Measures of Risk

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Contents

Contents **i**

1 Introduction **1**
1.1 Risk and risk measures 1
1.2 Valuations and acceptance sets 2
1.3 First examples of risk measures 5

2 Risk measurement over a single period **9**
2.1 Representation result for concave valuations 9
2.2 Continuity properties 11
2.3 Concave valuations on L^∞ 13
2.4 Shortfall risk, revisited 16
2.5 AV@R, revisited 17

3 Risk measurement over multiple periods **21**
3.1 Initial period valuations 21
3.2 Dynamic valuations, and further reading 24

A Numerical AV@R example **27**

References **29**

1 Introduction

1.1 Risk and risk measures

This essay discusses the problem of quantifying the *risk* of a financial position in a consistent way and surveys the literature in this area. Risk measurement over a single period and over multiple periods is discussed, with an emphasis on economic interpretations and tractable representations of risk measures. Many examples are given and special attention is given to the (currently popular) risk measure known as Value at Risk.

The problem of quantifying risk is of concern to the

> *regulator* who must disallow positions so risky that they may require the resources of the government in the event of catastrophe, e.g. as in the downfall of *Long-Term Capital Management*;
>
> *exchange's clearing firm* which must ensure that the promises of all parties of transactions are securely fulfilled;
>
> *investment manager* who must watch over his traders to be sure that they do not get in over their heads.

These regulating bodies will have some idea of *acceptable positions*, positions with an acceptable payoff. To remedy an unacceptable position, one could simply alter the position, or one could add to it some quantity of numéraire which would make the combined position acceptable. The cost of doing this is a good candidate for the measure of the risk of the position. Arguably a lot of information is lost in reducing the "risk" of a position down to a single number, but at the end of the day it is a binary "yes" or "no" decision on on whether to allow a risky position.

In the single period case a financial position is described by its corresponding *payoff profile*, a random variable on the set of possible *scenarios*. Let $\mathscr{X}$ be an $\mathbb{R}$-vector space of real valued functions on the set Ω of scenarios that contains the constant functions. The interpretation is that $X \in \mathscr{X}$ is the discounted final value of a *financial position* at the end of the time period. The measures of risk $\rho : \mathscr{X} \to \mathbb{R}$ that we will consider satisfy several natural axioms, originally put forth in [2] and [8].

If a position X is guaranteed to pay out more than a position Y, then surely X is less risky than Y. Whence we require risk measures to be *monotone*,

(M) $X \geq Y \implies \rho(X) \leq \rho(Y)$.

The perspective of a risk measure as a *capital requirement*, i.e. $\rho(X)$ being the least amount of cash which must be added to the position X to make it acceptable, leads to the requirement that risk measures are *translation invariant*, (TI) $\rho(X+m)=\rho(X)-m$ for $m\in\mathbb{R}$.

Discounted values are used (or equivalently, a risk free interest rate of zero is assumed) primarily so that the formula for translation invariance is uncomplicated. The m on the left hand side of (TI) represents an amount of cash at the end of the period, while the m on the right hand side represents the same amount at the beginning of the period. If an interest rate of $r>0$ is assumed and X were not discounted then (TI) would have to be expressed as $\rho(X+(1+r)m)=\rho(X)-m$.

It is well know that diversification leads to less risky portfolios. This is expressed mathematically by requiring that a risk measure is *convex*,

(C) $\rho(\lambda X+(1-\lambda)Y)\leq\lambda\rho(X)+(1-\lambda)\rho(Y)$ for $\lambda\in[0,1]$.

When measures of risk were investigated early on in [2], they considered only *coherent* risk measures, those convex risk measures which are also *positive homogeneous*,

(H) $\rho(\lambda X)=\lambda\rho(X)$ for $\lambda\geq 0$.

In many situations risk may grow in a non-linear way, so positive homogeneity is not necessary a realistic assumption. For example, in heavily leveraged positions there may be a significant risk due to lack of liquidity, which would not be a factor in smaller positions.

In the multi-period case information is revealed over time, and the financial position under consider may change as well. In this case the position is represented by a *stochastic process* and we take $\mathscr{X}$ to be the set of all such bounded processes. The axioms (M), (C), and (H) have obvious analogs in this setting, but (TI) may be extended consistently in different ways. Further, since information is revealed over time, our assessment of risk will likely change over time, so in this case we are really dealing with *risk measurement processes*. Multi-period risk measurement is currently an active area of research.

1.2 Valuations and acceptance sets

To eliminate excessive fiddling with signs, we consider instead the negatives of risk measures, called *valuations* or *risk-adjusted values*. In accordance with the monetary interpretation of risk measures, the valuation of a position $X\in\mathscr{X}$, $\pi(X)=-\rho(X)$, is the largest amount of cash we can remove from X and have it remain acceptable, with the understanding that a negative valuation implies that cash must be added. In this language, a *concave valuation* is one which satisfies

(M) $X\leq Y \implies \pi(X)\leq\pi(Y)$

(TI) $\pi(X+m)=\pi(Y)+m$ for all $m\in\mathbb{R}$

(C) $\pi(\lambda X+(1-\lambda)Y)\geq\lambda\pi(X)+(1-\lambda)\pi(Y)$ for $\lambda\in[0,1]$ (concavity).

And a *coherent valuation* is one which is also positive homogeneous

(H) $\pi(\lambda X)=\lambda\pi(X)$ for $\lambda\geq 0$.

It will be show that a concave valuation, under certain (weak) assumptions of continuity, can be represented as

$$\pi(X) = \inf_{Q \in \mathcal{Q}} \{\mathbb{E}^Q[X] + \alpha(Q)\} \tag{1.1}$$

where $\mathcal{Q}$ is some collection of probability measures and α is a "penalty" function on $\mathcal{Q}$, first defined in [8].

Lemma 1. *Any monotone, translation invariant valuation is Lipschitz with respect to the supremum norm.*

PROOF: Let π be such a valuation and $X, Y \in \mathcal{X}$. Then $X \le Y + \|X - Y\|$, so $\pi(X) \le \pi(Y) + \|X - Y\|$. Reversing the rôles of X and Y and combining the inequalities proves

$$|\pi(X) - \pi(Y)| \le \|X - Y\|. \qquad \square$$

Dual with the notion of a valuation is that of an *acceptance set*, a subset $\mathcal{A} \subseteq \mathcal{X}$ of *acceptable positions*. In their original paper [2], Artzner et al. preferred working with acceptance sets and derived the axioms of valuations from their properties. Given a valuation π, the associated acceptance set is

$$\mathcal{A}_\pi := \{X \in \mathcal{X} \mid \pi(X) \ge 0\},$$

the collection of positions which are acceptable because they do not require additional capital. If $\mathcal{A} \subseteq \mathcal{X}$ is a subset of acceptable positions then the associated valuation is given by

$$\pi_{\mathcal{A}}(X) = \sup\{m \in \mathbb{R} \mid X - m \in \mathcal{A}\},$$

making precise the idea that $\pi(X)$ is the largest amount of cash we can remove from X and still have it acceptable. There is a duality between valuations and their acceptance sets, outlined in the following two propositions, found in [9].

