Replacing the equality in Eq. (1) and (2) with \( \leq \) and \( \geq \) respectively leads to the notion of weak submajorization and weak supermajorization.

Definition 2. For \( x, y \in \mathbb{R}^n \), \( x \) is weakly submajorized by \( y \), denoted by \( x \prec_w y \), if

\[
\sum_{i=1}^{k} x[i] \leq \sum_{i=1}^{k} y[i] \quad \text{for} \quad k = 1, \ldots, n.
\]

\( x \) is weakly supermajorized by \( y \), denoted by \( x \succ_w y \), if

\[
\sum_{i=1}^{k} x(i) \geq \sum_{i=1}^{k} y(i) \quad \text{for} \quad k = 1, \ldots, n.
\]

Interchangeably, we denote \( x \succ_w y \) if \( y \prec_w x \) and \( x \succ_w y \) if \( y \prec_w x \).

We next explain how we express preference between losses using majorization. Let \( x \in \mathbb{R}^n \) be a vector, whose \( i \)-th component \( x_i \) is interpreted as the loss generated by entity \( i \) in a financial
network. We say that the loss vector \( x \) is preferred to the loss vector \( y \) if \( x <_w y \). Our choice is driven by the following consideration. Consider two networks \( a \) and \( b \) consisting of the same set of entities. We now think of \( x \) as the loss vector associated with network \( a \) and of \( y \) as the loss vector associated with network \( b \). The difference between \( a \) and \( b \) lies in the interbank structure. If, for any \( k \in \{1, \ldots, n\} \), the sum of the \( k \) largest losses generated by entities in network \( a \) never exceeds the corresponding quantity in network \( b \), then we prefer the interbank network \( a \) to \( b \). In particular, this means that the maximum loss generated by a node in the network \( a \) does not exceed the maximum loss generated by a node in the network \( b \) (\( k = 1 \)). Further, it also implies that the total loss in the network \( a \) never exceeds the corresponding quantity in the network \( b \) (\( k = n \)). Our preference criterion is also related to the monitoring mechanism proposed by Duffie (2011), where each systemically important entity is suggested to report the identities of the ten counterparties against which it has the largest gains or losses, under a set of stressful scenarios. While in Duffie (2011) the regulator is interested in monitoring losses on an individual basis, i.e. separately for each bank, in our case he would be concerned about the aggregate loss generated by those banks with the \( k \) highest shortfalls.

Notice that our objective is to measure the size of losses, and not in which specific nodes of the network they occurred. This property is preserved when weak submajorization is used to express preferences. Recall that weak submajorization is a preorder, i.e. \( x <_w y \) and \( y <_w x \) together imply that \( x = yP \) for some permutation matrix \( P \), but not that \( x = y \). Hence, a permutation of the loss vector is equally preferred to the original loss vector.

2.2 The Eisenberg-Noe Framework with Bankruptcy Costs

We define the loss vector associated with a financial system using an extended Eisenberg-Noe model, where losses due to bankruptcy are modeled as in Glasserman and Young (2014). We consider a network of interbank liabilities consisting of \( n \) nodes, where each node represents a financial institution. Let \( L \in \mathbb{R}^{n \times n}_\geq \) be the interbank liability matrix with \( l_{i,j} \) denoting the amount of liabilities owed by \( i \) to \( j \), and \( c \in \mathbb{R}^{1 \times n}_\geq \) be the outside asset vector, in which each component \( c_i \) represents the value of outside assets held by node \( i \). Each node also has liabilities towards entities which are not part of the interbank network. More specifically, we let \( e \in \mathbb{R}^{1 \times n}_\geq \) be the outside liability vector, where the entry \( e_i \) denotes the amount of liabilities of node \( i \) towards entities outside the network. These liabilities are assumed to have equal priority to the interbank liabilities.

The total liability vector is denoted by \( \ell \in \mathbb{R}^{1 \times n}_\geq \), with \( \ell_i = \sum_{j=1}^n l_{i,j} + e_i \) being the total amount of obligations from node \( i \) to all other nodes and to the outside network. Further, we denote by

\[
\pi_{i,j} = \begin{cases} 
\frac{l_{i,j}}{\ell_i} & \text{if } \ell_i > 0 \\
0 & \text{if } \ell_i = 0,
\end{cases}
\]

the relative size of liabilities owed by \( i \) to \( j \). Here, \( \Pi \) is the interbank relative liability matrix. Because \( i \) may also owe to entities outside the network, \( \sum_{j=1}^n \pi_{i,j} \leq 1 \) for each \( i \), i.e. \( \Pi \) is row substochastic.

The approach used by Glasserman and Young (2014) to model bankruptcy costs captures the fact that large shortfalls are more costly than small shortfalls. Concretely, when a node \( i \) defaults its assets are reduced by the amount

\[
\gamma \max \left\{ \ell_i - \left( \sum_{j=1}^n \pi_{j,i} p_{j}^* + c_i \right), 0 \right\}.
\]
Above, the term in curly brackets is the shortfall of node $i$ at default. Multiplying this quantity by the factor $\gamma$ gives the bankruptcy costs incurred by node $i$ at default. After accounting for these deadweight losses, the assets of node $i$ are distributed proportionally to its creditors. Hence, the clearing payment vector $p^* \in \mathbb{R}_{\geq 0}^n$ is a solution to the system

$$p^* = \left( [\ell \wedge (p^* \Pi + c)] - \gamma [\ell - (p^* \Pi + c)] \right)^+,$$

where for any two vectors $x, y \in \mathbb{R}^n$, $x \wedge y := (\min\{x_1, y_1\}, \min\{x_2, y_2\}, \ldots, \min\{x_n, y_n\})$, and $x^+ = (\max\{x_1, 0\}, \max\{x_2, 0\}, \ldots, \max\{x_n, 0\})$. Using the equality $(x - y)^+ = x - (x \wedge y)$, the above equation may be re-written as

$$p^* = \left( (1 + \gamma) [\ell \wedge (p^* \Pi + c)] - \gamma \ell \right)^+ = \ell \wedge [p^* (1 + \gamma) \Pi + (1 + \gamma) c - \gamma \ell]^+.$$

From the above expression, it is obvious that if $\gamma = 0$, then $p^*$ coincides with the clearing payment vector in the basic Eisenberg-Noe model.

We use the 4-tuple $(\Pi, \ell, c, \gamma)$ to identify the financial system. We make the following assumption, supported by empirical evidence provided in section 5.

**Assumption 1.** For any financial system $(\Pi, \ell, c, \gamma)$, $(1 + \gamma) c - \gamma \ell \geq 0$.

Glasserman and Young (2014) mention that the case to be expected in practice is $\gamma < 0.5$. When $\gamma = 0.5$, the above condition reduces to $c \geq \ell/3$. We conduct an empirical study in Section 5 showing that, in this case, the previous inequality always holds (see Table 2 for details). Then the fixed point equation yielding the clearing payment can be simplified to

$$p^* = \ell \wedge [p^* (1 + \gamma) \Pi + (1 + \gamma) c - \gamma \ell].$$

Glasserman and Young (2014) show that the clearing payment vector is uniquely determined if the spectral radius of the matrix $(1 + \gamma) \Pi$ is smaller than 1. This turns out to be the case in most financial networks. (For instance, Glasserman and Young (2014) use European Banking Authority’s 2011 stress test data and find that the fraction of liabilities within the interbank network is 0.43.)

In the sequel, we use $p^*(\Pi, \ell, c, \gamma)$ to emphasize the dependence of the clearing payment vector on the financial system $(\Pi, \ell, c, \gamma)$. We also distinguish between the asset value of a node before clearing and after clearing. The vector of asset values before clearing consists of the values of assets held by each node if all liabilities are repaid in full, and no bankruptcy costs are incurred. This is given by $\ell \Pi + c$. The vector of asset values after clearing has components given by the values of assets held by each node after interbank clearing occurred. This is given by $(p^* \Pi + c) - \gamma [\ell - (p^* \Pi + c)]^+.$

### 2.3 Objective of this Study

We aim at understanding how losses are affected by concentration of liabilities. The loss vector associated with the financial system $(\Pi, \ell, c, \gamma)$ is defined as the difference between the total liability and clearing payment vector, i.e.

$$s(\Pi, \ell, c, \gamma) := \ell - p^*(\Pi, \ell, c, \gamma).$$

The $i$-th component of the above vector denotes the amount of losses generated by node $i$. We illustrate in Figure 1 the behavior which we aim at capturing. The top graphs give interbank
networks where the node with the largest outstanding liabilities (node 4) has the smallest equity value. The bottom graphs give interbank networks where the node with the smallest outstanding amount of liabilities (node 1) has the smallest equity value. Both top and bottom panels have in common that the node with the smallest equity value generates the largest loss in the system. Then the network with the smallest net exposure to this node is always the most preferred in terms of losses. However, there is a distinguishing feature between the interbank network structures. In the top panels, the undesired system is the network whose liabilities are less concentrated. In the bottom panels, instead, the network with higher concentration of liabilities is the undesired one. Our objective is to capture this behavior quantitatively.

The next section defines balancing systems to capture the network behavior reported in the top panels of Figure 1, and unbalancing systems to capture the network behavior in the bottom panels.

3 Concentration of Liabilities

The objective of this section is to quantitatively analyze how concentration of liabilities affects the loss profile of a financial system. Preliminaries on majorization, along with related results that will be extensively used later, are summarized in §3.1. A relaxed equivalent version of the liability matrix is introduced in §3.2, and this proves to be the key to characterizing liability concentration via matrix majorization. The notions of balancing and unbalancing systems and results concerning loss preferences are presented in Section 3.3.

3.1 Preliminary Results

We start recalling the definition of similarly ordered vectors. Two vectors $x$ and $y$ are similarly ordered if $(x_i - x_j)(y_i - y_j) \geq 0$ for all $i, j$ (see also Arnold et al. (2011) Ch6 A.1.a).

Next, we define what it means for a matrix to be order preserving.

**Definition 3.** Let $D \in \mathbb{R}^{n \times n}$ and $A \subset \mathbb{R}^n$ be a subset of the space of $n$ dimensional real-valued vectors. $D$ is order preserving w.r.t $A$ if for $x \in A$, $xD$ and $x$ are similarly ordered.

The next lemma characterizes the set of matrices which are order preserving w.r.t. a set of positive similarly ordered vectors. We denote by $\nu(w)$ the mapping defined by $\nu_i(w) = j$ if $w_i = w_j$. Given a matrix $D = (d_{i,j})$ we use the abbreviated notation $d_{i,j}^{\nu(w)} := d_{\nu_i(w),\nu_j(w)}$.

**Lemma 1.** $D \in \mathbb{R}^{n \times n}$ is order preserving w.r.t. a set $A$ of positive vectors if and only if

$$\sum_{i=k}^{n} d_{i,j}^{\nu(x)} \leq \sum_{i=k}^{n} d_{i,j+1}^{\nu(x)}, k = 1, \ldots, n, j = 1, \ldots, n - 1, \text{ for any } x \in A.$$

Next, we define the class of matrices which preserves the weak majorization order.

**Definition 4.** Let $D \in \mathbb{R}^{n \times n}$ and $A \subset \mathbb{R}^n$ be a subset of the space of $n$ dimensional real-valued vectors.

- $D$ is weak submajorization preserving w.r.t $A$ if for $x, y \in A$,

  $$x \prec_w y \text{ implies } xD \prec_w yD.$$
(a) Networks in which the node with the largest outstanding liabilities (node 4) has the smallest equity value. Higher liability concentration generates smaller loss, hence it is preferred.

(b) Networks in which the node with the smallest outstanding liabilities (node 1) has the smallest equity value. Lower liability concentration generates smaller loss, hence is preferred.

Figure 1: Networks illustrating the objective of study. We set $\gamma = 0$. † The equity under the base-liability configuration is given by $\ell \Pi + c - \ell$. ‡ The equity under the reduced-liability configuration is given by $\ell \Pi + c - \ell$, where $\ell$ is later defined in Eq. (5).
• **D** is weak supermajorization preserving w.r.t. **A** if for \( x, y \in \mathcal{A} \),

\[
x <^w y \text{ implies } xD <^w yD.
\]

The set of order preserving matrices which are also weak submajorization or weak supermajorization preserving are characterized in the following lemma.

**Lemma 2.** Let \( \mathcal{A} \) be a set of positive similarly ordered vectors and \( D \in \mathbb{R}^{n \times n} \) an order preserving matrix w.r.t. \( \mathcal{A} \). The following statements hold:

- **D** is weak submajorization preserving w.r.t. \( \mathcal{A} \) if and only if for any \( x \in \mathcal{A} \),

\[
\sum_{j=k}^{n} d_{i,j}^\nu(x) \leq \sum_{j=k}^{n} d_{i+1,j}^\nu(x), \quad k = 1, \ldots, n, i = 1, \ldots, n-1.
\]

- **D** is weak supermajorization preserving w.r.t. \( \mathcal{A} \) if and only if for any \( x \in \mathcal{A} \),

\[
\sum_{j=1}^{k} d_{i,j}^\nu(x) \geq \sum_{j=1}^{k} d_{i+1,j}^\nu(x), \quad k = 1, \ldots, n, i = 1, \ldots, n-1.
\]

### 3.2 Liability Matrix: Relaxation and Concentration

The presence of zero elements on the diagonal of the matrix \( \Pi \) (i.e. \( \pi_{i,i} = 0 \)) restricts drastically the set of matrices \( \Pi \) that are order preserving. Indeed, multiplying a vector \( x \) by a relative liability matrix \( \Pi \), we can obtain a vector \( x\Pi \) that has the opposite ordering of \( x \). Such a situation can be avoided by replacing \( \Pi \) with a suitably chosen relaxed version defined as follows.

