Countability Axioms
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Definitions

Definition. A *topological space* is a pair $(\mathcal{X}, \mathscr{T})$ where $\mathcal{X}$ is a set and $\mathscr{T}$ is a collection of subsets of $\mathcal{X}$ that satisfies the following conditions

(i) $\emptyset, \mathcal{X} \in \mathscr{T}$,
(ii) if $\{U_\alpha\}_{\alpha \in A}$ is a collection of sets from $\mathscr{T}$ then $\bigcup_{\alpha \in A} U_\alpha \in \mathscr{T}$, and
(iii) if $U, V \in \mathscr{T}$ then $U \cap V \in \mathscr{T}$.

The sets in $\mathscr{T}$ are refered to as the *open* sets.

We need a concept that ties together the points in $\mathcal{X}$ with the open sets in $\mathscr{T}$. The concept that will be used here is the idea of a *neighbourhood*.

Definition. Let $x \in \mathcal{X}$. A set $N \subseteq \mathcal{X}$ is a *neighbourhood* (or simply *nhood*) of x if there is an open set $U \in \mathscr{T}$ such that $x \in U \subseteq N$.

Now of course, topologies can be described or defined in terms of nhoods.

Definition. Let $x \in \mathcal{X}$. A *nhood base* $\mathscr{B}_x$ at x is a collection of nhoods of x such that every nhood of x contains some nhood in $\mathscr{B}_x$. A *base* for the topology $\mathscr{T}$ on $\mathcal{X}$ is a collection of open sets $\mathscr{B}$ such that $\mathscr{T} = \{\bigcup_{B \in \mathscr{B}'} \mid \mathscr{B}' \subseteq \mathscr{B}\}$.

Definition. A topological space is

(i) *first countable* if every point has a countable nhood base,
(ii) *second countable* if it has a countable base,
(iii) *separable* if it has a countable dense subset, and
(iv) *Lindelöf* if every open cover has a countable subcover.

Implications

Second countability is a very strong property, and in fact it implies the other three. The proof is an easy application of the definitions.

Proposition. *Second countability implies first countability, separability, and the Lindelöf property.*

PROOF: Let $\mathscr{B}$ be a countable base for X.

For each $x \in X$, let $\mathscr{B}_x = \{B \in \mathscr{B} \mid x \in B\}$. If U is an open set containing x, then since $\mathscr{B}$ is a base for $\mathscr{T}$, $U = \bigcup_{B \in \mathscr{B}'}$ for some $\mathscr{B}' \subseteq \mathscr{B}$. Hence $x \in B \subseteq U$ for some $B \in \mathscr{B}'$, so necessarily $B \in \mathscr{B}_x$. Therefore $\mathscr{B}_x$ is a countable nhood base for x.

For each $B \in \mathscr{B}$, let $x_B \in B$. I claim that $\{x_B \mid B \in \mathscr{B}\}$ is dense in X. Indeed, for any $x \in X$ and any open set U containing x, there is $B \in \mathscr{B}$ such that $x \in B \subseteq U$. But $x_B \in B$ too, so $\{x_B \mid B \in \mathscr{B}\}$ is a countable dense set.

Let $\mathscr{U}$ be an open cover of X. For each $U \in \mathscr{U}$ and $x \in U$, there is some $B_{x,U} \in \mathscr{B}$ such that $x \in B_{x,U} \subseteq U$. But $\{B_{x,U} \mid U \in \mathscr{U}, x \in U\} \subseteq \mathscr{B}$, so this collection is countable. We may suppose

$$\{B_{x,U} \mid U \in \mathscr{U}, x \in U\} = \{B_{x_1,U_1}, B_{x_2,U_2}, \dots\},$$

so $U_1, U_2, \dots$ is a countable subcover of $\mathscr{U}$. □

Unfortunately, these are the only implications between the four properties.