Proposition 2. *Let π be a valuation on $\mathcal{X}$.*

(i) *$\mathcal{A} := \mathcal{A}_\pi$ is non-empty and satisfies*

$$\inf\{m \in \mathbb{R} \mid m \in \mathcal{A}\} > -\infty, \tag{1.2}$$

and

$$X \in \mathcal{A}, Y \in \mathcal{X}, Y \ge X \implies Y \in \mathcal{A}. \tag{1.3}$$

Moreover, for $X \in \mathcal{A}$ and $Y \in \mathcal{X}$,

$$\{\lambda \in [0,1] \mid \lambda X + (1-\lambda)Y \in \mathcal{A}\} \tag{1.4}$$

is closed in $[0,1]$.

(ii) *π is a concave valuation if and only if $\mathcal{A}_\pi$ is convex and positive homogeneous if and only if $\mathcal{A}_\pi$ is a cone. In particular π is coherent if and only if $\mathcal{A}_\pi$ is a convex cone.*

(iii) *π can be recovered from $\mathcal{A}_\pi$, that is to say $\pi_{\mathcal{A}_\pi} = \pi$.*

PROOF:

(i) Notice that $\pi(m) = \pi(0) + m$, so $m \in \mathscr{A}_\pi$ if and only if $m \geq -\pi(0)$. The second assertion is clear, and the third assertion follows because $\lambda \mapsto \pi(\lambda X + (1-\lambda)Y)$ is continuous by 1—(1.4) is the preimage of the closed set $[0, \infty)$ in this function.

(ii) Convexity of $\mathscr{A}_\pi$ is exactly concavity of π, and similarly for whether $\mathscr{A}_\pi$ is a cone and positive homogeneity of π. The converses follow from part (ii) of 3 and part (iii) of this proposition.

(iii) For $X \in \mathscr{X}$,

$$\begin{aligned}\pi_{\mathscr{A}_\pi}(X) &= \sup\{m \in \mathbb{R} \mid X - m \in \mathscr{A}_\pi\} \\ &= \sup\{m \in \mathbb{R} \mid \pi(X - m) \geq 0\} \\ &= \sup\{m \in \mathbb{R} \mid \pi(X) \geq m\} \\ &= \pi(X)\end{aligned}$$

□

Proposition 3. *Let $\mathscr{A} \subseteq \mathscr{X}$ be non-empty and satisfy* (1.2) *and* (1.3)*.*

(i) $\pi_{\mathscr{A}}$ is translation invariant, monotone, and takes only finite values.

(ii) If $\mathscr{A}$ is convex then $\pi_{\mathscr{A}}$ is concave and if $\mathscr{A}$ is a cone then $\pi_{\mathscr{A}}$ is positive homogeneous. In particular if $\mathscr{A}$ is a convex cone then $\pi_{\mathscr{A}}$ is a coherent valuation.

(iii) $\mathscr{A} \subseteq \mathscr{A}_{\pi_{\mathscr{A}}}$, and if $\{\lambda \in [0,1] \mid \lambda X + (1-\lambda)Y \in \mathscr{A}\}$ is closed in $[0,1]$ for all $X \in \mathscr{A}$ and $Y \in \mathscr{X}$ then $\mathscr{A} = \mathscr{A}_{\pi_{\mathscr{A}}}$.

PROOF:

(i) That $\pi_{\mathscr{A}}$ is monotone and translation invariant is clear. To show that it takes only finite values, fix any $Y \in \mathscr{A}$. For any X there is m such that $X + m \geq Y$ since they are both bounded. Then

$$0 \leq \pi_{\mathscr{A}}(Y) \leq \pi_{\mathscr{A}}(X + m) = \pi_{\mathscr{A}}(X) + m$$

so $\pi_{\mathscr{A}}(X) \geq -m$. Notice that (1.2) is equivalent to $\pi_{\mathscr{A}}(0) < \infty$, so since there is m' such that $X - m' \leq 0$, $\pi_{\mathscr{A}}(X) \leq \pi_{\mathscr{A}}(0) + m' < \infty$.

(ii) Suppose $X_1, X_2 \in \mathscr{X}$ and $m_1, m_2 \in \mathbb{R}$ are such that $X_i - m_i \in \mathscr{A}$ for $i = 1, 2$. Then if $\mathscr{A}$ is convex

$$\begin{aligned}0 &\leq \pi_{\mathscr{A}}(\lambda(X_1 - m_1) + (1-\lambda)(X_2 - m_2)) \\ &= \pi_{\mathscr{A}}(\lambda X_1 + (1-\lambda)X_2) - (\lambda m_1 + (1-\lambda)m_2)\end{aligned}$$

and $\pi_{\mathscr{A}}$ is seen to be concave. The proof that $\lambda \pi_{\mathscr{A}}(X) \leq \pi_{\mathscr{A}}(\lambda X)$ for $\lambda \geq 0$ when $\mathscr{A}$ is a cone is analogous. To prove the converse, if $m > \pi_{\mathscr{A}}(X)$ then $X - m \notin \mathscr{A}$ and so $\lambda X - \lambda m \notin \mathscr{A}$ for $\lambda > 0$ since $\mathscr{A}$ is cone. Hence $\lambda m > \pi_{\mathscr{A}}(\lambda X)$, and $\pi_{\mathscr{A}}$ is seen to be positive homogeneous.

(iii) Clearly $\mathscr{A} \subseteq \mathscr{A}_{\pi_{\mathscr{A}}}$. Assume that $\mathscr{A}$ satisfies the closure property. We must show that if $X \notin \mathscr{A}$ then $\pi_{\mathscr{A}}(X) < 0$. Take $m > \|X\|$ so large that $m \in \mathscr{A}$ (there is such an m by (1.3) and the fact that $\mathscr{A}$ is non-empty). By the closure property, there is $\varepsilon \in (0,1)$ such that $\varepsilon m + (1-\varepsilon)X \notin \mathscr{A}$, so

$$0 \geq \pi_{\mathscr{A}}(\varepsilon m + (1-\varepsilon)X) = \pi_{\mathscr{A}}((1-\varepsilon)X) + \varepsilon m.$$

1 shows that $|\pi_{\mathscr{A}}(X) - \pi_{\mathscr{A}}((1-\varepsilon)X)| \leq \varepsilon \|X\|$, so

$$\pi_{\mathscr{A}}(X) \leq \pi_{\mathscr{A}}((1-\varepsilon)X) + \varepsilon \|X\| < \pi_{\mathscr{A}}((1-\varepsilon)X) + \varepsilon m \leq 0. \qquad \square$$

1.3 First examples of risk measures

Example 4.

(i) Fix $m \in \mathbb{R}$ and consider $\pi(X) := \inf X - m$. Then π is a concave valuation and the associated acceptance set is

$$\mathscr{A}_\pi = \{X \in \mathscr{X} \mid X \geq m\}.$$

The financial interpretation is that there is a fixed absolute lower bound on the magnitude of possible losses. In this case π is coherent if and only if $m = 0$.

(ii) (Trading in an incomplete market model, [8]) Suppose that $\mathscr{A}$ is an acceptance set and $\mathscr{C}$ is the convex cone of bounded claims attainable at zero cost in an incomplete market model. Then

$$\mathscr{A}_{\mathscr{C}} := \{X \in \mathscr{X} \mid X + f \in \mathscr{A} \text{ for some } f \in \mathscr{C}\}$$

is the collection of claims by which trading on the market with zero initial cost can be made acceptable for $\mathscr{A}$.