**Definition 5.** A 3-tuple \( (\Pi_{\alpha,\gamma}, \ell, c, \gamma) \), \( \alpha \in [0,1) \), is called the \( \alpha \)-relaxed equivalent version of a financial system \( (\Pi, \ell, c, \gamma) \), if \( \Pi_{\alpha,\gamma} = (1-\alpha) [(1+\gamma)\Pi] + \alpha \mathbf{I} \) and \( c_{\alpha,\gamma} = (1-\alpha) [(1+\gamma)c - \gamma\ell] \). Here \( \mathbf{I} \) is the identity matrix.

As an example, consider the case of no bankruptcy costs. Choosing \( \alpha = 0.5 \), we obtain that the vector \( x\Pi_{0.5,0} \) preserves the ordering of \( x \).

\[
\begin{pmatrix}
1 & 2 & 3 \\
0.4 & 0 & 0.4 \\
0.4 & 0.4 & 0
\end{pmatrix}
\begin{pmatrix}
0.0 & 0.4 & 0.4 \\
0.4 & 0 & 0.4 \\
0.4 & 0.4 & 0
\end{pmatrix} =
\begin{pmatrix}
2 & 1.6 & 1.2 \\
1 & 3 & 2 \\
0.2 & 0.5 & 0.2 \\
0.2 & 0.2 & 0.5
\end{pmatrix} =
\begin{pmatrix}
1.5 & 1.8 & 2.1 \\
0 & 0.5 & 0.2 \\
0.2 & 0.2 & 0.5
\end{pmatrix} =
\begin{pmatrix}
1.5 & 1.8 & 2.1 \\
0.5 & 0.2 & 0.2 \\
0.2 & 0.2 & 0.5
\end{pmatrix}
\]

The clearing payment is invariant with respect to this relaxation, as stated in the following lemma.

**Lemma 3.** Let \( (\Pi, \ell, c, \gamma) \) be a financial system. Then it holds that \( p^*(\Pi, \ell, c, \gamma) = p^*(\Pi_{\alpha,\gamma}, \ell, c, \gamma) \) for any \( \alpha \in [0,1) \).

Last, in order to compare two network topologies with different liability configurations, the notion of majorization for vectors is generalized to majorization for matrices.
Definition 6. Let $X$ and $Y$ be $m \times n$ matrices. $X$ is said to be majorized by $Y$, $X < Y$, if there exists a doubly stochastic matrix $S$ such that $X = YS$.

If $X < Y$, each row in $X$ is more evenly distributed than the corresponding row in $Y$. Indeed, when $m = 1$, it is well known that this definition is equivalent to Definition 1 (see Arnold et al. (2011) Ch.2 Theorem B.2). This leads us to use the following criteria to compare networks in terms of liability concentration.

Definition 7. Given two financial systems $(\Pi^a, \ell, c, \gamma)$ and $(\Pi^b, \ell, c, \gamma)$, we say that $b$ has a higher liability concentration than $a$ if there exists $\alpha \in [0,1)$ such that $\Pi^a_{a,\gamma} < \Pi^b_{a,\gamma}$.

The above definition is consistent with intuition. All systems whose relative liability matrices are permutations of each other have the same liability concentration. In all other cases, if $b$ has higher liability concentration than $a$, then $a$ cannot have higher concentration than $b$. The precise statement and proof of this fact is given in Lemma A1 in the appendix.

Using the above definition of liability concentration, we can verify that the networks in the left panels of Figure 1, here denoted by $a$, have lower liability concentrations than those in the right panels, here denoted by $b$. For the top panel, we have that $\Pi^a_{0.2,0} < \Pi^b_{0.2,0}$, given that

$$
\begin{pmatrix}
0.2 & 0.2 & 0.2 & 0.2 \\
0.2 & 0.2 & 0.2 & 0.2 \\
0.2 & 0.2 & 0.2 & 0.2 \\
0.2 & 0.2 & 0.2 & 0.2
\end{pmatrix}
\begin{pmatrix}
0.2 \\
0 \\
0 \\
0.2
\end{pmatrix} =
\begin{pmatrix}
0.25 & 0.25 & 0.25 & 0.25 \\
0.25 & 0.25 & 0.25 & 0.25 \\
0.25 & 0.25 & 0.25 & 0.25 \\
0.25 & 0.25 & 0.25 & 0.25
\end{pmatrix}.
$$

For the bottom panels, we have that $\Pi^a_{0.14,0} < \Pi^b_{0.14,0}$, i.e.

$$
\begin{pmatrix}
0.14 & 0.14 & 0.14 & 0.14 \\
0.14 & 0.14 & 0.14 & 0.14 \\
0.14 & 0.14 & 0.14 & 0.14 \\
0.14 & 0.14 & 0.14 & 0.14
\end{pmatrix}
\begin{pmatrix}
0.14 \\
0 \\
0 \\
0.14
\end{pmatrix} =
\begin{pmatrix}
0.25 & 0.25 & 0.25 & 0.25 \\
0.25 & 0.25 & 0.25 & 0.25 \\
0.25 & 0.25 & 0.25 & 0.25 \\
0.25 & 0.25 & 0.25 & 0.25
\end{pmatrix}.
$$

Remark 1. Amini et al. (2010) use the proportion of edges which are contagious to assess the resilience of a financial network to shocks. The weight of the edge directed from $i$ to $j$ represents the exposure of $i$ to $j$. The edge from $i$ to $j$ is contagious if node $j$’s default will cause $i$ to default. Fix a level of total exposure within the network (given by the sum of bilateral exposures). In their model, losses are proportional to the size of exposures. Hence, a larger fraction of contagious edges implies a higher concentration of exposures within the network, and in turn a larger default cascade. Conversely, a smaller fraction of contagious edges implies a lower concentration of exposures and a smaller default cascade. Asymptotically, their analysis shows that a financial network with fewer contagious links is more resilient to shocks. Using the above mentioned implications, their results also indicate that a financial network with lower concentration of exposures is more resilient to shocks. Their results are consistent with ours for the case of unbalancing systems, where higher systemic losses arise in the network with higher concentration of liabilities (see Theorem 2 for details).
3.3 Loss preferences in Balancing and Unbalancing Systems

This section defines the class of balancing and unbalancing financial systems and provides quantitative statements relating liability concentration to loss profiles. We make the following empirically supported assumption (see Section 5, Table 2, for details).

**Assumption 2.** For any financial system \((\Pi, \ell, c, \gamma)\), \(\ell\) and \([(1 + \gamma)c - \gamma \ell]\) are similarly ordered.

This assumption directly implies that \(\ell\) and \(c\) are similarly ordered, i.e., that nodes with larger liabilities also have larger outside asset values.

The definition of balancing and unbalancing systems depends on the nodes’ equity under the base and reduced-liability configurations. In the balancing system, the equity of node \(i\) is

\[
\left[\ell \Pi + c\right]_i - \ell_i. \\
\text{assets under base-liability configuration}
\]

The above can be viewed as the pre-clearing equity at node \(i\) (or equivalently, its post-clearing equity assuming all nodes pay their full liabilities). In the unbalancing system, the equity is computed by reducing each node’s liability to a level as follows:

\[
\ell := \ell - \delta \mathbf{1}, \quad \text{where} \quad \delta := \max \left\{ \ell(1) - \left[ (1 + \gamma)c - \gamma \ell \right](1), 0 \right\}, \\
\text{and } \mathbf{1} \text{ denotes a row vector with all entries equal to one. Thus, the equity of node } i \text{ is given by}
\]

\[
\left[\ell \Pi + c\right]_i - \ell_i. \\
\text{assets under reduced-liability configuration}
\]

We show in the following lemma 4 that \(\ell\) coincides with the maximum liability vector under which all nodes can repay their liabilities in full.

Given a financial system \((\Pi, \ell, c, \gamma)\), we set \(\mu := \nu(\ell)\), so that \(\mu_i = j\) if \(\ell(i) = \ell_j\). We define balancing and unbalancing financial systems and illustrate their structural properties in Figure 2.

**Definition 8.** Let \((\Pi, \ell, c, \gamma)\) be a financial system.

(I) It is balancing if, for \(j = 1, \ldots, n - 1,\)

\[
\sum_{i=1}^{n} \ell(i) \pi_{i,j+1}^\mu + c_{i+1} - \ell(j+1) \leq \sum_{i=1}^{n} \ell(i) \pi_{i,j}^\mu + c_{i} - \ell(j) \\
\text{equity of node } (j+1) \leq \text{equity of node } (j) \text{ under the base-liability configuration}
\]

(II) It is unbalancing if, for \(j = 1, \ldots, n - 1,\)

\[
\sum_{i=1}^{n} \ell(i) \pi_{i,j+1}^\mu + c_{i+1} - \ell(j+1) \geq \sum_{i=1}^{n} \ell(i) \pi_{i,j}^\mu + c_{i} - \ell(j) \\
\text{equity of node } (j+1) \geq \text{equity of node } (j) \text{ under the reduced-liability configuration}
\]

where we recall that \(\ell\) is defined in Eq. (5).
Figure 2: Balancing: if a node has larger liabilities than another, then its equity is smaller under the base-liability configuration. Unbalancing: if a node has larger liabilities than another, then its equity is larger under the reduced-liability configuration.

Intuitively, in a balancing system, a node with a larger liability ($\ell_{(j+1)}$) has a smaller equity before clearing, as represented by the inequality in (6); in an unbalancing system, if a node has a larger liability then it also has a larger equity before clearing under the reduced-liability configuration, per the inequality in (7). This captures precisely the behavior of the networks illustrated in Figure 1, where the node with the largest liability has the smallest equity value in the top two graphs (balancing system), while the same node has the largest equity value in the bottom two graphs (unbalancing system).

The next lemma shows that under the reduced-liability configuration, all nodes repay their liabilities in full; and the reduction as specified in (5) is minimal — short of which the absence of default is no longer guaranteed.

**Lemma 4.** For any unbalancing system $(\Pi, \ell, c, \gamma)$, it must hold that $p^*(\Pi, \ell, c, \gamma) = \ell$. For any vector $\epsilon$ with strictly positive entries, there exists at least one unbalancing system $(\Pi, \ell, c, \gamma)$ such that $p^*(\Pi, \ell + \epsilon, c, \gamma) \neq \ell + \epsilon$.

To proceed further, we need to first specify the set of payment vectors used to analyze the loss generated in balancing and unbalancing systems:

$$\mathcal{P} = \{p | p \text{ is similarly ordered to } \ell, 0 \leq p \leq \ell\}.$$ 

Note that in characterizing $\mathcal{P}$, the condition of $p$ being similarly ordered to $\ell$ is the more substantive one. It requires that the nodes making larger payments are also those with larger outstanding liabilities; and Proposition 1 gives a sufficient condition for this to hold. The other condition $0 \leq p \leq \ell$ says that a node does not pay more than its nominal liabilities, but allows for a large class of payment vectors including those for which absolute priority or limited liability is violated.

The following proposition characterizes the relations between clearing payments, liabilities and loss vectors, both in balancing and unbalancing financial systems.

**Proposition 1.** Let $(\Pi, \ell, c, \gamma)$ be a financial system. Suppose there exists $\alpha \in [0, 1)$ such that $\Pi_{\alpha, \gamma}$ is order preserving w.r.t. to $\mathcal{P}$. Then,

(1) $p^*$ is similarly ordered to $\ell$. 

If \((\Pi, \ell, c, \gamma)\) is balancing, then
\[\ell_1 - p_1^* \geq \ell_2 - p_2^* \geq \cdots \geq \ell_n - p_n^*.\]

If \((\Pi, \ell, c, \gamma)\) is unbalancing, then
\[\ell_1 - p_1^* \geq \ell_2 - p_2^* \geq \cdots \geq \ell_n - p_n^*.\]

The above proposition indicates that when there exists an \(\alpha\)-relaxed equivalent version that preserves the order of payments, the nodes making larger payments are also those with larger liabilities. Moreover, larger shortfalls occur at nodes with larger liabilities in balancing systems, and at those with smaller liabilities in unbalancing systems.

The next theorem compares clearing payment vectors and loss profiles in balancing systems based on liability concentration. It concludes that the system with a higher concentration is preferred as it results in a smaller loss.

**Theorem 1.** Let \((\Pi^a, \ell, c, \gamma)\) and \((\Pi^b, \ell, c, \gamma)\) be two balancing financial systems. Suppose there exists \(\alpha \in [0, 1]\) such that both \(\Pi^a_{\alpha, \gamma}\) and \(\Pi^b_{\alpha, \gamma}\) are order preserving w.r.t. \(\mathcal{P}\) and

1. \(\Pi^a_{\alpha, \gamma} \) or \(\Pi^b_{\alpha, \gamma}\) is weak submajorization preserving w.r.t. \(\mathcal{P}\),

2. \(\Pi^a_{\alpha, \gamma}\) and \(\Pi^b_{\alpha, \gamma}\) are order preserving w.r.t. \(\mathcal{P}\), and \(\Pi^a_{\alpha, \gamma}\) weak submajorization preserving w.r.t. \(\mathcal{P}\).