Counterexamples

Proposition. *First countability does not imply second countability, separability, or the Lindelöf property.*

PROOF: Consider the discrete topology on the real line. Each point $x \in \mathbb{R}$ has an nhood base consisting of just one set, namely $\{x\}$, so this space is first countable. But clearly it is not second countable, separable, or Lindelöf. □

Proposition. *The Lindelöf property does not imply second countability or separability.*

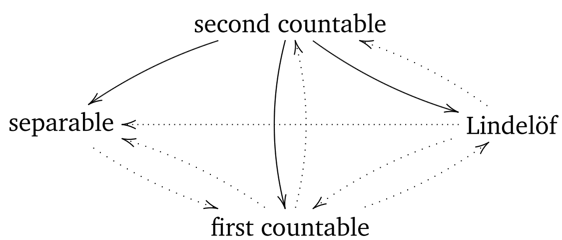
PROOF: Again consider the discrete topology on $\mathbb{R}$. Adjoin a new point $*$ and define the nhoods of $*$ to be of the form $\{*\} \cup F$, for finite sets F. The resulting space X^* is Lindelöf (indeed, it is compact; X^* is the one-point compactification of the discrete topology on $\mathbb{R}$), but it is not separable since all of the sets $\{x\}$ remain open for $x \neq *$. □

Proposition. *Neither separability nor the Lindelöf property imply first countability.*

PROOF: This time consider the cofinite topology on the real line, call this space X, where the open sets are exactly the ones with finite complement. It is separable since the rationals are dense (in fact, every countable subset of $\mathbb{R}$ is dense). X is Lindelöf since if $\mathscr{U}$ is any open cover then there is some open set $U \in \mathscr{U}$, and the complement of U is finite. Finitely many further sets from $\mathscr{U}$ cover the remaining finite number of missed points, so $\mathscr{U}$ has a countable (actually finite) subcover.

Finally, X is not first countable. If $\mathscr{B}$ is any countable collection of nhoods of 0 then $\{0\} \cup \bigcup_{B \in \mathscr{B}} \mathbb{R} \setminus B$ is countable, so there is some x in its complement. Then $\mathbb{R} \setminus \{x\}$ is an open set containing zero that does not contain any element of $\mathscr{B}$ as a subset, so $\mathscr{B}$ cannot be an nhood base of 0. □

We can summarize our results succinctly in diagrammatic form, where solid lines indicate implication and dotted lines indicate lack of implication. We must still investigate the implications of separability on first countability and the Lindelöf property.



Moore Plane

Let $B(x, r)$ denote the open ball centered at x of radius r (where the metric will be clear from the discussion). The *Moore plane* $\mathbf{\Gamma}$ is the closed upper half plane $\mathbb{R} \times \mathbb{R}_{\geq 0}$ with the following topology. For points (x, y) with $y > 0$, take

$$\{B((x, y); t) \cap \mathbf{\Gamma} \mid t > 0\}$$

as an nhood base. For points of the form $(x, 0)$, take

$$\{B((x, t); t) \cup \{(x, 0)\} \mid t > 0\}$$

as an nhood base. To reiterate, the open upper half plane gets the induced topology from $\mathbb{R}^2$, while nhood bases for the points z on the x-axis are given by circles tangent to the x-axis at z.

The Moore plane is interesting in part because it is a separable space with a nonseparable subspace. We exploit this fact in the proof of the following theorem.

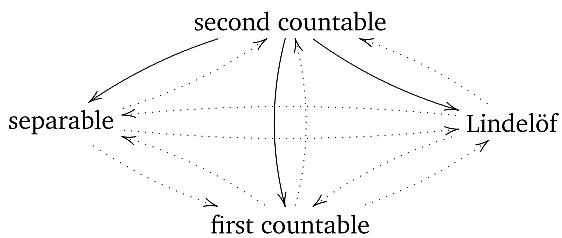
Theorem. *The Moore plane* $\mathbf{\Gamma}$ *is separable and first countable, but not Lindelöf or second countable.*

PROOF: It is clear from the definition that the Moore plane is first countable. Points with rational coordinates form a dense set, so $\mathbf{\Gamma}$ is separable. To finish the proof it suffices to prove that the Moore plane is not Lindelöf, since if $\mathbf{\Gamma}$ were second countable it would be Lindelöf. Consider

$$\mathscr{B} = \{B((x,y); \tfrac{y}{2}) \mid x, y \in \mathbb{R}, y > 0\} \cup \{B((x,1);1) \cup \{x\} \mid x \in \mathbb{R}\},$$

an open cover of $\mathbf{\Gamma}$. None of the open sets around the points with postive y-coordinate contain any point on the x-axis, and no set in $\mathscr{B}$ contains more than point on the x-axis, so $\mathscr{B}$ cannot have a countable subcover. $\square$

The complete diagram of implications is seen to be as follows, where solid lines denote implication and dotted lines indicate lack of implication.



References

[1] Willard, Stephen. *General Topology.* Dover Publications, Inc. N.Y. (2004)