(iii) Consider the acceptance set

$$\mathscr{A} = \{X \in \mathscr{X} \mid \mathbb{E}^{Q_1}[X] \geq F(Q_1), \dots, \mathbb{E}^{Q_n}[X] \geq F(Q_n)\},$$

where the Q_i are *test measures* with corresponding *floors* $F(Q_i) \leq 0$. Then $\mathscr{A}$ is seen to be the intersection of affine subspaces of $\mathscr{X}$ each containing the positive orthant, so $\mathscr{A}$ satisfies (1.2), (1.3), and is convex. The associated valuation is coherent if and only if each of the floors are zero, making $\mathscr{A}$ a positive cone.

Example 5 (General capital requirements, [10]).
Let $\mathscr{A}$ be an acceptance set, $\mathscr{C} \subseteq \mathscr{X}$ be a set of *strategies*, $v : \mathscr{C} \to \mathbb{R}$, and let

$$\pi_{\mathscr{A},\mathscr{C},v}(X) := -\inf\{v(Y) \mid Y \in \mathscr{C}, X + Y \in \mathscr{A}\}.$$

Here v is the *cost* of putting a given strategy into action. Valuations of this type are translation invariant in that $\pi_{\mathscr{A},\mathscr{C},v}(X+Y) = \pi_{\mathscr{A},\mathscr{C},v}(X) + v(Y)$ for $Y \in \mathscr{C}$ when $\mathscr{C}$ is a subspace of $\mathscr{X}$. The entire theory presented in Chapter 2 of this essay may be extended to valuations of this general type, but for elegance of presentation, we consider only $\mathscr{C} = \mathbb{R}$ and $v = \mathrm{id}$.

For the next examples, fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Example 6 ($\mathbf{V@R}_\lambda$). Fix $0 < \lambda < 1$ and let

$$\mathscr{A}_\lambda := \{X \in L^\infty(\Omega, \mathscr{F}, \mathbb{P}) \mid \mathbb{P}(X < 0) \leq \lambda\}.$$

Then the associated risk measure is known as *value at risk*,

$$\mathrm{V@R}_\lambda(X) := -\sup\{m \in \mathbb{R} \mid \mathbb{P}[X < m] \leq \lambda\}.$$

V@R is the most widely used risk measure in current practise. It tries to answer the question "How bad can things get?" by noting that with confidence $1-\lambda$ the position X will not lose more than $\mathrm{V@R}_\lambda(X)$ over the time period. Unfortunately, V@R is not a convex measure of risk. It only takes into account the probability of loss and not the magnitude of possible losses that could occur. Also, careless use of V@R can discourage diversification—see Section 2.5 for a numerical example illustrating this phenomenon.

Example 7 ($\mathbf{AV@R}_\lambda$). *Average value at risk* is defined in terms of V@R and instead tries to answer the question "If things get bad, how much can we expect to lose?"

$$\mathrm{AV@R}_\lambda(X) := \frac{1}{\lambda}\int_0^\lambda \mathrm{V@R}_\gamma(X)d\gamma$$

It is also called *tail conditional expectation* in [2], *tail value at risk* in [4], *conditional value at risk* in [11], or also sometimes *expected shortfall*. AV@R is a coherent measure of risk, and so has a representation as in (1.1),

$$\mathrm{AV@R}_\lambda(X) := -\inf_{Q \in \mathscr{Q}_\lambda} \mathbb{E}^Q[X].$$

and we will show that the set of probability measures may be taken to be

$$\mathscr{Q}_\lambda = \left\{Q \ll \mathbb{P} \mid \|\tfrac{dQ}{d\mathbb{P}}\|_\infty \leq \tfrac{1}{\lambda}\right\},$$

Example 8 ($\mathbf{WCE}_\lambda$, [2]). Consider the following set of probability measures absolutely continuous with respect to $\mathbb{P}$,

$$\mathscr{Q}_\lambda := \{\mathbb{P}[\cdot \mid A] \mid A \in \mathscr{F}, \mathbb{P}[A] > \lambda\},$$

Then the associated coherent risk measure is the *worst conditional expectation* at level λ,

$$\mathrm{WCE}_\lambda(X) := -\inf_{Q \in \mathscr{Q}_\lambda} \mathbb{E}^Q[X] = -\inf_{\mathbb{P}[A] > \lambda} \mathbb{E}[X \mid A].$$

When the probability space is "rich enough" the worst conditional expectation is equal to the average value at risk.

Example 9 (Shortfall risk). Let $u : \mathbb{R} \to \mathbb{R}$ be a *utility function*, i.e. a concave increasing function. Then $\ell(x) = -u(-x)$ is a convex increasing function, which may be interpreted as representing an investor's aversion to loss. The *expected loss* of a position X is $\mathbb{E}[\ell(-X)]$. If $\ell \equiv 0$ on $(-\infty, 0]$ (so $u \equiv 0$ on $[0, \infty)$) then

$$\mathbb{E}[\ell(-X)] = \mathbb{E}[\ell(X^-)],$$

the *shortfall risk*. For generality we will not assume this of ℓ. For x_0 in the interior of $\ell(\mathbb{R})$ define the acceptance set

$$\mathscr{A}(x_0) := \{X \in \mathscr{X} \mid \mathbb{E}[\ell(-X)] \leq x_0\} = \{X \in \mathscr{X} \mid \mathbb{E}[u(X)] \geq -x_0\},$$

the collection of positions with expected loss under Q bounded by x_0. The associated measure of risk is also known as *shortfall risk*, and it will be discussed in greater detail in Section 2.4.

2 Risk measurement over a single period

At first we discuss risk measurement over a single time period, where no changes to the position may be made nor any new information is revealed until the end of the period. Fix a σ-algebra $\mathscr{F}$ on Ω. The σ-algebra is interpreted as the collection of events which can be distinguished at the end of the period. Take $\mathscr{X}$ to be the collection of all bounded measurable functions on the measurable space $(\Omega, \mathscr{F})$. Equipped with the uniform norm, $\mathscr{X}$ is a Banach space with dual space $\mathscr{M}$, the collection of all finitely additive signed measures on $(\Omega, \mathscr{F})$ of bounded variation (see Chapter 13 of [1]) i.e. functions $Q : \mathscr{F} \to \mathbb{R}$ such that

(i) $Q[\varnothing] = 0$,

(ii) $Q[A \cup B] = Q[A] + Q[B]$ for every pair of disjoint sets $A, B \subseteq \Omega$, and

(iii) $|Q|[\Omega] := \sup \sum_i |Q[E_i]| < \infty$, where the supremum is taken over all partitions of Ω into finitely many measurable subsets.

Let $\mathscr{M}_{1,f}$ denote the collection of non-negative, normalized, finitely additive set functions, or "finitely additive probabilities," and let $\mathscr{M}_1$ denote the collection of probability measures.