Then
\[p^a(\Pi^a, \ell, c, \gamma) \prec_w p^b(\Pi^b, \ell, c, \gamma) \text{ and } s(\Pi^a, \ell, c, \gamma) \succ_w s(\Pi^b, \ell, c, \gamma).\]

Here is the intuition behind the above results. First, from the earlier Proposition 1, we know that in the balancing case larger shortfalls occur at nodes with larger liabilities; and this holds for both systems \(a\) and \(b\). Moreover, in one of the systems, those nodes with larger liabilities will make larger payments due to the weak submajorization preserving condition in (I) (this can be seen from Lemma 2 setting \(D = \Pi^a_{\alpha, \gamma}\)), and they will receive smaller payments in system \(a\) than in system \(b\), as the payments in system \(a\) are more evenly distributed due to the majorization order in (II). Consequently, nodes with large liabilities in system \(a\) will make less payments and have larger shortfalls relative to those in system \(b\).

Next, we give the corresponding result for unbalancing systems. In this case, the system with a lower concentration is preferred.

**Theorem 2.** Let \((\Pi^a, \ell, c, \gamma)\) and \((\Pi^b, \ell, c, \gamma)\) be two unbalancing financial systems. Suppose there exists \(\alpha \in [0, 1]\) such that both \(\Pi^a_{\alpha, \gamma}\) and \(\Pi^b_{\alpha, \gamma}\) are order preserving w.r.t. \(\mathcal{P}\) and

1. \(\Pi^a_{\alpha, \gamma} \) or \(\Pi^b_{\alpha, \gamma}\) is weak supermajorization preserving w.r.t. \(\mathcal{P}\),

2. \(\Pi^a_{\alpha, \gamma}\) and \(\Pi^b_{\alpha, \gamma}\) are order preserving w.r.t. \(\mathcal{P}\), and \(\Pi^a_{\alpha, \gamma}\) weak supermajorization preserving w.r.t. \(\mathcal{P}\).

Then
\[p^a(\Pi^a, \ell, c, \gamma) \prec_w p^b(\Pi^b, \ell, c, \gamma) \text{ and } s(\Pi^a, \ell, c, \gamma) \succ_w s(\Pi^b, \ell, c, \gamma).\]

Here the intuition parallels the balancing case. In the unbalancing case larger shortfalls occur at nodes with smaller liabilities, per Proposition 1; and the losses are exacerbated by the fact that those nodes tend to make larger payments, due to the weak supermajorization preserving condition in (I) (this can be seen from Lemma 2 setting \(D = \Pi^b_{\alpha, \gamma}\)). Moreover, those same nodes in system \(b\) will receive smaller payments than in system \(a\), due to the majorization order in (II). Consequently, nodes with smaller liabilities in system \(b\) can only make less payments and have larger shortfalls relative to those in system \(a\).
Figure 3: Two financial systems consisting of six nodes whose liabilities are $\ell_6 > \ell_5 > \cdots > \ell_1$. The graphs illustrate the relation between asset values of defaulted nodes after clearing (when a node $i$ defaults, it must hold that $p^*_i = [p^*+c-\gamma(e-\Pi+c)]^*$ and liability concentration. If both systems are balancing, losses occur at the nodes with larger liabilities and larger losses are generated in system $a$ where liabilities are less concentrated. The right panel shows that if both systems are unbalancing, losses occur at the nodes with smaller liabilities and larger losses occur in system $b$, where liabilities are more concentrated.

Figure 3 illustrates how liability concentration affects the asset values of nodes after clearing, and consequently the loss profile in both types of systems.

We conclude this section with a result indicating that systemic losses would be reduced if the financial system is both balancing and unbalancing. As the next lemma shows, if at least one node does not default and order preserving relations of the relative liability matrix are maintained, there would be zero loss in such a financial system. From a regulatory perspective, this suggests that it is beneficial to drive the network towards such a state.

**Proposition 2.** Let $(\Pi, \ell, c, \gamma)$ be a financial system such that $\sum_{i=1}^n \pi_{i,j+1}^\mu - \sum_{i=1}^n \pi_{i,j}^\mu \geq 0$ for $j = 1, \ldots, n-1$, and at least one node repays its liabilities in full. If $(\Pi, \ell, c, \gamma)$ is both balancing and unbalancing, then $s(\Pi, \ell, c, \gamma) = 0$.

We next provide an example showing the existence of a system satisfying the conditions in the above proposition.

**Example 1.** Consider a balancing (respectively unbalancing) system $(\Pi, \ell, c, \gamma)$ where the column sums of $\Pi$ are identical and the inequalities in (6) (respectively (7)) become equalities. Then the system is both balancing and unbalancing. One instance of such a system is given by

$$
\Pi = \begin{pmatrix}
0 & 0.27 & 0.27 & 0.24 \\
0.26 & 0 & 0.25 & 0.28 \\
0.26 & 0.25 & 0 & 0.28 \\
0.28 & 0.28 & 0.28 & 0
\end{pmatrix}, \quad \ell = (4 \ 6 \ 8 \ 50), \quad c = (1 \ 3.6 \ 6.1 \ 58.8), \quad \gamma = 0.
$$

Clearly, the difference between two consecutive rank ordered components in $\ell \Pi + c = (18.6 \ 20.6 \ 22.6 \ 64.6)$ and $\ell$ are the same. The same applies to $\ell \Pi + c = (16.2 \ 18.2 \ 20.2 \ 62.2)$ and $\ell = (1 \ 3 \ 5 \ 47)$.
4 Application to Core-Periphery Network

This section develops numerical examples of financial networks which mimic the structure of tiered systems. Such systems have been identified by empirical research as good descriptors of interbank activity, see the study of Craig and Von Peter (2014) on the German banking system from 1999 to 2007, and of Fricke and Lux (2013) on the overnight interbank transactions in the Italian market from 1999 to 2010. Our objective is to apply the results developed in the previous sections to analyze which tiered structure is preferred depending on whether the system is balancing or unbalancing.

In a tiered financial system the pattern of interbank liabilities follows a core-periphery structure. The network is centered around a set of core nodes which intermediate between numerous smaller nodes in the periphery. In particular, such a network has following properties:

(I) The size of the core node is significantly larger than that of peripheral nodes. Craig and Von Peter (2014) find that the average size of core banks is 51 times that of peripheral banks.

(II) The peripheral nodes do not borrow from or lend to other peripheral nodes.

A financial system satisfying (I) and (II) is called perfectly tiered, while a financial system which only satisfies (I) is called imperfectly tiered. For illustration purposes, we set the number of nodes \( n = 4 \), with nodes 1, 2, 3 being peripheral and node 4 core. This means that the relative liability matrix \( \Pi^a \) of an imperfectly tiered financial system is given by

\[
\Pi^a = \begin{pmatrix}
0 & \pi_{1,2}^a & \pi_{1,3}^a & \pi_{1,4}^a \\
\pi_{2,1}^a & 0 & \pi_{2,3}^a & \pi_{2,4}^a \\
\pi_{3,1}^a & \pi_{3,2}^a & 0 & \pi_{3,4}^a \\
\pi_{4,1}^a & \pi_{4,2}^a & \pi_{4,3}^a & 0
\end{pmatrix},
\]

where \( \pi_{i,j}^a \geq 0 \) for \( i, j = 1, \ldots, 3 \), and at least one entry \( \pi_{i,j}^a > 0 \). The relative liability matrix \( \Pi^b \) of a perfectly tiered financial system, instead, has the form given by

\[
\Pi^b = \begin{pmatrix}
0 & 0 & 0 & \pi_{1,4}^b \\
0 & 0 & 0 & \pi_{2,4}^b \\
0 & 0 & 0 & \pi_{3,4}^b \\
\pi_{4,1}^b & \pi_{4,2}^b & \pi_{4,3}^b & 0
\end{pmatrix},
\]

where \( \pi_{i,4}^b \geq 0 \) for \( i = 1, \ldots, 3 \), and \( \pi_{4,j}^b \geq 0 \) for \( j = 1, \ldots, 3 \).

We next develop numerical examples of networks that are balancing or unbalancing; and for both cases, we analyze whether the tiered or imperfectly tiered structure is preferred.

4.1 Balancing Networks

Consider the three financial systems, \( (\Pi^a, \ell, c, \gamma) \), \( (\tilde{\Pi}^a, \ell, c, \gamma) \), \( (\Pi^b, \ell, c, \gamma) \) given by

\[
\Pi^a = \begin{pmatrix}
0 & 0.25 & 0.25 & 0.24 \\
0.23 & 0 & 0.23 & 0.28 \\
0.23 & 0.23 & 0 & 0.28 \\
0.28 & 0.28 & 0.28 & 0
\end{pmatrix}, \quad \tilde{\Pi}^a = \begin{pmatrix}
0 & 0.24 & 0.25 & 0.25 \\
0.18 & 0 & 0.28 & 0.28 \\
0.18 & 0.28 & 0 & 0.28 \\
0.28 & 0.28 & 0.28 & 0
\end{pmatrix}, \quad \Pi^b = \begin{pmatrix}
0 & 0.25 & 0.25 & 0.25 \\
0.23 & 0 & 0.23 & 0.28 \\
0.23 & 0.23 & 0 & 0.28 \\
0.28 & 0.28 & 0.28 & 0
\end{pmatrix},
\]
\[ \Pi^b = \begin{pmatrix} 0 & 0 & 0 & 0.74 \\ 0 & 0 & 0 & 0.74 \\ 0 & 0 & 0 & 0.74 \\ 0.28 & 0.28 & 0.28 & 0 \end{pmatrix}, \]

\[ \ell = (4g, 6g, 8g, 50g), \ c = (3g, 4g, 5g, 25g), \ and \ \gamma = 1/5. \ Here, \ g \ is \ a \ positive \ constant. \ Choosing \ \alpha = 1/4, \ we \ obtain\]

\[ \Pi^{a}_{1/4,1/5} = \begin{pmatrix} 0.25 & 0.23 & 0.23 & 0.21 \\ 0.21 & 0.25 & 0.21 & 0.25 \\ 0.21 & 0.21 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{pmatrix}, \quad \tilde{\Pi}^{a}_{1/4,1/5} = \begin{pmatrix} 0.25 & 0.21 & 0.23 & 0.23 \\ 0.17 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{pmatrix}, \]

\[ \Pi^{b}_{1/4,1/5} = \begin{pmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{pmatrix}, \]

where the relaxed versions \( \Pi^{a}_{1/4,1/5} \) and \( \tilde{\Pi}^{a}_{1/4,1/5} \) are both order and weak submajorization preserving w.r.t. \( \mathcal{P} \). This is because both systems satisfy the assumptions in lemmas 1 and 2. Moreover, \( \Pi^{b}_{1/4,1/5} \) is order preserving w.r.t. \( \mathcal{P} \) given that it satisfies the conditions of Lemma 1. Moreover, we can find doubly stochastic matrices

\[ S = \begin{pmatrix} 0.35 & 0.27 & 0.27 & 0.11 \\ 0.2 & 0.31 & 0.17 & 0.32 \\ 0.2 & 0.17 & 0.31 & 0.32 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{pmatrix}, \quad \tilde{S} = \begin{pmatrix} 0.47 & 0.11 & 0.21 & 0.21 \\ 0.17 & 0.31 & 0.26 & 0.26 \\ 0.17 & 0.31 & 0.26 & 0.26 \\ 0.19 & 0.27 & 0.27 & 0.27 \end{pmatrix}, \]

such that \( \Pi^{b}_{1/4,1/5}S = \Pi^{a}_{1/4,1/5} \) and \( \Pi^{b}_{1/4,1/5}\tilde{S} = \tilde{\Pi}^{a}_{1/4,1/5} \).

We next show that \((\Pi^a, \ell, c, \gamma)\), \((\Pi^\alpha, \ell, c, \gamma)\) and \((\Pi^b, \ell, c, \gamma)\) are balancing. For these systems, the equity vectors under the base-liability configuration are given by

\[ \ell\Pi^a + c - \ell = (16g, 15g, 13g, -20g) \]
\[ \ell\Pi^\alpha + c - \ell = (16g, 15g, 14g, -20g) \]
\[ \ell\Pi^b + c - \ell = (13g, 12g, 11g, -12g) \]

These vectors are all reversely ordered to the liability vector \( \ell \), which implies that the three systems are balancing from Definition 8.

By taking convex combinations of the financial systems \((\Pi^a, \ell, c, \gamma)\) and \((\Pi^\alpha, \ell, c, \gamma)\), we can generate a large class of balancing systems. Concretely, let \((\Pi^{\beta}, \ell, c, \gamma) = \lambda(\Pi^a, \ell, c, \gamma) + (1 - \lambda)(\Pi^\alpha, \ell, c, \gamma)\) for some \( \lambda \in [0, 1] \). Then it is clear that \((\Pi^{\beta}, \ell, c, \gamma)\) is balancing because the class of balancing systems is closed under convex combination. Moreover, \( \Pi^{\beta}_{1/4,1/5} \) is order and weak submajorization preserving w.r.t. \( \mathcal{P} \) since both of these relations are preserved for convex combinations. Additionally \( \Pi^{\beta}_{1/4,1/5} \) is majorized by \( \Pi^{b}_{1/4,1/5} \) given that \( \Pi^{\alpha}_{1/4,1/5} = \lambda\Pi^{b}_{1/4,1/5}S + (1 - \lambda)\Pi^{b}_{1/4,1/5}\tilde{S}, \) and \( \lambda S + (1 - \lambda)\tilde{S} \) is doubly stochastic.

