Definitions

**Definition.** A topological space is a pair \((X, \mathcal{T})\) where \(X\) is a set and \(\mathcal{T}\) is a collection of subsets of \(X\) that satisfies the following conditions

(i) \(\emptyset, X \in \mathcal{T}\),
(ii) if \(\{U_a\}_{a \in A}\) is a collection of sets from \(\mathcal{T}\) then \(\bigcup_{a \in A} U_a \in \mathcal{T}\), and
(iii) if \(U, V \in \mathcal{T}\) then \(U \cap V \in \mathcal{T}\).

The sets in \(\mathcal{T}\) are referred to as the open sets.

We need a concept that ties together the points in \(X\) with the open sets in \(\mathcal{T}\). The concept that will be used here is the idea of a *neighbourhood*.

**Definition.** Let \(x \in X\). A set \(N \subseteq X\) is a *neighbourhood* (or simply nhood) of \(x\) if there is an open set \(U \in \mathcal{T}\) such that \(x \in U \subseteq N\).

Now of course, topologies can be described or defined in terms of nhoods.

**Definition.** Let \(x \in X\). A nhood base \(\mathcal{B}_x\) at \(x\) is a collection of nhoods of \(x\) such that every nhood of \(x\) contains some nhood in \(\mathcal{B}_x\). A base for the topology \(\mathcal{T}\) on \(X\) is a collection of open sets \(\mathcal{B}\) such that \(\mathcal{T} = \{\bigcup_{B \in \mathcal{B}} | \mathcal{B}' \subseteq \mathcal{B}\}\).

**Definition.** A topological space is

(i) *first countable* if every point has a countable nhood base,
(ii) *second countable* if it has a countable base,
(iii) *separable* if it has a countable dense subset, and
(iv) *Lindelöf* if every open cover has a countable subcover.
Implications

Second countability is a very strong property, and in fact it implies the other three. The proof is an easy application of the definitions.

**Proposition.** Second countability implies first countability, separability, and the Lindelöf property.

**Proof:** Let $\mathscr{B}$ be a countable base for $X$. For each $x \in X$, let $\mathscr{B}_x = \{ B \in \mathscr{B} \mid x \in B \}$. If $U$ is an open set containing $x$, then since $\mathscr{B}$ is a base for $\mathcal{T}$, $U = \bigcup_{B \in \mathscr{B}'}$ for some $\mathscr{B}' \subseteq \mathscr{B}$. Hence $x \in B \subseteq U$ for some $B \in \mathscr{B}'$, so necessarily $B \in \mathscr{B}_x$. Therefore $\mathscr{B}_x$ is a countable nhood base for $x$.

For each $B \in \mathscr{B}$, let $x_B \in B$. I claim that $\{ x_B \mid B \in \mathscr{B} \}$ is dense in $X$. Indeed, for any $x \in X$ and any open set $U$ containing $x$, there is $B \in \mathscr{B}$ such that $x \in B \subseteq U$. But $x_B \in B$ too, so $\{ x_B \mid B \in \mathscr{B} \}$ is a countable dense set.

Let $\mathscr{U}$ be an open cover of $X$. For each $U \in \mathscr{U}$ and $x \in U$, there is some $B_{x,U} \in \mathscr{B}$ such that $x \in B_{x,U} \subseteq U$. But $\{ B_{x,U} \mid U \in \mathscr{U}, x \in U \} \subseteq \mathscr{B}$, so this collection is countable. We may suppose $\{ B_{x,U} \mid U \in \mathscr{U}, x \in U \} = \{ B_{x_1,U_1}, B_{x_2,U_2}, \ldots \}$, so $U_1, U_2, \ldots$ is a countable subcover of $\mathscr{U}$.

Unfortunately, these are the only implications between the four properties.

Counterexamples

**Proposition.** First countability does not imply second countability, separability, or the Lindelöf property.

**Proof:** Consider the discrete topology on the real line. Each point $x \in \mathbb{R}$ has an nhood base consisting of just one set, namely $\{x\}$, so this space is first countable. But clearly it is not second countable, separable, or Lindelöf.

**Proposition.** The Lindelöf property does not imply second countability or separability.

**Proof:** Again consider the discrete topology on $\mathbb{R}$. Adjoin a new point $*$ and define the nhoods of $*$ to be of the form $\{x\} \cup F$, for finite sets $F$. The resulting space $X^*$ is Lindelöf (indeed, it is compact; $X^*$ is the one-point compactification of the discrete topology on $\mathbb{R}$), but it is not separable since all of the sets $\{x\}$ remain open for $x \neq *$.

**Proposition.** Neither separability nor the Lindelöf property imply first countability.
PROOF: This time consider the cofinite topology on the real line, call this space $X$, where the open sets are exactly the ones with finite complement. It is separable since the rationals are dense (in fact, every countable subset of $\mathbb{R}$ is dense). $X$ is Lindelöf since if $\mathcal{U}$ is any open cover then there is some open set $U \in \mathcal{U}$, and the complement of $U$ is finite. Finitely many further sets from $\mathcal{U}$ cover the remaining finite number of missed points, so $\mathcal{U}$ has a countable (actually finite) subcover.

Finally, $X$ is not first countable. If $\mathcal{B}$ is any countable collection of nhoods of 0 then $\{0\} \cup \bigcup_{B \in \mathcal{B}} \mathbb{R} \setminus B$ is countable, so there is some $x$ in its complement. Then $\mathbb{R} \setminus \{x\}$ is an open set containing zero that does not contain any element of $\mathcal{B}$ as a subset, so $\mathcal{B}$ cannot be an nhhood base of 0.

We can summarize our results succinctly in diagrammatic form, where solid lines indicate implication and dotted lines indicate lack of implication. We must still investigate the implications of separability on first countability and the Lindelöf property.

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**Moore Plane**

Let $B(x, r)$ denote the open ball centered at $x$ of radius $r$ (where the metric will be clear from the discussion). The Moore plane $\Gamma$ is the closed upper half plane $\mathbb{R} \times [0, \infty)$ with the following topology. For points $(x, y)$ with $y > 0$, take

$$\{B((x, y); t) \cap \Gamma \mid t > 0\}$$

as an nhhood base. For points of the form $(x, 0)$, take

$$\{B((x, t); t) \cup \{(x, 0)\} \mid t > 0\}$$

as an nhhood base. To reiterate, the open upper half plane gets the induced topology from $\mathbb{R}^2$, while nhhood bases for the points $z$ on the $x$-axis are given by circles tangent to the $x$-axis at $z$.

The Moore plane is interesting in part because it is a separable space with a nonseparable subspace. We exploit this fact in the proof of the following theorem.

**Theorem.** The Moore plane $\Gamma$ is separable and first countable, but not Lindelöf or second countable.

**Proof:** It is clear from the definition that the Moore plane is first countable. Points with rational coordinates form a dense set, so $\Gamma$ is separable. To finish the proof
it suffices to prove that the Moore plane is not Lindelöf, since if $\Gamma$ were second countable it would be Lindelöf. Consider

$$\mathcal{B} = \{B((x, y); \frac{y}{2}) \mid x, y \in \mathbb{R}, y > 0\} \cup \{B((x, 1); 1) \cup \{x\} \mid x \in \mathbb{R}\},$$

an open cover of $\Gamma$. None of the open sets around the points with positive $y$-coordinate contain any point on the $x$-axis, and no set in $\mathcal{B}$ contains more than one point on the $x$-axis, so $\mathcal{B}$ cannot have a countable subcover. □

The complete diagram of implications is seen to be as follows, where solid lines denote implication and dotted lines indicate lack of implication.

![Diagram](image)

**References**