2.1 Representation result for concave valuations

We prove a representation result for concave valuations in terms of $\mathscr{M}_{1,f}$, first investigated in [8]. Let $\alpha : \mathscr{M}_{1,f} \to \mathbb{R} \cup \{\infty\}$ be a function for which

$$\inf_{Q \in \mathscr{M}_{1,f}} \alpha(Q) \in \mathbb{R}.$$

For each $Q \in \mathscr{M}_{1,f}$, $X \mapsto \mathbb{E}^Q[X] + \alpha(Q)$ is monotone, translation invariant, and concave since expectation is linear. These properties are preserved upon taking infima, so

$$\pi(X) := \inf_{Q \in \mathscr{M}_{1,f}} \{\mathbb{E}^Q[X] + \alpha(Q)\}$$

is a concave valuation on $\mathscr{X}$ for which

$$\pi(0) = \inf_{Q \in \mathscr{M}_{1,f}} \alpha(Q). \tag{2.1}$$

Such a function α is called a *penalty function* for π on $\mathscr{M}_{1,f}$, and we say what π is *represented* by α on $\mathscr{M}_{1,f}$. Note that this representation is far from unique.

Theorem 10. *Any concave valuation $\pi : \mathscr{X} \to \mathbb{R}$ is of the form*

$$\pi(X) = \inf_{Q \in \mathscr{M}_{1,f}} \{\mathbb{E}^Q[X] + \alpha_{min}(Q)\} \tag{2.2}$$

where the penalty function $\alpha_{min} : \mathscr{M}_{1,f} \to \mathbb{R} \cup \{\infty\}$ is given by

$$\alpha_{min}(Q) := \sup_{X \in \mathscr{A}_\pi} \mathbb{E}^Q[-X]. \tag{2.3}$$

Moreover, α_{min} is the minimal penalty function which represents π.

PROOF: By translation invariance, $\pi(X - \pi(X)) = 0$ so $X - \pi(X) \in \mathscr{A}_\pi$. Therefore for each $Q \in \mathscr{M}_{1,f}$,

$$\alpha_{min}(Q) \geq \mathbb{E}^Q[-X + \pi(X)] = -\mathbb{E}^Q[X] + \pi(X),$$

so

$$\pi(X) \leq \inf_{Q \in \mathscr{M}_{1,f}} \{\mathbb{E}^Q[X] + \alpha_{min}(Q)\}.$$

Fix $X \in \mathscr{X}$. We now construct $Q_X \in \mathscr{M}_{1,f}$ such that

$$\pi(X) \geq \mathbb{E}^{Q_X}[X] + \alpha_{min}(Q_X),$$

which will complete the proof of the representation (2.2). By translation invariance and linearity of expectation, we may assume that $\pi(X) = 0$, so X is not contained in the non-empty, convex, open set

$$\mathscr{B} := \{Y \in \mathscr{X} \mid \pi(Y) > 0\} = \pi^{-1}((0, \infty)).$$

Furthermore, we may assume without loss of generality that $\pi(0) = 0$. (It suffices to show the minimal penalty functions are what they should be, and this is clear.) By the geometric version of the Hahn-Banach Separation Theorem (see [15]), there is a continuous linear functional $\varphi \in \mathscr{M}$ such that

$$\varphi(Y) > b \geq \varphi(X)$$

for all $Y \in \mathscr{B}$, where $b = \inf_{Y \in \mathscr{B}} \varphi(Y)$. If $Y \geq 0$ then for any $\lambda > 0$ we have

$$\pi(1 + \lambda Y) = 1 + \pi(\lambda Y) \geq 1 > 0$$

so $1 + \lambda Y \in \mathscr{B}$. Therefore

$$\varphi(X) \leq \varphi(1 + \lambda Y) = \varphi(1) + \lambda\varphi(Y),$$

which could not hold if $\varphi(Y) < 0$, so $\varphi(Y) \geq 0$ for all $Y \geq 0$. Now since φ is not identically zero, there is $Z \in B_1(0)$ such that $\varphi(Z) > 0$. If $Z = Z^+ - Z^-$ in terms of its positive and negative parts, by positivity of φ we have $0 \leq \varphi(Z^-) < \varphi(Z^+)$ and $0 \leq \varphi(1 - Z^+)$, so

$$\varphi(1) = \varphi(1 - Z^+) + \varphi(Z^+) > 0.$$

Whence $\frac{1}{\varphi(1)}\varphi$ is a positive, normalized linear functional, so there is $Q_X \in \mathscr{M}_{1,f}$ such that

$$\mathbb{E}^{Q_X}[Z] = \frac{\varphi(Z)}{\varphi(1)}$$

for all $Z \in \mathscr{X}$. Now $\mathscr{B} \subseteq \mathscr{A}_\pi$, so

$$\alpha_{min}(Q_X) = \sup_{Z \in \mathscr{A}_\pi} \mathbb{E}^{Q_X}[-Z] \geq \sup_{Y \in \mathscr{B}} \mathbb{E}^{Q_X}[-Y] = -\frac{b}{\varphi(1)}.$$

But for any $Z \in \mathscr{A}_\pi$, $Z + \varepsilon \in \mathscr{B}$ for every $\varepsilon > 0$, so

$$\alpha_{min}(Q_X) \leq \sup_{Y \in \mathscr{B}} \mathbb{E}^{Q_X}[-(Y - \varepsilon)] = -\frac{b}{\varphi(1)} + \varepsilon,$$

which proves that $\alpha_{min}(Q_X) = -\frac{b}{\varphi(1)}$. Therefore

$$\mathbb{E}^{Q_X}[X] + \alpha_{min}(Q_X) = \frac{1}{\varphi(1)}(\varphi(X) - b) \leq 0 = \pi(X),$$

as required.

Finally, suppose that α is another penalty function that represents π. Then for every $X \in \mathscr{X}$ and $Q \in \mathscr{M}_{1,f}$,

$$\pi(X) \leq \mathbb{E}^Q[X] + \alpha(Q)$$

so

$$\alpha(Q) \geq \sup_{X \in \mathscr{X}} \{\mathbb{E}^Q[-X] + \pi(X)\} \geq \sup_{X \in \mathscr{A}_\pi} \{\mathbb{E}^Q[-X] + \pi(X)\} \geq \alpha_{min}(Q). \qquad \square$$

Remark. We have proved further that the infimum in the representation (2.2) of π is obtained (on $\mathscr{M}_{1,f}$).

Example 11. Let $\{\pi_i\}_{i \in I}$ be a collection of convex valuations represented by penalty functions $\{\alpha_i\}_{i \in I}$. If $\inf_{i \in I} \pi_i(0) > -\infty$ then

$$\pi(X) := \inf_{i \in I} \pi_i(X)$$

is a convex valuation represented by

$$\alpha(Q) := \inf_{i \in I} \alpha_i(Q).$$

Notice that α is not necessarily the minimal penalty function of π.

2.2 Continuity properties

Next we derive some properties of those convex valuations which admit a representation as in (2.2) in terms of *probability measures* only, i.e. those which can be represented by a penalty function which takes the value ∞ outside of the set $\mathscr{M}_1$ of probability measures on $(\Omega, \mathscr{F})$. This section follows [9].

A necessary condition is that the valuation is *continuous from above*, i.e. $X_n \searrow X$ point-wise implies that $\pi(X_n) \to \pi(X)$.