Since both \((\Pi^a, \ell, c, \gamma)\) and \((\Pi^b, \ell, c, \gamma)\) satisfy the assumptions of Theorem 1, we deduce that \( s(\Pi^a, \ell, c, \gamma) >_w s(\Pi^b, \ell, c, \gamma) \). Therefore, the perfectly tiered network is preferred to the imperfectly tiered network if both are balancing.
This can be understood as follows. When a network is balancing, nodes with larger liabilities will incur larger losses, i.e. the core nodes, see also left panel of Figure 3. Moreover, from Theorem 1 we obtain that the clearing payments in the imperfectly tiered network are more evenly distributed than the corresponding payments in the perfectly tiered network. This is because in the imperfectly tiered structure, larger payments are made to periphery as opposed to core nodes, while in the perfectly tiered structure all payments from periphery nodes are directed to core nodes. Hence, core nodes are more likely to default and generate larger losses in the imperfectly tiered network.

4.2 Unbalancing Networks

Let the three financial systems, \((\Pi^{a}, \ell, c, \gamma)\), \((\Pi^{a}, \ell, c, \gamma)\), \((\Pi^{b}, \ell, c, \gamma)\) be specified by

\[
\Pi^{a} = \begin{pmatrix}
0 & 0.32 & 0.35 & 0.23 \\
0.27 & 0 & 0.33 & 0.3 \\
0.27 & 0.25 & 0 & 0.38 \\
0.25 & 0.27 & 0.27 & 0
\end{pmatrix}, \quad \tilde{\Pi}^{a} = \begin{pmatrix}
0 & 0.35 & 0.33 & 0.22 \\
0.25 & 0 & 0.32 & 0.33 \\
0.26 & 0.3 & 0 & 0.34 \\
0.25 & 0.27 & 0.27 & 0
\end{pmatrix},
\]

\[
\Pi^{b} = \begin{pmatrix}
0 & 0 & 0 & 0.9 \\
0 & 0 & 0 & 0.9 \\
0 & 0 & 0 & 0.9 \\
0.19 & 0.3 & 0.3 & 0
\end{pmatrix},
\]

\[\ell = (24g, 26g, 28g, 60g)\] and \(c = (3g, 8g, 13g, 56g)\), and \(\gamma = 1/10\). Choosing \(\alpha = 1/4\), we obtain

\[
\Pi^{a}_{1/4,1/10} = \begin{pmatrix}
0.25 & 0.26 & 0.29 & 0.19 \\
0.22 & 0.25 & 0.27 & 0.25 \\
0.22 & 0.21 & 0.25 & 0.31 \\
0.21 & 0.22 & 0.22 & 0.25
\end{pmatrix}, \quad \tilde{\Pi}^{a}_{1/4,1/10} = \begin{pmatrix}
0.25 & 0.29 & 0.27 & 0.18 \\
0.21 & 0.25 & 0.26 & 0.27 \\
0.21 & 0.25 & 0.25 & 0.28 \\
0.21 & 0.22 & 0.22 & 0.25
\end{pmatrix},
\]

\[
\Pi^{b}_{1/4,1/10} = \begin{pmatrix}
0.25 & 0 & 0 & 0.74 \\
0 & 0 & 0.25 & 0.74 \\
0 & 0 & 0 & 0.74 \\
0.15 & 0.25 & 0.25 & 0.25
\end{pmatrix}.
\]

The relaxed versions \(\Pi^{a}_{1/4,1/10}\) and \(\Pi^{b}_{1/4,1/10}\) are both order and weak supermajorization preserving w.r.t. \(P\) because they satisfy the assumptions of lemmas 1 and 2. Moreover, \(\Pi^{b}_{1/4,1/10}\) is order preserving w.r.t. \(P\) because it satisfies the conditions in Lemma 1. The doubly stochastic matrices

\[
S = \begin{pmatrix}
0.34 & 0.33 & 0.33 & 0 \\
0.22 & 0.29 & 0.23 & 0.26 \\
0.22 & 0.15 & 0.15 & 0.48 \\
0.22 & 0.23 & 0.29 & 0.26
\end{pmatrix}, \quad \tilde{S} = \begin{pmatrix}
0.37 & 0.33 & 0.30 & 0 \\
0.21 & 0.2 & 0.23 & 0.36 \\
0.21 & 0.2 & 0.20 & 0.39 \\
0.21 & 0.27 & 0.27 & 0.25
\end{pmatrix},
\]

are such that \(\Pi^{b}_{1/4,1/10}S = \Pi^{a}_{1/4,1/10}\) and \(\Pi^{b}_{1/4,1/10}\tilde{S} = \tilde{\Pi}^{a}_{1/4,1/10}\).

We next verify that \((\Pi^{a}, \ell, c, \gamma), \ (\Pi^{a}, \ell, c, \gamma)\) and \((\Pi^{b}, \ell, c, \gamma)\) are unbalancing. The equity vectors under the reduced-liability configuration are given by

\[
\ell^{a} = (14g, 16g, 19g, 22g) \quad \ell^{a} = (13g, 17g, 19g, 22g)
\]
\[ \ell + c - \ell = (9g, 16g, 19g, 27g), \]

and are all similarly ordered to the liability vector \( \ell \). This implies that all three systems are unbalancing.

Consider a convex combination of the two imperfectly tiered systems, given by \( (\hat{\Pi}^a, \ell, c, \gamma) = \lambda(\Pi^a, \ell, c, \gamma) + (1 - \lambda)(\tilde{\Pi}^a, \ell, c, \gamma), \) where \( \lambda \in [0, 1] \). Since the class of unbalancing systems is closed under convex combinations, we have that \( (\hat{\Pi}^a, \ell, c, \gamma) \) is also unbalancing, while \( \Pi^a_{1/4,1/10} \) is order and weak supermajorization preserving w.r.t. \( \mathcal{P} \) and majorized by \( \Pi^b_{1/4,1/10} \). The majorization relation follows from the fact that \( \hat{\Pi}^a_{1/4,1/10} = \lambda\Pi^b_{1/4,1/10} + (1 - \lambda)\Pi^b_{1/4,1/10} \).

Since both \( (\hat{\Pi}^a, \ell, c, \gamma) \) and \( (\Pi^b, \ell, c, \gamma) \) satisfy the assumptions of Theorem 2, we deduce that \( s(\hat{\Pi}^a, \ell, c, \gamma) \preceq_w s(\Pi^b, \ell, c, \gamma) \). Hence, if both networks are unbalancing, the imperfectly tiered is preferred to the perfectly tiered network. This can be explained in intuitive terms as follows. When a network is unbalancing, larger losses are incurred by nodes with smaller liabilities, i.e. peripheral nodes, see also right panel of Figure 3. From Theorem 2, we obtain that the clearing payment in the perfectly tiered network is less evenly distributed than the corresponding payments in the imperfectly tiered network. This is because in the latter, larger payments are directed to periphery as opposed to core nodes, while in the perfectly tiered network peripheral nodes only receive payments from the core node. Hence, they are more likely to default and generate larger losses.

5 Empirical Analysis and Policy Implications

The objective of this section is to provide empirical evidence to (1) support the two assumptions made earlier in the paper and (2) show that real-world financial networks often tend to be in an unbalancing state. We consider the system consisting of banking sectors in eight European countries for seven years, starting from 2008 and ending in 2014. These countries are well representative of interbank activities in the European market as their liabilities account for 80% of the total liabilities of the European banking sector.

We use consolidated banking data released from the European Central Bank, reported in Table 2, and foreign claims data from the BIS (Bank for International Settlements) Quarterly Review, summarized in Table 1, to estimate the various parameters of the financial system.

Tables 1 and 2 show that both assumptions are satisfied in December 2009 and June 2010. Assumption 1 holds because the values in fourth column of Table 2 are all positive. Moreover, such column is similarly ordered to the second column, hence showing that Assumption 2 also holds. Table 2 indicates that the financial system is in the unbalancing state at these two time points. This is because the column corresponding to the equity under the reduced-liability configuration is similarly ordered to the total liability vector in the second column.

We next analyze if the unbalancing state of the system is persistent over time. To this purpose, we recall that a financial system is unbalancing if the inequalities given in Eq. (7) are satisfied for \( i = 1, \ldots, n - 1 \). Table 3 computes the degree of unbalance of the system, estimated as the ratio between the number of satisfied inequalities over the total number of inequalities. Large values of this ratio are indicative of a financial system which is close to an unbalancing state. The results in Table 3 confirm the findings from Tables 1 and 2 that the system is unbalancing in December 2009 and June 2010, and additionally indicate it is close to being unbalancing at the remaining points in time.
Our results indicate that an imperfectly tiered is preferred to a perfectly tiered structure if the systems, identified by recent empirical studies as good descriptors of real-world financial networks, would result in reduced losses. We show how our framework can be specialized to capture tiered it is desirable for the network to be in a state which is both balancing and unbalancing as this unbalancing financial networks, while the opposite is true in balancing systems. For regulatory purposes, this brings out the qualitatively different implications of liability concentration. Policies of this type are already in place, see for instance the supervisory framework put forward by the Basel Committee (BCBS (2014)) which aims at limiting the size of gross exposures to individual counterparties.

Since higher concentration of liabilities induce larger systemic losses in unbalancing systems, the theoretical findings of our study indicate that it is desirable for regulatory purposes to reduce gross exposures to individual counterparties and drive the network towards a state of smaller concentration. Policies of this type are already in place, see for instance the supervisory framework put forward by the Basel Committee (BCBS (2014)) which aims at limiting the size of gross exposures to individual counterparties.

6 Concluding Remarks

This study is what we believe to be the first to provide an analytical tool to compare financial networks. Our main focus is on the implications of liability concentration on the loss profile of the system. Specifically, we use vector majorization to express preferences between losses and quantify liability concentration by applying the majorization order to the relatively liability matrix, capturing the interconnectedness of banks in the network. We then develop notions of balancing and unbalancing networks to bring out the qualitatively different implications of liability concentration on the system’s loss profile.

Our main result is that higher liability concentration leads to larger systemic losses in unbalancing financial networks, while the opposite is true in balancing systems. For regulatory purposes, it is desirable for the network to be in a state which is both balancing and unbalancing as this would result in reduced losses. We show how our framework can be specialized to capture tiered systems, identified by recent empirical studies as good descriptors of real-world financial networks. Our results indicate that an imperfectly tiered is preferred to a perfectly tiered structure if the

<table>
<thead>
<tr>
<th>December 2009</th>
<th>United Kindom</th>
<th>Germany</th>
<th>France</th>
<th>Spain</th>
<th>Netherland</th>
<th>Ireland</th>
<th>Belgium</th>
<th>Portugal</th>
</tr>
</thead>
<tbody>
<tr>
<td>United Kindom</td>
<td>0.00</td>
<td>500.62</td>
<td>341.62</td>
<td>409.36</td>
<td>189.95</td>
<td>231.97</td>
<td>36.22</td>
<td>10.43</td>
</tr>
<tr>
<td>Germany</td>
<td>172.97</td>
<td>0.00</td>
<td>292.94</td>
<td>51.02</td>
<td>176.58</td>
<td>20.35</td>
<td>30.23</td>
<td>26.56</td>
</tr>
<tr>
<td>France</td>
<td>239.17</td>
<td>195.64</td>
<td>0.00</td>
<td>50.42</td>
<td>92.73</td>
<td>20.60</td>
<td>32.57</td>
<td>8.08</td>
</tr>
<tr>
<td>Spain</td>
<td>114.14</td>
<td>237.98</td>
<td>219.64</td>
<td>0.00</td>
<td>119.73</td>
<td>30.23</td>
<td>26.56</td>
<td>28.08</td>
</tr>
<tr>
<td>Netherland</td>
<td>96.69</td>
<td>155.65</td>
<td>150.57</td>
<td>22.82</td>
<td>0.00</td>
<td>15.47</td>
<td>28.11</td>
<td>11.39</td>
</tr>
<tr>
<td>Ireland</td>
<td>187.51</td>
<td>183.76</td>
<td>60.33</td>
<td>15.66</td>
<td>30.82</td>
<td>0.00</td>
<td>64.50</td>
<td>21.52</td>
</tr>
<tr>
<td>Belgium</td>
<td>30.72</td>
<td>40.68</td>
<td>301.37</td>
<td>9.12</td>
<td>131.55</td>
<td>6.11</td>
<td>0.00</td>
<td>1.17</td>
</tr>
<tr>
<td>Portugal</td>
<td>24.26</td>
<td>47.38</td>
<td>44.74</td>
<td>86.08</td>
<td>12.41</td>
<td>5.43</td>
<td>3.14</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Table 1: Banks’ consolidated foreign claims (in USD billion). Data source: BIS Quarterly Review Table 9B. The $ij$-th entry of each matrix denotes the interbank liabilities from the banking sector of country $i$ to the banking sector of country $j$. 
Table 2: Consolidated banking sector data of European countries. Data source: European Central Bank. The first two columns report the values of assets and liabilities of each banking sector. In our model, these correspond respectively to $\ell \Pi + c$ and $\ell$. We set $\gamma = 0.5$. The values reported in the third and fourth columns are computed as follows. We first estimate the relative liability matrix, $\Pi$, whose entries are obtained from the interbank liability matrix in Table 1 and the vector $\ell$ of total liabilities. We then compute $c$ by subtracting $\ell \Pi$ from the asset values vector given in the first column. The equity under the reduced-liability configuration is estimated using Eq. (7).