Proposition 12. *Let π be a concave valuation that can be represented by a penalty function taking the value ∞ outside of $\mathscr{M}_1$. Then π is continuous from above.*

Proof: For $X_n \searrow X$ and any $Q \in \mathcal{M}_1$, by the bounded convergence theorem $\mathbb{E}^Q[X_n] \to \mathbb{E}^Q[X]$. Therefore

$$\begin{aligned}\pi(X) &= \inf_{Q\in\mathcal{M}_1}\{\mathbb{E}^Q[X] + \alpha(Q)\}\\ &= \inf_{Q\in\mathcal{M}_1}\{\lim_{n\to\infty}\mathbb{E}^Q[X_n] + \alpha(Q)\}\\ &\geq \limsup_{n\to\infty}\inf_{Q\in\mathcal{M}_1}\{\mathbb{E}^Q[X_n] + \alpha(Q)\}\\ &= \limsup_{n\to\infty}\pi(X_n)\end{aligned}$$

By monotonicity $\pi(X_n) \geq \pi(X_{n+1}) \geq \pi(X)$ for all n, so

$$\pi(X) \geq \limsup_{n\to\infty}\pi(X_n) = \lim_{n\to\infty}\pi(X_n) \geq \pi(X).$$

□

A (strong) sufficient condition is that the valuation be *continuous from below*, i.e. $X_n \nearrow X$ point-wise implies that $\pi(X_n) \to \pi(X)$.

Theorem 13. *Let π be a concave valuation that is continuous from below. Let α be any penalty function representing π on $\mathcal{M}_{1,f}$. Then α takes the value ∞ outside of $\mathcal{M}_1$.*

Proof: For $c > \pi(0) = \inf_{Q\in\mathcal{M}_{1,f}}\alpha(Q)$, let

$$\Lambda_c := \{Q \in \mathcal{M}_{1,f} \mid \alpha(Q) \leq c\}.$$

We will show that the following are equivalent for any sequence (X_n) in $\mathcal{X}$ with $0 \leq X_n \leq 1$ for all n.

(i) $\pi(\lambda X_n) \to \pi(\lambda)$ for all $\lambda \geq 1$.

(ii) $\inf_{Q\in\Lambda_c}\mathbb{E}^Q[X_n] \to 1$ for all $c > \pi(0)$.

Given this equivalence, recall that $Q \in \mathcal{M}_{1,f}$ is σ-additive if and only if $Q(A_n) \nearrow 1$ for any increasing sequence of events $A_n \in \mathcal{F}$ such that $\bigcup_n A_n = \Omega$ (see [14] Lemma II.4.2). Letting A_n be such sequence and $X_n := \mathbf{1}_{A_n}$, we have $\lambda X_n \nearrow \lambda$ point-wise, so $\pi(\lambda X_n) \to \pi(\lambda)$ by continuity from below. By the equivalence, $\inf_{Q\in\Lambda_c} Q(X_n) \to 1$, so $\Lambda_c \subseteq \mathcal{M}_1$. Therefore

$$\{Q \in \mathcal{M}_{1,f} \mid \alpha(Q) < \infty\} = \bigcup_{n>\pi(0)}\Lambda_n \subseteq \mathcal{M}_1,$$

and α is concentrated on the set of probability measures.

Let (X_n) be a sequence in $\mathcal{X}$ with $0 \leq X_n \leq 1$. Notice that for all $Q \in \Lambda_c$, since α represents π, $c \geq \alpha(Q) \geq \pi(\lambda X) - \mathbb{E}^Q[\lambda X]$, so

$$\inf_{Q\in\Lambda_c}\mathbb{E}^Q[X] \geq \frac{1}{\lambda}(\pi(\lambda X) - c).$$

Assuming (X_n) satisfies (i),

$$\liminf_{n\to\infty}\inf_{Q\in\Lambda_c}\mathbb{E}^Q[X_n] \geq \lim_{n\to\infty}\frac{1}{\lambda}(\pi(\lambda X_n) - c) = \frac{\pi(\lambda) - c}{\lambda} = 1 + \frac{\pi(0) - c}{\lambda},$$

so letting $\lambda \to \infty$ proves (ii).

Conversely, notice that for all n,

$$\pi(\lambda) \geq \pi(\lambda X_n) = \inf_{Q \in \mathcal{M}_{1,f}} \{\mathbb{E}^Q[\lambda X_n] + \alpha(Q)\}.$$

Since $\mathbb{E}^Q[\lambda X_n] \geq 0$ for all Q, those Q which matter in the infimum satisfy

$$\alpha(Q) \leq \pi(\lambda) = \pi(0) + \lambda =: c.$$

Hence, $\pi(\lambda X_n) = \inf_{Q \in \Lambda_c} \{\mathbb{E}^Q[\lambda X_n] + \alpha(Q)\} \to \lambda + \pi(0) = \pi(\lambda)$ as $n \to \infty$, recalling (2.1). □

Remark.

(i) It follows that a concave valuation π that is continuous from below is continuous from above, and so $\pi(X_n) \to \pi(X)$ whenever $X_n \to X$ is a bounded sequence converging point-wise.

(ii) The condition of continuity from below can be weakened considerably to continuity from below on $C_b(\Omega) \subset \mathcal{X}$ when Ω is a complete separable metric space and $\mathcal{F}$ is the Borel σ-algebra.

2.3 Concave valuations on L^∞

As before, the interpretation of Ω is as the collection of all possible scenarios, and the interpretation of $\mathcal{F}$ is as the collection of events that can be distinguished at the time of observation. However, the interpretation of a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is not so clear. In finance and economics probabilities are subjective. Regulators and the firms they supervise will likely have different ideas of these probabilities, and there may be disagreement between various branches of the same firm, e.g. between the risk manager and the trading desk. In [6], it is argued that it is reasonable to believe that a consensus can be reached regarding which events are impossible. Mathematically, this gives us an interpretation of the collection of $\mathbb{P}$-null sets, and we are justified to work in spaces which are invariant under replacing $\mathbb{P}$ by an equivalent probability measure, such as $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, the space of equivalence classes of bounded measurable functions, and $L^0(\Omega, \mathcal{F}, \mathbb{P})$, the space of equivalence classes of finite-valued measurable functions. $L^p(\Omega, \mathcal{F}, \mathbb{P})$ for $p \in (1, \infty)$ is not such a space, since $\mathbb{E}[X] < \infty$ does not imply that $\mathbb{E}^Q[X] < \infty$ even for an equivalent probability measure Q.

For the rest of this section we consider only those valuations π for which $\pi(X) = \pi(Y)$ if $X = Y$ a.s., i.e. those valuations that may be taken to be functions on $L^\infty := L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. In [6], Delbaen shows that there are no real-valued coherent valuations defined on L^0 (as a consequence of the fact that L^0 has trivial dual space, as a topological vector space equipped with convergence in probability). He suggests that the domain of coherent valuations be enlarged to $\mathbb{R} \cup \{-\infty\}$, where the interpretation of $\pi(X) = -\infty$ is that a position is absolutely unacceptable. The theory of these extended valuations is an extension of the L^∞ theory and is not discussed here. Let $\mathcal{M}_1(\mathbb{P})$ denote the collection of probability measures absolutely continuous with respect to $\mathbb{P}$. This section follows [6] and [9].

Lemma 14. *Let π be a concave valuation on L^∞, represented by a penalty function α as in* (2.2)*. Then $\alpha(Q) = \infty$ for any probability measure Q not absolutely continuous with respect to $\mathbb{P}$.*

PROOF: Let $Q \in \mathcal{M}_1 \setminus \mathcal{M}_1(\mathbb{P})$. Then there is $A \in \mathcal{F}$ such that $Q[A] > 0$ but $\mathbb{P}[A] = 0$. Take any $X \in \mathcal{A}_\pi$ and define $X_n := X - n\mathbf{1}_A$. Then clearly $\pi(X_n) = \pi(X) \geq 0$, so $X_n \in \mathcal{A}_\pi$ for all n, so

$$\alpha(Q) \geq \alpha_{min}(Q) \geq \mathbb{E}^Q[-X_n] = \mathbb{E}^Q[-X] + nQ[A] \to \infty$$

as $n \to \infty$, so $\alpha(Q) = \infty$. □

A valuation π on L^∞ is said to have the *Fatou property* if for any uniformly bounded sequence $(X_n)_{n\geq 1} \subseteq L^\infty$ which converges a.s. to $X \in L^\infty$ we have

$$\pi(X) \geq \limsup_n \pi(X_n).$$

Theorem 15. *Let π a concave valuation on L^∞. The following are equivalent.*

(i) π is a represented by a penalty function on $\mathcal{M}_1(\mathbb{P})$.

(ii) π has the Fatou property.

(iii) π is continuous from above, i.e. $X_n \searrow X$ a.s. implies $\pi(X_n) \to \pi(X)$.

(iv) The acceptance set $\mathcal{A}_\pi$ is weak-closed in L^∞, i.e. $\mathcal{A}_\pi$ is closed in the $\sigma(L^\infty, L^1)$ topology.*

PROOF: The implications (i) implies (ii) and (ii) implies (iii) are proved as in the proof of 12, replacing point-wise convergence with a.s.-convergence. In fact, (iii) implies (ii) as well, since if $X_n \to X$ a.s. then $Y_n := \sup_{m\geq n} X_m$ is decreasing and converges a.s. to X. We have $\pi(X_n) \leq \pi(Y_n)$ for all n, so

$$\limsup_{n\to\infty} \pi(X_n) \leq \lim_{n\to\infty} \pi(Y_n) = \pi(X).$$

To prove (iii) implies (iv), recall that the Krein-Smulian Theorem says that it suffices to show that $C_r := \mathcal{A}_\pi \cap B_r$ is $\sigma(L^\infty, L^1)$-closed for all $r > 0$, where B_r is the closed ball of radius r in L^∞. Consider C_r as a subset of L^1. If (X_n) is a sequence in C_r converging to $X \in L^1$ in the norm topology then it has a subsequence that converges a.s., so the Fatou property implies that $X \in C_r$, and C_r is closed in $(L^1, \|\cdot\|_1)$. But C_r is convex, so C_r is closed in $(L^1, \sigma(L^1, L^\infty))$ (see Chapter II of [15]). Further, since B_r is compact in $(L^1, \sigma(L^1, L^\infty))$, C_r is compact as well. To conclude, notice that L^∞ embeds into $(L^\infty)^*$ via $Y \mapsto$ "$Z \mapsto \mathbb{E}[YZ]$", and as a family of linear functionals it separates the points of L^∞. Whence we may consider the $\sigma(L^\infty, L^\infty)$ topology on L^∞. Now the natural injection of B_r, considered as a subspace of $(L^1, \sigma(L^1, L^\infty))$, into $(L^\infty, \sigma(L^\infty, L^\infty))$ is continuous, so C_r (as a subspace of $(L^\infty, \sigma(L^\infty, L^\infty))$) is compact and hence closed. Finally, $\sigma(L^\infty, L^\infty)$ is contained in $\sigma(L^\infty, L^1)$, so C_r is $\sigma(L^\infty, L^1)$-closed in L^∞.

To prove (iv) implies (i) we argue from the Hahn-Banach Theorem, as in the proof of 10. From the first few lines of that proof,

$$\pi(X) \leq \inf_{Q\in\mathcal{M}_1(\mathbb{P})} \{\mathbb{E}^Q[X] + \alpha_{min}(Q)\},$$

so it suffices to show that whenever $m < \inf_{Q\in\mathcal{M}_1(\mathbb{P})}\{\mathbb{E}^Q[X] + \alpha_{min}(Q)\}$ we have $m \leq \pi(X)$, or equivalently that $X - m \in \mathcal{A}_\pi$. Suppose for contradiction that $X - m \notin \mathcal{A}_\pi$. Then by a geometric version of the Hahn-Banach theorem on the locally convex space $(L^\infty, \sigma(L^\infty, L^1))$ applied to the non-empty convex closed set $\mathscr{A}_\pi$ and the point $X - m$ (see [15]), there is a linear functional φ such that

$$b := \inf_{Y \in \mathscr{A}_\pi} \varphi(Y) > \varphi(X - m).$$

Now the dual of $(L^\infty, \sigma(L^\infty, L^1))$ is L^1 (see Chapter IV of [15]), so φ is of the form $Y \mapsto \mathbb{E}[YZ]$ for some $Z \in L^1$. Furthermore, $Z \geq 0$. Indeed, for $Y \geq 0$, $\pi(\lambda Y - \pi(0)) \in \mathscr{A}_\pi$ for $\lambda \geq 0$. Therefore

$$-\infty < \varphi(X - m) < \varphi(\lambda Y - \pi(0)) = \lambda\varphi(Y) - \varphi(\pi(0)),$$

and taking $\lambda \to \infty$ shows that $\varphi(Y) \geq 0$, and hence $Z \geq 0$ since $\mathbb{E}[YZ] \geq 0$ for every $Y \geq 0$. Moreover, φ is not identically zero, so $\mathbb{P}[Z > 0] > 0$. Thus

$$\frac{dQ_m}{d\,\mathbb{P}} = \frac{Z}{\mathbb{E}[Z]}$$

defines a probability measure absolutely continuous with respect to $\mathbb{P}$. Then

$$\alpha_{min}(Q_m) = \sup_{Y \in \mathscr{A}_\pi} \mathbb{E}^{Q_m}[-Y] = -\frac{b}{\mathbb{E}[Z]},$$

but

$$\mathbb{E}^{Q_m}[X] - m = \frac{\varphi(X - m)}{\mathbb{E}[Z]} < \frac{b}{\mathbb{E}[Z]} = -\alpha_{min}(Q_m)$$

so $m > \mathbb{E}^{Q_m}[X] + \alpha_{min}(Q_m)$, a contradiction. Whence $X - m \in \mathscr{A}_\pi$, and π is represented by a penalty function on $\mathscr{M}_1(\mathbb{P})$. □

Remark. In [4] the elements of $\mathscr{Q}$ in the representation

$$\pi(X) = \inf_{Q \in \mathscr{Q}} \{\mathbb{E}^Q[X] + \alpha_{min}(Q)\}$$

with $\mathscr{Q} \subseteq \mathscr{M}_1(\mathbb{P})$ are referred to as *test probabilities* or *generalized scenarios*.