Table 3: The degree of unbalance of the financial system consisting of the eight representative banking sectors at ten different time points. In each time point, we sort the eight banking sectors by increasing liabilities size. We then check if the inequalities in Eq. (7) are satisfied for each pair of adjacent banking sectors. This yields a total of seven inequalities. We compute the degree of unbalance as the number of satisfied inequalities divided by the total number of inequalities.
state of the financial network is unbalancing, whereas the opposite preference relation holds if the state is balancing.

We conduct an empirical analysis of the network formed by the eight largest European banking sectors and find that the state of the financial network is either unbalancing or close to it, consistently over a period of seven years. Such an analysis, along with the theoretical predictions of our study, indicate that it is advisable to avoid concentration of gross exposures as they would have serious systemic effects in unbalancing systems.

Acknowledgments

We gratefully acknowledge J. George Shanthikumar for discussions on the concept of majorization.

Appendix

We first provide the proofs for the lemmas which characterize order preserving, weak submajorization preserving, and weak supermajorization preserving matrices.

Proof of Lemma 1. We first prove the sufficiency. Let \( x \in \mathcal{A} \). For notational convenience, we omit the dependence of \( \nu(x) \) on \( x \). For \( j = 1, \ldots, n-1 \),

\[
[xD]_{\nu_{j+1}} - [xD]_{\nu_j} = \sum_{i=1}^{n} x(i) (d_{i,j+1}^\nu - d_{i,j}^\nu)
\]

\[
= x(n) (d_{n,j+1}^\nu - d_{n,j}^\nu) + \sum_{i=1}^{n-1} x(i) (d_{i,j+1}^\nu - d_{i,j}^\nu)
\]

\[
\geq x(n-1) \sum_{i=n-1}^{n} (d_{i,j+1}^\nu - d_{i,j}^\nu) + \sum_{i=1}^{n-2} x(i) (d_{i,j+1}^\nu - d_{i,j}^\nu)
\]

\[
\geq x(n-2) \sum_{i=n-2}^{n} (d_{i,j+1}^\nu - d_{i,j}^\nu) + \sum_{i=1}^{n-3} x(i) (d_{i,j+1}^\nu - d_{i,j}^\nu)
\]

\[
\geq \cdots \geq x(1) \sum_{i=1}^{n} (d_{i,j+1}^\nu - d_{i,j}^\nu) \geq 0
\]

where each inequality above follows from the inequality \( \sum_{i=k}^{n} d_{i,j}^\nu \leq \sum_{i=k}^{n} d_{i,j+1}^\nu \) for \( k = 1, \ldots, n \). This shows that \( xD \) is similarly ordered to \( x \) and \( D \) is order preserving with respect to \( A \).

Next, we prove the necessity. Choose \( x \in \mathcal{A} \) such that \( x(n) = x(n-1) = \cdots = x(k) = z > 0 \) and \( x(k-1) = x(k-2) = \cdots = x(1) = 0 \). Because \( xD \) is similarly ordered to \( x \), it holds that for \( j = 1, \ldots, n-1 \),

\[
0 \leq [xD]_{\nu_{j+1}} - [xD]_{\nu_j} = \sum_{i=1}^{n} x(i) d_{i,j+1}^\nu - \sum_{i=1}^{n} x(i) d_{i,j}^\nu = \left( \sum_{i=k}^{n} d_{i,j+1}^\nu - \sum_{i=k}^{n} d_{i,j}^\nu \right) z.
\]

Because \( z > 0 \), the above inequality implies that \( \sum_{i=k}^{n} d_{i,j+1}^\nu - \sum_{i=k}^{n} d_{i,j}^\nu \geq 0 \) holds for \( j = 1, \ldots, n-1 \). 

\( \square \)
Proof of Lemma 2. First, we prove the sufficient condition in the first statement. Let \(x, y \in \mathcal{A}\), \(x <_w y\). For notational convenience, we omit the dependence of \(\nu(x)\) on \(x\). Because \(x, y \in \mathcal{A}\), we also have \(\nu(y) = \nu\). For \(k = 1, \ldots, n\),
\[
\sum_{j=k}^{n} [xD]_{\nu_j} - \sum_{j=k}^{n} [yD]_{\nu_j} = \sum_{j=k}^{n} \left( \sum_{i=1}^{n} x(i)d_{i,j}^{\nu} - \sum_{i=1}^{n} y(i)d_{i,j}^{\nu} \right) \\
= \sum_{i=1}^{n} \left( (x(i) - y(i)) \sum_{j=k}^{n} d_{1,j}^{\nu} + \sum_{j=k}^{n} (x(i) - y(i)) \sum_{j=k}^{n} (d_{2,j}^{\nu} - d_{1,j}^{\nu}) \right) \\
+ \cdots + \left( x(n) - y(n) \right) \sum_{j=k}^{n} (d_{n,j}^{\nu} - d_{n-1,j}^{\nu}) \\
\leq 0,
\]
where the first equality follows because \(D\) is order preserving w.r.t. \(\mathcal{A}\) and the last inequality is due to \(x <_w y\) and to the assumption that \(\sum_{j=k}^{n} d_{i-1,j}^{\nu} \leq \sum_{j=k}^{n} d_{i,j}^{\nu}\) for \(i = 2, \ldots, n\). Hence, using the definition of weak submajorization, \(xD <_w yD\).

We then prove the reverse implication in the first statement. Choose \(x, y \in \mathcal{A}\) such that \(0 \leq x(t) < y(t), y(t) - x(t) = x(t-1) - y(t-1)\) for some \(t \in \{2, \ldots, n\}\) and \(x(u) = y(u)\) for \(u = 1, \ldots, n, u \notin \{t-1, t\}\). Clearly,
\[
\sum_{i=t}^{n} x(i) < \sum_{i=t}^{n} y(i), \quad \sum_{i=u}^{n} x(i) = \sum_{i=u}^{n} y(i) \quad \text{for} \quad u = 1, \ldots, n, u \notin t, \quad (9)
\]
and \(x <_w y\). Because \(D\) is order and weak submajorization preserving w.r.t. \(\mathcal{A}\), Eq. (8) must hold for \(x\) and \(y\), and by Eq. (9) the last inequality can be simplified to
\[
\sum_{i=t}^{n} (x(i) - y(i)) \sum_{j=k}^{n} (d_{i,j}^{\nu} - d_{i-1,j}^{\nu}) \leq 0.
\]
Because \(\sum_{i=t}^{n} (x(i) - y(i)) < 0\), we obtain that \(\sum_{j=k}^{n} (d_{i,j}^{\nu} - d_{i-1,j}^{\nu}) \geq 0\) for \(k = 1, \ldots, n\). This concludes the proof.

The proof for the second statement follows similar arguments used above. To prove the sufficiency, we obtain \(\sum_{j=1}^{k-1} [xD]_{\nu_j} - \sum_{j=1}^{k-1} [yD]_{\nu_j} \geq 0\) for \(k = 1, \ldots, n, x, y \in \mathcal{A}\) by expanding vector-matrix products and combining terms in a similar way as done for the inequality (8). To prove the necessity of the condition, we choose \(x, y \in \mathcal{A}\) such that \(x(t) > y(t) \geq 0, x(t) - y(t) = y(t+1) - x(t+1)\) for some \(t \in \{1, \ldots, n-1\}\) and \(x(u) = y(u)\) for \(u = 1, \ldots, n, u \notin \{t, t+1\}\). Then we obtain
\[
\sum_{i=1}^{t} (x(i) - y(i)) \sum_{j=1}^{k} (d_{i,j}^{\nu} - d_{i+1,j}^{\nu}) \geq 0,
\]
for \(k = 1, \ldots, n\). Then, we conclude the proof following similar arguments as in the proof of the first statement.

Proof of Lemma 3. For notational convenience, we set \(\bar{\Pi} := (1 + \gamma)\Pi\) and \(\bar{c} := (1 + \gamma)c - \gamma\ell\). As pointed out in Eisenberg and Noe (2001), the clearing payment vector \(p^*\) is obtained as the solution to the following optimization problem:
\[
\max_x f(x), \quad \text{s.t.} \quad x(I - \bar{\Pi}) \leq \bar{c}, \quad 0 \leq x \leq \ell,
\]

22
where the objective function $f$ is any real valued increasing function of the vector $x$. Multiplying both sides of the first constraint by $(1 - \alpha)$, with $\alpha \in [0, 1)$, will lead to an equivalent optimization problem. But this leads to:

$$x [I - (1 - \alpha)\bar{\Pi} - \alpha I] \leq (1 - \alpha)\bar{c}.$$ 

That is, if we replace $\bar{\Pi}$ by the matrix $\Pi_{\alpha,\gamma} := (1 - \alpha)\bar{\Pi} + \alpha I$ and $\bar{c}$ by $c_{\alpha,\gamma} := (1 - \alpha)\bar{c}$, the clearing payment vector stays the same. \hfill \Box

**Lemma A1.** Let $(\Pi^a, \ell, c)$ and $(\Pi^b, \ell, c)$ be two financial systems. If there exists $\alpha$ so that $\Pi^a_{\alpha,\gamma} < \Pi^b_{\alpha,\gamma}$ and $\Pi^a_{\alpha,\gamma} \neq \Pi^b_{\alpha,\gamma}$, then it must hold that $\Pi^a_{\beta,\gamma} < \Pi^b_{\beta,\gamma}$ for all $\beta \in [0, 1), \beta \neq \alpha$.

**Proof.** Assume the existence of $A := \Pi^a_{\alpha,\gamma}, B := \Pi^b_{\alpha,\gamma}, C := \Pi^a_{\beta,\gamma}, D := \Pi^b_{\beta,\gamma}, \beta \neq \alpha$, so that $A < B$ and $A \neq B$, but $C > D$. Denote by $X^k$ the $k$-th row of the matrix $X$. Because $A < B$ and $A \neq B$, there must exist $g$ and $i$ such that

$$\sum_{j=1}^{g} A^{i}_{(j)} > \sum_{j=1}^{g} B^{i}_{(j)}. \quad (10)$$

Moreover, since $C > D$, from the definition of majorization the following inequality must hold:

$$\sum_{j=1}^{k} C^{i}_{(j)} \leq \sum_{j=1}^{k} D^{i}_{(j)} \text{ for any } k = 1, \ldots, n. \quad (11)$$

Next, we show that Eq. (10) and Eq. (11) cannot hold simultaneously. Let $h, m, w, z$ be such that $A^{i}_{(h)} = B^{i}_{(m)} = \alpha$ and $C^{i}_{(w)} = D^{i}_{(z)} = \beta$. We first discuss the implications of Eq. (10) using a case-by-case analysis based on $g$.

- $g > \max\{h, m\}$. Using the definition of relaxed equivalent version, we obtain

$$\sum_{j=1}^{g} A^{i}_{(j)} + \alpha = \sum_{j=1}^{g} A^{i}_{(j)} > \sum_{j=1}^{g} B^{i}_{(j)} = \sum_{j=1, j \neq m}^{g} B^{i}_{(j)} + \alpha$$

$$\Rightarrow \sum_{j=1}^{g} (1 - \alpha)(1 + \gamma)\Pi^{a,i}_{(j)} > \sum_{j=1}^{g} (1 - \alpha)(1 + \gamma)\Pi^{b,i}_{(j)}. \quad (12)$$

- $h \geq g \geq m$. We obtain

$$\sum_{j=1}^{g} A^{i}_{(j)} + \alpha \geq \sum_{j=1}^{g} A^{i}_{(j)} > \sum_{j=1}^{g} B^{i}_{(j)} = \sum_{j=1, j \neq m}^{g} B^{i}_{(j)} + \alpha$$

leading to the inequality (12).

- $h \leq g \leq m$. Eq. (10) implies that

$$\sum_{j=1}^{m} A^{i}_{(j)} = \sum_{j=1, j \neq h}^{g} A^{i}_{(j)} + \alpha + \sum_{j=g+1}^{m} A^{i}_{(j)} > \sum_{j=1}^{g} B^{i}_{(j)} + \sum_{j=g+1}^{m-1} B^{i}_{(j)} + \alpha = \sum_{j=1}^{m} B^{i}_{(j)}$$

$$\Rightarrow \sum_{j=1}^{m} (1 - \alpha)(1 + \gamma)\Pi^{a,i}_{(j)} > \sum_{j=1}^{m} (1 - \alpha)(1 + \gamma)\Pi^{b,i}_{(j)}. \quad (13)$$

23
• $g < \min\{h, m\}$. Eq. (10) directly leads to

$$\sum_{j=1}^{g+1} (1 - \alpha)(1 + \gamma) \Pi_{a,j}^{k} > \sum_{j=1}^{g+1} (1 - \alpha)(1 + \gamma) \Pi_{b,j}^{k}, \quad g < \min\{h, m\}. \tag{14}$$

Next, we discuss the implications of Eq. (11) and show that it leads to

$$\sum_{j=1}^{k} (1 - \beta)(1 + \gamma) \Pi_{a,j}^{k} \leq \sum_{j=1}^{k} (1 - \beta)(1 + \gamma) \Pi_{b,j}^{k}, \quad k = 1, \ldots, n \tag{15}$$

This is done via a case-by-case analysis based on $k$.

• $k > \max\{w, z\}$. We obtain

$$\sum_{j=1, j \neq w}^{k} C_{i,j}^{k} + \beta = \sum_{j=1}^{k} C_{i,j}^{k} \leq \sum_{j=1}^{k} D_{i,j}^{k} = \sum_{j=1, j \neq z}^{k} D_{i,j}^{k} + \beta$$

hence implying the inequality (15).

• $k < \min\{w, z\}$. Eq. (11) directly leads to inequality (15).

• $w \geq k \geq z$. Eq. (11) implies the following inequality

$$\sum_{j=1}^{z-1} C_{i,j}^{k} + \beta \leq \sum_{j=1}^{z-1} D_{i,j}^{k} + \beta \leq \sum_{j=1}^{k} D_{i,j}^{k},$$

which further leads to the inequality (15).

• $w \leq k \leq z$. We obtain

$$\sum_{j=1, j \neq w}^{k} C_{i,j}^{k} + \beta = \sum_{j=1}^{k} C_{i,j}^{k} \leq \sum_{j=1}^{k} D_{i,j}^{k} \leq \sum_{j=1}^{k} D_{i,j}^{k} + \beta \geq D_{i,k}$$

which again leads to the inequality (15).

Setting $k = g$ in Eq. (15) shows that Eq. (12) and Eq. (15) cannot hold simultaneously. Setting $k = m$, we obtain that Eq. (13) and Eq. (15) cannot hold simultaneously. Setting $k = g + 1$, we obtain that Eq. (14) and Eq. (15) cannot hold simultaneously. This ends the proof. \hfill \Box

**Proof of Lemma 4.** We prove the first statement by showing that $\ell \Pi + c \geq \ell$. The asset value of the node with the smallest liability is given by

$$\sum_{i=1}^{n} \ell_{(i)} \pi_{c,1}^{h,i} + c(1) \geq c(1) \geq \mathbb{1}_{c(1) < \ell(1)} \left[ c(1) + \gamma \left( c(1) - \ell(1) \right) \right] + \mathbb{1}_{c(1) \geq \ell(1)} \ell(1)$$

$$\geq \mathbb{1}_{c(1) < \ell(1)} \left[ \ell(1) - \max \left\{ \ell(1) - [(1 + \gamma)c - \gamma \ell(1)], 0 \right\} \right] + \mathbb{1}_{c(1) \geq \ell(1)} \ell(1)$$

$$\geq \mathbb{1}_{c(1) < \ell(1)} \ell(1) + \mathbb{1}_{c(1) \geq \ell(1)} \ell(1) \geq \ell(1),$$

24
where \( 1 \) denotes the indicator function. Because \((\Pi, \ell, c, \gamma)\) is unbalancing, it must satisfy the inequalities (7) by definition. Combining them with the above inequality leads to

\[
\sum_{i=1}^{n} \ell_{(i)}^{\mu} \pi_{i,1} + c(n) - \ell_{(n)}^{(1)} \geq \cdots \geq \sum_{i=1}^{n} \ell_{(i)}^{\mu} \pi_{i,2} + c(2) - \ell_{(2)}^{(2)} \geq \sum_{i=1}^{n} \ell_{(i)}^{\mu} \pi_{i,1} + c(1) - \ell_{(1)}^{(1)} \geq 0,
\]

which proves the first statement.

Given \( \epsilon \), we can choose the entries of \( \Pi \) to be small enough so that \( \sum_{i=1}^{n} (\ell_{(i)} + \epsilon_{\mu,i}) \pi_{i,1}^{\mu} + c(1) < \ell_{(1)}^{(1)} + \epsilon_{\mu_1} \) and \( c(j_{(j+1)}) - c(j_{(j)}) \geq \ell_{(j+1)}^{(j)} - \ell_{(j)}^{(j)} + \max_{k=1,\ldots,n} \{ \sum_{i=1}^{n} \ell_{(i)}^{(k)} \pi_{i,k}^{\mu} \} \) for \( j = 1, \ldots, n-1 \) (recall here that \( \mu \) has been defined above definition (8)). Such a system \((\Pi, \ell, c, \gamma)\) satisfies the inequalities (7), but \((\ell + \epsilon)\Pi + c \not\leq \ell + \epsilon\). Hence, this system is unbalancing but \( p^* (\Pi, \ell + \epsilon, c, \gamma) \not\leq \ell + \epsilon \).

The following lemma provides sufficient conditions under which the minimum operation preserves the weak majorization relation. This is needed in the following proofs, given that the computation of the clearing payment vector requires taking the minimum between two vectors.

**Lemma A2.** Let \( x, y, z \in \mathbb{R}_{\geq 0}^n \) such that \( x \) and \( y \) are similarly ordered to \( z \).

(I) If \( z[i] \leq a[i] \) implies \( z[k] \leq a[k] \) for \( k > i, a \in \{x, y\} \), then \( x <_w y \) implies \( (x \land z) <_w (y \land z) \).

(II) If \( z(i) \leq a(i) \) implies \( z(k) \leq a(k) \) for \( k > i, a \in \{x, y\} \), then \( x <^w y \) implies \( (x \land z) <^w (y \land z) \).

**Proof.** (I) Because \( x \) and \( y \) are similarly ordered to \( z \), clearly, \( (x \land z) \) and \( (y \land z) \) are similarly ordered to \( z \). Hence, proving \( (x \land z) <_w (y \land z) \) is equivalent to show that

\[
\sum_{i=1}^{k} \min \{ x[i], z[i] \} \leq \sum_{i=1}^{k} \min \{ y[i], z[i] \} \quad \text{for } k = 1, \ldots, n. \tag{16}
\]

Let \( m_x = \min \{ i = 1, \ldots, n | z[i] \leq x[i] \} \) and \( m_y = \min \{ i = 1, \ldots, n | z[i] \leq y[i] \} \). It must hold that for \( k = 1, \ldots, m_y - 1 \),

\[
\sum_{i=1}^{k} x[i] \leq \sum_{i=1}^{k} y[i] \leq \sum_{i=1}^{k} y[i] = \sum_{i=1}^{k} \min \{ y[i], z[i] \}, \tag{17}
\]

where the second inequality holds because \( x <_w y \). Moreover, for \( k = m_y, \ldots, n \),

\[
\sum_{i=1}^{k} \min \{ x[i], z[i] \} = \mathbb{1} \{ m_x < m_y \} \left( \sum_{i=1}^{m_y - 1} \min \{ x[i], z[i] \} + \sum_{i=m_y}^{k} z[i] \right) + \mathbb{1} \{ m_x \geq m_y \} \left( \sum_{i=1}^{m_y - 1} x[i] + \sum_{i=m_y}^{k} \min \{ x[i], z[i] \} \right) \leq \sum_{i=1}^{m_y - 1} y[i] + \sum_{i=m_y}^{k} z[i] = \sum_{i=1}^{k} \min \{ y[i], z[i] \}, \tag{18}
\]

where \( \mathbb{1} \) denotes the indicator function and the above inequality follows from \( x <_w y \). Using equations (17) and (18), we obtain the inequality in (16).

(II) Because \( x \) and \( y \) are similarly ordered to \( z \), clearly, \( (x \land z) \) and \( (y \land z) \) are similarly ordered to \( z \). Hence, proving \( (x \land z) <^w (y \land z) \) is equivalent to show

\[
\sum_{i=1}^{k} \min \{ x(i), z(i) \} \geq \sum_{i=1}^{k} \min \{ y(i), z(i) \} \quad \text{for } k = 1, \ldots, n. \tag{19}
\]

25
Let $m_x = \min\{i = 1, \ldots, n | z(i) \leq x(i)\}$, $m_y = \min\{i = 1, \ldots, n | z(i) \leq y(i)\}$. For $k = 1, \ldots, m_x - 1$, we must have

$$
\sum_{i=1}^{k} \min\{x(i), z(i)\} = \sum_{i=1}^{k} x(i) \geq \sum_{i=1}^{k} y(i) \geq \sum_{i=1}^{k} \min\{y(i), z(i)\},
$$

where the first inequality follows from $x \prec w y$. Moreover, for $k = m_x, \ldots, n$,

$$
\sum_{i=1}^{k} \min\{x(i), z(i)\} = \sum_{i=1}^{m_x-1} x(i) + \sum_{i=m_x}^{k} z(i)
\geq \mathbb{1}(m_x < m_y) \left( \sum_{i=1}^{m_x-1} y(i) + \sum_{i=m_x}^{k} \min\{y(i), z(i)\} \right) + \mathbb{1}(m_x \geq m_y) \left( \sum_{i=1}^{m_x-1} \min\{y(i), z(i)\} + \sum_{i=m_x}^{k} z(i) \right)
= \sum_{i=1}^{k} \min\{y(i), z(i)\},
$$

where the inequality follows from $x \prec w y$. Using equations (20) and (21), we obtain the inequality (19). \hfill \Box

For any vector $x \in \mathbb{R}^n$, define the operation

$$
\Delta x := (0, \ x(2) - x(1), \ x(3) - x(2), \ \cdots \ x(n) - x(n-1)),
$$

which gives the increment from one component to the next rank ordered component in the vector.

**Lemma A3.** Let $x, y \in \mathbb{R}^n$ such that $x$ and $y$ are similarly ordered. If $\Delta x \preceq \Delta y$, then $\Delta (x \wedge y) \preceq \Delta y$. Vice versa, if $\Delta x \succeq \Delta y$, then $\Delta (x \wedge y) \succeq \Delta y$.

**Proof.** We first prove that $\Delta x \preceq \Delta y \Rightarrow \Delta (x \wedge y) \preceq \Delta y$. For $i = 1, \ldots, n - 1$,

$$
\Delta (x \wedge y)_{i+1} = (x \wedge y)_{(i+1)} - (x \wedge y)_{(i)}
= \min\{x_{(i+1)}, y_{(i+1)}\} - \min\{x_{(i)}, y_{(i)}\}
= \begin{cases} x_{(i+1)} - x_{(i)} & \text{if } x_{(i+1)} \leq y_{(i+1)} \text{ and } x_{(i)} \leq y_{(i)} \\ y_{(i+1)} - y_{(i)} & \text{if } x_{(i+1)} \leq y_{(i+1)} \text{ and } x_{(i)} \geq y_{(i)} \end{cases}
\leq y_{(i+1)} - y_{(i)}
= \Delta y_{i+1},
$$

where we have used the assumption that $x$ is similarly ordered to $y$. Notice that the case $x_{(i+1)} \geq y_{(i+1)}$ and $x_{(i)} \leq y_{(i)}$ is not listed, because it violates the assumption that $\Delta x \preceq \Delta y$. Hence, $\Delta (x \wedge y) \preceq \Delta y$.

We next prove the reverse implication by showing that $\Delta x \succeq \Delta y \Rightarrow \Delta (x \wedge y) \succeq \Delta y$. This is because

$$
\Delta (x \wedge y)_{i+1} = (x \wedge y)_{(i+1)} - (x \wedge y)_{(i)}
= \min\{x_{(i+1)}, y_{(i+1)}\} - \min\{x_{(i)}, y_{(i)}\}
$$

26
\[
(x_{i+1} - x_i) \text{ if } x_{i+1} \leq y_{i+1} \text{ and } x_i \leq y_i \\
y_{i+1} - x_i \text{ if } x_{i+1} \geq y_{i+1} \text{ and } x_i \leq y_i \\
y_{i+1} - y_i \text{ if } x_{i+1} \geq y_{i+1} \text{ and } x_i \geq y_i
\]

where the second equality and the inequality follow from the assumption that \( x \) is similarly ordered to \( y \). The third equality does not include the case \( x_{i+1} \leq y_{i+1} \) and \( x_i \geq y_i \) because such a case violates the assumption that \( \Delta x \geq \Delta y \). Hence, we can conclude that \( \Delta (x \land y) \geq \Delta y \).

The next lemma shows that if a payment vector is more evenly distributed than the liability vector, such a relation is preserved if both vectors are multiplied by the relaxed equivalent version of the relative liability matrix if the latter is order preserving.

**Lemma A.4.** If \( \Pi_{a,\gamma} := [\pi_{\alpha,\gamma,i,j}]_{i,j=1,...,n} \) is order preserving w.r.t. \( \mathcal{P} \), for any \( p \in \mathcal{P} \), it must hold that

(I) \( \Delta p \leq \Delta \ell \) implies that \( \Delta (p \Pi_{a,\gamma}) \leq \Delta (\ell \Pi_{a,\gamma}) \).

(II) \( \Delta p \geq \Delta \ell \) implies that \( \Delta (p \Pi_{a,\gamma}) \geq \Delta \left[ \ell - (\ell(1) - p(1)) \right] \Pi_{a,\gamma} \).