In practise we are mostly concerned with those valuations that are *relevant*, i.e. those π for which $\pi(-\mathbf{1}_A) < \pi(0)$ for all $A \in \mathscr{F}$ with $\mathbb{P}[A] > 0$.

Corollary 16. *A coherent valuation π on L^∞ can be represented on a set $\mathscr{Q} \subseteq \mathscr{M}_1(\mathbb{P})$ if and only if the equivalent conditions of 15 are satisfied. Furthermore, the maximal representing subset of $\mathscr{M}_1(\mathbb{P})$ is*

$$\mathscr{Q}_{max} := \{Q \in \mathscr{M}_1(\mathbb{P}) \mid \alpha_{min}(Q) = 0\},$$

and π is relevant if and only if $\mathscr{Q}_{max} \approx \mathbb{P}$, i.e. $\mathbb{P}[A] = 0$ if and only if $Q[A] = 0$ for all $Q \in \mathscr{Q}_{max}$, for all $A \in \mathscr{F}$.

Proof: Recall that the acceptance set of a coherent valuation is a (convex) cone, so by (2.3), for any $\lambda > 0$ and $Q \in \mathscr{M}_{1,f}$,

$$\alpha_{min}(Q) = \sup_{X \in \mathscr{A}_\pi} \mathbb{E}^Q[-X] = \sup_{\lambda X \in \mathscr{A}_\pi} \mathbb{E}^Q[-\lambda X] = \lambda\alpha_{min}(Q).$$

Therefore α_{min} takes only the values 0 and ∞. The relevance condition is clear. □

2.4 Shortfall risk, revisited

We revisit 9 in greater detail and give an explicit formula for the minimal penalty function in terms of the *conjugate function* of the *loss function* ℓ. For this section ℓ is a convex, increasing, non-constant function, x_0 is a fixed interior point of $\ell(\mathbb{R})$, and

$$\mathscr{A} := \mathscr{A}(x_0) = \{X \in \mathscr{X} \mid \mathbb{E}[\ell(-X)] \le x_0\}.$$

Proposition 17. *The acceptance set $\mathscr{A}$ defines a concave valuation $\pi := \pi_{\mathscr{A}}$ that is continuous from below.*

PROOF: $\mathscr{A}$ is convex and satisfies the conditions of 3 so π is a concave valuation. To show that π is continuous from below, first notice that

$$\pi(X) = \sup\{m \in \mathbb{R} \mid \mathbb{E}[\ell(m - X)]\}$$

is the unique solution to the equation

$$\mathbb{E}[\ell(z - X)] = x_0. \tag{2.4}$$

Indeed, the finite-value convex function ℓ is necessarily continuous, so

$$z \mapsto \mathbb{E}[\ell(z - X)]$$

is a continuous function for fixed $X \in \mathscr{X}$ (by dominated convergence). X is bounded, so there is a solution to (2.4) (by the intermediate value theorem) and by continuity $\pi(X)$ is a solution. It is the unique solution since ℓ is strictly increasing on $(x_0 - \varepsilon, \infty)$ for some $\varepsilon > 0$.

Suppose that $X_n \nearrow X$ point-wise. Then $\pi(X_n)$ increases to some (finite) limit L. By continuity of ℓ and dominated convergence

$$\mathbb{E}[\ell(\pi(X_n) - X_n)] \to \mathbb{E}[\ell(L - X)].$$

But each of the approaching expectations is x_0, so

$$\mathbb{E}[\ell(L - X)] = x_0$$

and by uniqueness $L = \pi(X)$. □

Assuming ℓ is not too pathological, π is well-defined on L^∞, so any penalty function representing π is concentrated on $\mathscr{M}_1(\mathbb{P})$, by 13 since π is continuous from below and 14. In this case we have the following theorem.

Theorem 18. *The minimal penalty function representing $\pi_{\mathscr{A}}$ is given by*

$$\alpha_{min}(Q) = \inf_{\lambda > 0} \frac{1}{\lambda} \left(x_0 + \mathbb{E}\left[\ell^* \left(\lambda \frac{dQ}{d\mathbb{P}} \right) \right] \right)$$

for $Q \in \mathscr{M}_1(\mathbb{P})$, where $\ell^*(z) := \sup_{x \in \mathbb{R}}\{zx - \ell(x)\}$.

PROOF: See [8] for the proof of this fact. □

Example 19. Take $\ell(x) = e^{\gamma x}$. Then $\ell^*(z) = \frac{z}{\gamma}\log\frac{z}{\gamma} - \frac{z}{\gamma}$ and by 18 the minimal penalty function giving the associated shortfall risk measure is

$$\begin{aligned}
\alpha_{min}(Q) &= \inf_{\lambda>0}\frac{1}{\lambda}\left(x_0 + \mathbb{E}\left[\frac{\lambda}{\gamma}\frac{dQ}{dP}\log\frac{\lambda}{\gamma}\frac{dQ}{dP} - \frac{\lambda}{\gamma}\frac{dQ}{dP}\right]\right)\\
&= \inf_{\lambda>0}\frac{1}{\lambda}\left(x_0 + \frac{\lambda}{\gamma}\mathbb{E}\left[\frac{dQ}{dP}\log\frac{dQ}{dP}\right] + \left(\frac{\lambda}{\gamma}\log\frac{\lambda}{\gamma} - \frac{\lambda}{\gamma}\right)\right)\\
&= \inf_{\lambda>0}\frac{1}{\gamma}\left(\frac{\gamma x_0}{\lambda} + H(Q\mid P) + \log\frac{\lambda}{\gamma} - 1\right)\\
&= \frac{1}{\gamma}\left(H(Q\mid P) + \log x_0 + (\gamma - \log\gamma - 1)\right),
\end{aligned}$$

where $H(Q \mid P)$ is the *relative entropy*.

Example 20. For $p > 1$ take $\ell(x) = \frac{1}{p}x^p$ on $[0,\infty)$ and $\ell \equiv 0$ on $(-\infty,0]$. Then $\ell^*(z) = \frac{1}{q}z^q$ on $[0,\infty)$ and $\ell^* \equiv \infty$ on $(-\infty,0]$. For any $Q \ll \mathbb{P}$, let $\varphi = \frac{dQ}{d\mathbb{P}}$. The minimal penalty function takes the value ∞ if $\varphi \notin L^q(\mathbb{P})$, and otherwise

$$\begin{aligned}
\alpha^p_{min}(Q) &= \inf_{\lambda>0}\frac{1}{\lambda}\left(x_0 + \mathbb{E}[\tfrac{1}{q}\lambda^q\varphi^q]\right)\\
&= \inf_{\lambda>0}\frac{1}{\lambda}\left(x_0 + \tfrac{1}{q}\lambda^q\|\varphi\|_q^q\right)\\
&= \|\varphi\|_q(px_0)^{\frac{1}{p}}
\end{aligned}$$

Letting $p \searrow 1$ we also get a formula for the minimal penalty function associated with $\ell(x) = x^+$, namely $\alpha^1_{min}(Q) = x_0\|\varphi\|_\infty$.