**Proof.** For \( j = 1, \ldots, n-1, p \in \mathcal{P} \),

\[
\Delta (\ell \Pi_{a,\gamma}) \]_{j+1} - \Delta (p \Pi_{a,\gamma}) \]_{j+1} = \sum_{i=1}^{n} (\ell(i) - p(i)) \left( \pi_{a,\gamma,i,j+1} - \pi_{a,\gamma,i,j} \right) \\
= (\ell(1) - p(1)) \left( \pi_{a,\gamma,1,j+1} - \pi_{a,\gamma,1,j} \right) + \left( (\ell(1) + \Delta \ell_k) - (p(1) + \Delta p_k) \right) \left( \pi_{a,\gamma,2,j+1} - \pi_{a,\gamma,2,j} \right) + \ldots \\
+ \left( (\ell(1) + \sum_{k=2}^{n} \Delta \ell_k) - (p(1) + \sum_{k=2}^{n} \Delta p_k) \right) \left( \pi_{a,\gamma,n,j+1} - \pi_{a,\gamma,n,j} \right) \\
= (\ell(1) - p(1)) \left( \sum_{i=1}^{n} \pi_{a,\gamma,i,j+1} - \pi_{a,\gamma,i,j} \right) + (\Delta \ell_2 - \Delta p_2) \left( \sum_{i=2}^{n} \pi_{a,\gamma,i,j+1} - \pi_{a,\gamma,i,j} \right) + \ldots \\
+ (\Delta \ell_n - \Delta p_n) \left( \pi_{a,\gamma,n,j+1} - \pi_{a,\gamma,n,j} \right).
\]

Applying Lemma 1 with \( \mathcal{A} = \mathcal{P} \) we deduce that \( \sum_{i=k}^{n} \pi_{a,\gamma,i,j+1} - \pi_{a,\gamma,i,j} \geq 0 \) for \( k = 1, \ldots, n \). If \( \Delta p \leq \Delta \ell \), then \( \Delta (\ell \Pi_{a,\gamma}) \]_{i+1} - \Delta (p \Pi_{a,\gamma}) \]_{i+1} \geq 0 because \( \ell \geq p \) and \( \sum_{i=k}^{n} \pi_{a,\gamma,i,j+1} - \pi_{a,\gamma,i,j} \geq 0 \) for \( k = 1, \ldots, n \). This proves (I). Vice versa, if \( \Delta p \geq \Delta \ell \), then

\[
\Delta (\ell \Pi_{a,\gamma}) \]_{j+1} - \Delta (p \Pi_{a,\gamma}) \]_{j+1} - (\ell(1) - p(1)) \left( \sum_{i=1}^{n} \pi_{a,\gamma,i,j+1} - \pi_{a,\gamma,i,j} \right) \leq 0
\]

because \( \sum_{i=k}^{n} \pi_{a,\gamma,i,j+1} - \pi_{a,\gamma,i,j} \geq 0 \) for \( k = 1, \ldots, n \). This proves (II).

The next lemma shows that if the financial system is balancing (unbalancing), the vector of asset values before clearing under the base (reduced) liability configuration has smaller (larger) variation than the liability vector. Moreover, if the payment vector has smaller variation than the liability vector, then the assets after payments are settled also have smaller variation than the liabilities.
Lemma A5. Let \((\Pi, \ell, c, \gamma)\) be a financial system and \(\Pi_{\alpha, \gamma}\) be an \(\alpha\)-relaxed equivalent version which is order preserving w.r.t. \(P\).

(I) If \((\Pi, \ell, c, \gamma)\) is balancing, then

- \(\Delta(\ell \Pi_{\alpha, \gamma} + c_{\alpha, \gamma}) \leq \Delta \ell\).
- \(\Delta p \leq \Delta \ell\) implies that \(\Delta(p \Pi_{\alpha, \gamma} + c_{\alpha, \gamma}) \leq \Delta \ell\) for \(p \in P\).

(II) If \((\Pi, \ell, c, \gamma)\) is unbalancing, then

- \(\Delta(\ell \Pi_{\alpha, \gamma} + c_{\alpha, \gamma}) \geq \Delta \ell\).
- \(\Delta p \geq \Delta \ell\) implies that \(\Delta(p \Pi_{\alpha, \gamma} + c_{\alpha, \gamma}) \geq \Delta \ell\) for \(p \in P\), \(p \geq [(1 + \gamma)c - \gamma \ell] \land \ell\).

Proof. (I)

\[
[\Delta(\ell \Pi_{\alpha, \gamma} + c_{\alpha, \gamma})]_j = [\Delta(\alpha \ell + (1 - \alpha)[\ell(1 + \gamma)\Pi + (1 + \gamma)c - \gamma \ell])]_j
= \alpha(\ell(j) - \ell(j-1))
+ (1 - \alpha) \sum_{i=1}^{n} \ell(i)(1 + \gamma)(c_{ij} - c_{j-1}) - \gamma(\ell(j) - \ell(j-1))
\]

\[
\leq \alpha(\ell(j) - \ell(j-1)) + (1 - \alpha)[(1 + \gamma)(\ell(j) - \ell(j-1)) - \gamma(\ell(j) - \ell(j-1))] = [\Delta \ell]_j,
\]
for \(j = 2, \ldots, n\). The second equality holds because \(\Pi_{\alpha, \gamma}\) is order preserving w.r.t. \(P\), \(c\) and \(\ell\) are similarly ordered to \(\ell\) and \(\Delta \ell = \Delta \ell\). The inequality follows from the fact that the system is balancing.

Next, by Lemma A4 (I), \(\Delta p \leq \Delta \ell\) implies that

\[
\Delta(p \Pi_{\alpha, \gamma} + c_{\alpha, \gamma}) \leq \Delta(\ell \Pi_{\alpha, \gamma} + c_{\alpha, \gamma}) \leq \Delta \ell,
\]
where the first inequality holds because \(\Pi_{\alpha, \gamma}\) is order preserving w.r.t. \(P\), and \(c_{\alpha, \gamma}\) and \(p\) are similarly ordered to \(\ell\).

(II)

\[
[\Delta(\ell \Pi_{\alpha, \gamma} + c_{\alpha, \gamma})]_j = [\Delta(\alpha \ell + (1 - \alpha)[\ell(1 + \gamma)\Pi + (1 + \gamma)c - \gamma \ell])]_j
= \alpha(\ell(j) - \ell(j-1))
+ (1 - \alpha) \sum_{i=1}^{n} \ell(i)(1 + \gamma)(c_{ij} - c_{j-1}) - \gamma(\ell(j) - \ell(j-1))
\]

\[
\geq \alpha(\ell(j) - \ell(j-1)) + (1 - \alpha)[(1 + \gamma)(\ell(j) - \ell(j-1)) - \gamma(\ell(j) - \ell(j-1))]\]

\[
\geq \alpha(\ell(j) - \ell(j-1)) + (1 - \alpha)[(1 + \gamma)(\ell(j) - \ell(j-1)) - \gamma(\ell(j) - \ell(j-1))] = [\Delta \ell]_j,
\]
for \(j = 2, \ldots, n\). The second equality is implied by \(\Pi_{\alpha, \gamma}\) being order preserving w.r.t. \(P\), along with the fact \(c\) and \(\ell\) are similarly ordered to \(\ell\) and that \(\Delta \ell = \Delta \ell\). The inequality follows because the system is unbalancing.

Next, by Lemma A4 (II), \(\Delta p \geq \Delta \ell\) implies

\[
\Delta(p \Pi_{\alpha, \gamma} + c_{\alpha, \gamma}) \geq \Delta((\ell - (\ell(1) - p(1))) \Pi_{\alpha, \gamma} + c_{\alpha, \gamma})) \geq \Delta(\ell \Pi_{\alpha, \gamma} + c_{\alpha, \gamma}) \geq \Delta \ell,
\]

28
where the first inequality follows because \( \mathbf{P}_{\alpha,\gamma} \) is order preserving w.r.t. \( \mathcal{P} \), and \( c_{\alpha,\gamma}, p \), and \( \ell - \ell'(\ell(1) - p(1)) \) are similarly ordered to \( \ell \). The second inequality follows from the assumption that \( p \geq [(1 + \gamma)c - \gamma\ell] \wedge \ell \) because

\[
\ell(1) - p(1) \leq \ell(1) - [(1 + \gamma)c - \gamma\ell] \wedge \ell \leq \max \left\{ \ell(1) - [(1 + \gamma)c - \gamma\ell], 0 \right\},
\]

and then \( \Delta((\ell - (\ell(1) - p(1)))\mathbf{P}_{\alpha,\gamma}) \geq \Delta((\ell\mathbf{P}_{\alpha,\gamma})) \) implied by Lemma 1 upon choosing \( \mathbf{D} = \mathbf{P}_{\alpha,\gamma} \) and \( \mathcal{A} = \mathcal{P} \).

The following lemma gives the properties of the converging sequence of vectors to the clearing payment vector.

**Lemma A6.** Let \( (\mathbf{P}, \ell, c, \gamma) \) be a financial system. Suppose there exists \( \alpha \in (0, 1) \) such that \( \mathbf{P}_{\alpha,\gamma} \) is order preserving w.r.t. to \( \mathcal{P} \). Define the vector valued function \( F(p; \mathbf{P}_{\alpha,\gamma}, \ell, c, \gamma) := \ell \wedge (p\mathbf{P}_{\alpha,\gamma} + c_{\alpha,\gamma}) \), and a sequence of vectors \( \{f_u\}_{u=0}^\infty \) given by \( f_u := F(f_{u-1}) \) and \( f_0 := \ell \). The following holds:

(I) \( f_u \) is similarly ordered to \( \ell \) and \( \{f_u\}_{u=0}^\infty \) decreasingly converges to \( p^*(\mathbf{P}, \ell, c, \gamma) \).

(II) If \( (\mathbf{P}, \ell, c, \gamma) \) is balancing, then \( \Delta f_u \leq \Delta \ell \) and \( \Delta f_u \mathbf{P}_{\alpha,\gamma} + c_{\alpha,\gamma} \leq \Delta \ell \).

(III) If \( (\mathbf{P}, \ell, c, \gamma) \) is unbalancing, then \( \Delta f_u \geq \Delta \ell \) and \( \Delta f_u \mathbf{P}_{\alpha,\gamma} + c_{\alpha,\gamma} \geq \Delta \ell \).

**Proof.** (I) It has been proven in Eisenberg and Noe (2001), Lemma 5, that \( f_u \) decreasingly converges to \( p^*(\mathbf{P}, \ell, c, \gamma) \). The statement that \( f_u \) is similarly ordered to \( \ell \) follows from the fact that \( \mathbf{P}_{\alpha,\gamma} \) is order preserving and from the assumption that \( c_{\alpha,\gamma} \) is similarly ordered to \( \ell \).

(II) We prove that \( \Delta(f_u \mathbf{P}_{\alpha,\gamma} + c_{\alpha,\gamma}) \leq \Delta \ell \) by induction. For \( u = 0 \), we know that \( f_0 = \ell \). Clearly, \( \Delta f_0 \leq \Delta \ell \). From the assumptions that \( (\mathbf{P}, \ell, c, \gamma) \) is balancing and \( \mathbf{P}_{\alpha,\gamma} \) is order preserving w.r.t. \( \mathcal{P} \), it follows that \( \Delta((\ell\mathbf{P}_{\alpha,\gamma} + c_{\alpha,\gamma}) \leq \Delta \ell \) from Lemma A5 (I) being \( \ell \in \mathcal{P} \). Hence, \( \Delta(f_0 \mathbf{P}_{\alpha,\gamma} + c_{\alpha,\gamma}) \leq \Delta \ell \).

Next, we prove the statement for \( u + 1 \) assuming that it holds for \( u \). Since \( \mathbf{P}_{\alpha,\gamma} \) is order preserving w.r.t. \( \mathcal{P} \), \( f_u \in \mathcal{P} \), and \( c_{\alpha,\gamma} \) is similarly ordered to \( \ell \), it follows that \( f_u \mathbf{P}_{\alpha,\gamma} + c_{\alpha,\gamma} \) is similarly ordered to \( \ell \). Hence \( f_{u+1} = \ell \wedge (f_u \mathbf{P}_{\alpha,\gamma} + c_{\alpha,\gamma}) \) is similarly ordered to \( \ell \). Using the induction hypothesis that \( \Delta(f_u \mathbf{P}_{\alpha,\gamma} + c_{\alpha,\gamma}) \leq \Delta \ell \) and Lemma A3, we obtain

\[
\Delta f_{u+1} = \Delta [\ell \wedge (f_u \mathbf{P}_{\alpha,\gamma} + c_{\alpha,\gamma})] \leq \Delta \ell.
\]

Since \( (\mathbf{P}, \ell, c, \gamma) \) is balancing, \( \mathbf{P}_{\alpha,\gamma} \) is order preserving w.r.t. \( \mathcal{P} \), and \( f_{u+1} \in \mathcal{P} \), we can apply Lemma A5 (I) and deduce that \( \Delta(f_{u+1} \mathbf{P}_{\alpha,\gamma} + c_{\alpha,\gamma}) \leq \Delta \ell \). This concludes the induction.

(III) Using Lemma A5 (II), the fact that \( \{f_u\} \) is a decreasingly sequence converging to \( p^* \) and the lower bound given in Lemma A5, we deduce that \( f_u \geq [(1 + \gamma)c - \gamma\ell] \wedge \ell \). Then we can use similar arguments as in (II) to conclude the proof.

**Proof of Proposition 1.** Letting \( u \to \infty \) in Lemma A6 (I) leads to \( p^* \) being similarly ordered to \( \ell \), hence yielding (I). Moreover, we obtain \( \Delta p^* \leq \Delta \ell \) for balancing financial systems and \( \Delta p^* \geq \Delta \ell \) for unbalancing financial systems from (II) and (III) in Lemma A6. It then follows that

\[
\ell[i] - p^*[i] \geq \ell[i] - p^*[i] - (\Delta \ell)[n-1-1] - [\Delta p^*][n-1] = \ell[i+1] - p^*[i+1], \quad i = 1, \ldots, n-1
\]

for balancing financial systems, and

\[
\ell(i) - p^*(i) \leq \ell(i) - p^*(i) + (\Delta \ell)[i+1] - [\Delta p^*][i+1] = \ell(i+1) - p^*(i+1), \quad i = 1, \ldots, n-1
\]

for unbalancing financial systems.
**Proof of Theorem 1.** Recall from Lemma A6 that a sequence of vectors \( \{f_u(\Pi_{\alpha,\gamma})\}_{u=0}^{\infty} \), where \( F(p, \Pi_{\alpha,\gamma}; k, c_{\alpha,\gamma}) := k \wedge (p\Pi_{\alpha,\gamma} + c_{\alpha,\gamma}) \), \( f_u(\Pi_{\alpha,\gamma}) := F(f_{u-1}(\Pi_{\alpha,\gamma}), \Pi_{\alpha,\gamma}) \), and \( f_0(\Pi_{\alpha,\gamma}) := k \), converges to the clearing payment vector \( p^* \). Hence, proving that \( p^{a*}(\Pi^a_{\alpha,\gamma}, k, c_{\alpha,\gamma}) <_w \ p^{b*}(\Pi^b_{\alpha,\gamma}, k, c_{\alpha,\gamma}) \) is equivalent to showing that \( f_u(\Pi^a_{\alpha,\gamma}) <_w f_u(\Pi^b_{\alpha,\gamma}) \) for \( u = 1, 2, \ldots \). For brevity, we denote hereafter \( f_u(\Pi^a_{\alpha,\gamma}) \) by \( f_u^a \) and \( p^*(\Pi_{\alpha,\gamma}, k, c_{\alpha,\gamma}) \) by \( p^* \).

Next, we prove that \( f_u^a <_w f_u^b \) by induction. Without loss of generality, we take \( \Pi^a_{\alpha,\gamma} \) to be submajorization preserving w.r.t. \( \mathcal{P} \). (If it were the case that \( \Pi^b_{\alpha,\gamma} \) is submajorization preserving w.r.t. \( \mathcal{P} \), we would obtain the same result and the proof would proceed in a symmetric fashion by interchanging the roles of \( a \) and \( b \).) For \( u = 0 \), we have \( f_0^a = \ell \leq f_0^b = \ell \). Assume \( f_u^a <_w f_u^b \). Then we want to prove the statement for \( u + 1 \). We first deduce that

\[
(f_u^a \Pi_{\alpha,\gamma} + c_{\alpha,\gamma}) <_w (f_u^b \Pi_{\alpha,\gamma} + c_{\alpha,\gamma}) <_w (f_{u+1}^b \Pi_{\alpha,\gamma} + c_{\alpha,\gamma}),
\]

where the first inequality follows from the assumption that \( \Pi^a_{\alpha,\gamma} \) is weak submajorization preserving, the fact that \( f_u^a, f_u^b \) are similarly ordered to \( k \) by Lemma A6 and the assumption 2 that \( c_{\alpha,\gamma} \) is similarly ordered to \( k \); the second inequality follows because \( f_{u+1}^b \Pi_{\alpha,\gamma} + c_{\alpha,\gamma} \) is equivalent to showing that \( f_{u+1}^b \Pi_{\alpha,\gamma} + c_{\alpha,\gamma} \) is submajorization preserving w.r.t. \( f_{u+1}^b \Pi_{\alpha,\gamma} + c_{\alpha,\gamma} \). Hence, proving that \( f_u^a <_w f_u^b \).

By definition of weak submajorization, this means that

\[
f_{u+1}^a = [k \wedge (f_u^a \Pi_{\alpha,\gamma} + c_{\alpha,\gamma})] <_w [k \wedge (f_u^b \Pi_{\alpha,\gamma} + c_{\alpha,\gamma})] = f_u^b
\]

by taking \( x = f_u^a \Pi_{\alpha,\gamma} + c_{\alpha,\gamma}, y = f_u^b \Pi_{\alpha,\gamma} + c_{\alpha,\gamma}, \) and \( z = \ell \). This concludes the proof that \( f_u^a <_w f_u^b \) for \( u = 1, 2, \ldots \).

By definition of weak submajorization, this means that

\[
\sum_{i=1}^{k} f_{u+[i]}^b - \sum_{i=1}^{k} f_{u+[i]}^a \geq 0 \text{ for } k = 1, \ldots, n.
\]

Letting \( u \to \infty \), the above inequality leads to

\[
\sum_{i=1}^{k} f_{[i]}^b - \sum_{i=1}^{k} f_{[i]}^a \geq 0 \text{ for } k = 1, \ldots, n, \quad \text{hence,} \quad p^{a*} = f^a <_w f^b = p^{b*}.
\]

Together with Eq. (22), Proposition 1 (II) leads to

\[
\sum_{i=1}^{k} [k - p^{a*}] = \sum_{i=1}^{k} [k - p^{b*}] = \sum_{i=1}^{k} [k - p^{b*}] = \sum_{i=1}^{k} [k - p^{b*}]
\]

for \( k = 1, \ldots, n \), or equivalently \( s(\Pi^a, k, c_{\alpha,\gamma}) >_w s(\Pi^b, k, c_{\alpha,\gamma}) \).

**Proof of Theorem 2.** Similarly to the proof for Theorem 1, proving that \( p^{a*}(\Pi^a_{\alpha,\gamma}, k, c_{\alpha,\gamma}) <_w p^{b*}(\Pi^b_{\alpha,\gamma}, k, c_{\alpha,\gamma}) \) is equivalent to showing that \( f_u(\Pi^a_{\alpha,\gamma}) <_w f_u(\Pi^b_{\alpha,\gamma}) \) for \( u = 1, 2, \ldots \). For brevity, we denote hereafter \( f_u(\Pi^a_{\alpha,\gamma}) \) by \( f_u^a \) and \( p^*(\Pi_{\alpha,\gamma}, k, c_{\alpha,\gamma}) \) by \( p^* \).

Next, we prove \( f_u^a <_w f_u^b \) by induction. Without loss of generality, we take \( \Pi^a_{\alpha,\gamma} \) to be supermajorization preserving w.r.t. \( \mathcal{P} \). (If \( \Pi^b_{\alpha,\gamma} \) were to be to be supermajorization preserving w.r.t. \( \mathcal{P} \), we
would obtain the same result and the proof would proceed in a symmetric fashion by interchanging the role of \( a \) and \( b \). For \( u = 0 \), by definition, \( f_u^a = \ell \prec w \ell = f_u^0 \). Assume \( f_u^a \prec w f_u^b \). Then we want to prove the statement for \( u + 1 \). First, we notice that the following majorization inequalities hold:

\[
(f_{u + 1}^a \Pi^a_{\alpha, \gamma} + c_{\alpha, \gamma}) \prec w (f_{u + 1}^b \Pi^a_{\alpha, \gamma} + c_{\alpha, \gamma}) - (f_{u + 1}^b \Pi^b_{\alpha, \gamma} + c_{\alpha, \gamma}),
\]

where the first inequality follows from the assumption that \( \Pi^a_{\alpha, \gamma} \) is weak supermajorization preserving w.r.t. \( P \) and the fact that \( f_u^a \) and \( f_u^b \) are similarly ordered to \( \ell \) by Lemma A6 and \( c_{\alpha, \gamma} \) is similarly ordered to \( \ell \) in light of the assumption 2; the second inequality is due to that \( \Pi^a_{\alpha, \gamma} \prec \Pi^b_{\alpha, \gamma} \), and \( \Pi^a_{\alpha, \gamma} \) and \( \Pi^b_{\alpha, \gamma} \) are order preserving w.r.t. \( P \). For \( z \in \{a, b\} \), because \( \Delta(f_u^z \Pi^z_{\alpha, \gamma} + c_{\alpha, \gamma}) \geq \Delta \ell \) (by Lemma A6), \( \ell_{(i)} \leq (f_u^z \Pi^z_{\alpha, \gamma} + c_{\alpha, \gamma})_{(i)} \) must imply that \( \ell_{(k)} \leq (f_u^z \Pi^z_{\alpha, \gamma} + c_{\alpha, \gamma})_{(k)} \) for \( k > i \). Moreover, \( f_u^z \Pi^z_{\alpha, \gamma} + c_{\alpha, \gamma} \) is similarly ordered to \( \ell \). Applying Lemma A2 (II) with \( x = f_u^a \Pi^a_{\alpha, \gamma} + c_{\alpha, \gamma}, y = f_u^b \Pi^b_{\alpha, \gamma} + c_{\alpha, \gamma} \), and \( z = \ell = \ell \), we deduce

\[
f_{u + 1}^a = [\ell \wedge (f_u^a \Pi^a_{\alpha, \gamma} + c_{\alpha, \gamma})] \prec w [\ell \wedge (f_u^b \Pi^a_{\alpha, \gamma} + c_{\alpha, \gamma})] = f_{u + 1}^b.
\]

This concludes the proof that \( f_u^a \prec w f_u^b \) for \( u = 1, 2, \ldots \).

By definition of weak supermajorization,

\[
\sum_{i=1}^{k} f_{u,(i)}^b - \sum_{i=1}^{k} f_{u,(i)}^a \leq 0 \text{ for } k = 1, \ldots, n.
\]

Letting \( u \rightarrow \infty \), the above inequality leads to

\[
\sum_{i=1}^{k} f_{(i)}^b - \sum_{i=1}^{k} f_{(i)}^a \leq 0 \text{ for } k = 1, \ldots, n, \quad \text{hence, } \quad p^{a_+} = f^a \prec w f^b = p^{b_+}. \tag{23}
\]

Together with Eq. (23), Proposition 1 (III) leads to

\[
\sum_{i=1}^{k} [\ell - p^{a_+}]_{(i)} = \sum_{i=1}^{k} \ell_{(i)} - p^{a_+}_{(i)} \leq \sum_{i=1}^{k} \ell_{(i)} - p^{b_+}_{(i)} = \sum_{i=1}^{k} [\ell - p^{b_+}]_{(i)}
\]

for \( k = 1, \ldots, n \), or equivalently \( s(P^a, \ell, c, \gamma) \prec w s(P^b, \ell, c, \gamma) \).

**Proof of Proposition 2.** Because \((P, \ell, c, \gamma)\) is unbalancing, it must hold that for \( j = 1, \ldots, n-1 \),

\[
\left[ \sum_{i=1}^{n} \ell_{(i)} \pi_{i,j+1}^{\mu} + c_{(j+1)} \right] - \left[ \sum_{i=1}^{n} \ell_{(i)} \pi_{i,j}^{\mu} + c_{(j)} \right] \geq \ell_{(j+1)} - \ell_{(j)} = \ell_{(j+1)} - \ell_{(j)}
\]

\[
\Rightarrow \left[ \sum_{i=1}^{n} \ell_{(i)} \pi_{i,j+1}^{\mu} + c_{(j+1)} \right] - \left[ \sum_{i=1}^{n} \ell_{(i)} \pi_{i,j}^{\mu} + c_{(j)} \right]
\]

\[
- \max \left\{ \ell_{(1)-(1+\gamma)c-\gamma} \right\} \left[ \sum_{i=1}^{n} \pi_{i,j}^{\mu} - \sum_{i=1}^{n} \pi_{i,j+1}^{\mu} \right] \geq \ell_{(j+1)} - \ell_{(j)}
\]

\[
\Rightarrow \left[ \sum_{i=1}^{n} \ell_{(i)} \pi_{i,j+1}^{\mu} + c_{(j+1)} \right] - \left[ \sum_{i=1}^{n} \ell_{(i)} \pi_{i,j}^{\mu} + c_{(j)} \right] \geq \ell_{(j+1)} - \ell_{(j)}
\]

where the last inequality follows from the assumption that \( \sum_{i=1}^{n} \pi_{i,j+1}^{\mu} - \sum_{i=1}^{n} \pi_{i,j}^{\mu} \) is nonnegative. Moreover, since \((P, \ell, c, \gamma)\) is balancing, we have

\[
\left[ \sum_{i=1}^{n} \ell_{(i)} \pi_{i,j+1}^{\mu} + c_{(j+1)} \right] - \left[ \sum_{i=1}^{n} \ell_{(i)} \pi_{i,j}^{\mu} + c_{(j)} \right] = \ell_{(j+1)} - \ell_{(j)} \tag{24}
\]
for $j = 1, \ldots, n - 1$. By the assumption of the lemma, at least one node repays its liabilities in full, hence it must hold that

$$\sum_{i=1}^{n} \ell(i) \pi_{i,k}^\mu + c(k) \geq \sum_{i=1}^{n} p_{pi}^* \pi_{i,k}^\mu + c(k) \geq \ell(k)$$

for some $k \in \{1, \ldots, n\}$,

where we recall that $p_{pi}^*$ is the clearing payment made by node $j$ if $\ell(i) = \ell(j)$. Together with the above Eq. (24), this inequality leads to

$$\left[ \sum_{i=1}^{n} \ell(i) \pi_{i,j}^\mu + c(j) \right] - \ell(j) \geq 0 \text{ for } j = 1, \ldots, n,$$

and further implies that $p^* = \ell = \ell \wedge (\ell\Pi + c)$. Hence, it must hold that $s(\Pi, \ell, c, \gamma) = 0$. 

Bibliography