2.5 AV@R, revisited

We revisit AV@R and V@R in greater detail, give equivalent formulations in terms of quantiles, and discuss a specific numerical example. This section follows [9].

A λ-*quantile* of X on $(\Omega,\mathscr{F},\mathbb{P})$ is a real number q such that

$$\mathbb{P}[X < q] \le \lambda \le \mathbb{P}[X \le q].$$

The collection of λ-quantiles forms a closed interval $[q^-_\lambda(X), q^+_\lambda(X)]$, and V@R$_\lambda$ may alternatively be defined as V@R$_\lambda(X) := -q^+_\lambda(X)$. In practise this is how the V@R is usually computed, using historical data (see [11]).

Lemma 21. *Let $X \in L^\infty$, $\lambda \in (0,1)$, and q be any λ-quantile. Then*

$$AV@R_\lambda(X) = \frac{1}{\lambda}\mathbb{E}[(q-X)^+] - q \tag{2.5}$$

and further,

$$AV@R_\lambda(X) = \frac{1}{\lambda}(\mathbb{E}[-X\mathbf{1}_{\{X\le q\}}] + q(\mathbb{P}[X \le q] - \lambda)). \tag{2.6}$$

PROOF: Define $\tilde{q} : [0,1] \to \mathbb{R}$ by

$$\tilde{q}(x) := \begin{cases} q & \text{if } x = \lambda \\ q_x^+(X) & \text{otherwise.} \end{cases}$$

Lebesgue measure on $[0,1]$ has no atoms, so $\tilde{q}$ has the same distribution as X, and

$$-\int_0^\lambda \mathrm{V@R}_\gamma(X) d\gamma = \int_0^\lambda \tilde{q}(x) dx.$$

Now $\tilde{q}$ is increasing and $\tilde{q}(\lambda) = q$, so

$$[0,\lambda] \subseteq \{\tilde{q} \le q\}.$$

Notice that $x > \lambda$ and $\tilde{q}(x) \le q$ together imply that $\tilde{q}(x) = q$. Putting it all together,

$$\begin{aligned}
&\lambda \mathrm{AV@R}_\lambda(X) \\
&= -\int_0^1 \tilde{q}(x) \mathbf{1}_{\{\tilde{q}(x) \le q\}} dx + q \mathrm{Leb}[\{x \mid \tilde{q}(x) \le q, x > \lambda\}] \\
&= \mathbb{E}[-X \mathbf{1}_{\{X \le q\}}] + q(\mathrm{Leb}[\{x \mid \tilde{q}(x) \le q\}] - \mathrm{Leb}[\{x \mid \tilde{q}(x) \le q, x \le \lambda\}]) \\
&= \mathbb{E}[-X \mathbf{1}_{\{X \le q\}}] + q(\mathbb{P}[X \le q] - \lambda),
\end{aligned}$$

which is (2.6). Finally, (2.5) follows from the fact that

$$\mathbb{E}[(q-X)^+] = \mathbb{E}[-X \mathbf{1}_{\{X \le q\}}] + q\,\mathbb{P}[X \le q]. \qquad \square$$

Theorem 22. *For $\lambda \in (0,1)$, $AV@R_\lambda$ is a coherent measure of risk that has a representation*

$$AV@R_\lambda(X) = -\inf_{Q \in \mathcal{Q}_\lambda} \mathbb{E}^Q[X]$$

where $\mathcal{Q}_\lambda = \{Q \ll \mathbb{P} \mid \|\frac{dQ}{d\mathbb{P}}\|_\infty \le \frac{1}{\lambda}\}$. Moreover, the optimizer is obtained.

PROOF: Fix $X \in L^\infty$. Take $Q \in \mathcal{Q}_\lambda$ and let $\varphi := \frac{dQ}{d\mathbb{P}}$. If q is any λ-quantile of X then from (2.6),

$$\begin{aligned}
\mathrm{AV@R}_\lambda(X) + E^Q[X] &= \frac{1}{\lambda}(\mathbb{E}[-X \mathbf{1}_{\{X \le q\}}] + q(\mathbb{P}[X \le q] - \lambda)) + \mathbb{E}^Q[X] \\
&= \frac{1}{\lambda} \mathbb{E}[-X \mathbf{1}_{\{X \le q\}} + q \mathbf{1}_{\{X \le q\}} - q\lambda\,\mathbb{E}[\varphi] + \lambda \varphi X] \\
&= \frac{1}{\lambda} \mathbb{E}[(X-q)(\lambda\varphi - \mathbf{1}_{\{X \le q\}})] \ge 0
\end{aligned}$$

since $(X-q)(\lambda\varphi - \mathbf{1}_{\{X \le q\}})$ is non-negative because $0 \le \lambda\varphi \le 1$. Therefore

$$\mathrm{AV@R}_\lambda(X) \ge -\mathbb{E}^Q[X].$$

To prove the converse, let q be a λ-quantile of X and

$$\kappa := \begin{cases} 0 & \text{if } \mathbb{P}[X = q] = 0 \\ \frac{\lambda - \mathbb{P}[X < q]}{\mathbb{P}[X = q]} & \text{otherwise.} \end{cases}$$

Then $0 \le \kappa \le 1$ and $\varphi := \frac{1}{\lambda}(\mathbf{1}_{\{X<q\}} + \kappa \mathbf{1}_{\{X=q\}}$ is such that $\mathbb{E}[\varphi] = 1$, so φ is the density of a probability measure $Q \in \mathcal{Q}_\lambda$. Then

$$\begin{aligned}
-\mathbb{E}^Q[X] &= \frac{1}{\lambda}(\mathbb{E}[-X\mathbf{1}_{\{X<q\}}] + \kappa\,\mathbb{E}[-X\mathbf{1}_{\{X=q\}}]) \\
&= \frac{1}{\lambda}(\mathbb{E}[-X\mathbf{1}_{\{X<q\}}] - q\lambda + q\,\mathbb{P}[X<q]) \\
&= \frac{1}{\lambda}\,\mathbb{E}[(q-X)^+] - q \\
&= \mathrm{AV@R}_\lambda(X) \qquad \text{by (2.5)} \qquad \square
\end{aligned}$$

Corollary 23. *For all $X \in \mathcal{X}$,*

$$AV@R_\lambda(X) \ge WCE_\lambda(X) \ge V@R_\lambda(X).$$

PROOF: If $\mathbb{P}[A] > \lambda$ then the density of $\mathbb{P}[\cdot \mid A]$ with respect to $\mathbb{P}$ is bounded by $\frac{1}{\lambda}$, so 22 implies the first inequality. By definition,

$$\mathbb{P}[-X \ge \mathrm{V@R}_\lambda(X) - \varepsilon] > \lambda$$

for all small $\varepsilon > 0$, so $\mathrm{WCE}_\lambda(X) \ge \mathbb{E}[-X \mid -X \ge \mathrm{V@R}(X) - \varepsilon]$ and the second inequality follows by taking $\varepsilon \to 0$. $\square$

Remark. If X has a continuous distribution then $\mathbb{P}[X \le q_\lambda^+(X)] = \lambda$, so (2.6) and the proof of the corollary together imply that in this case

$$\mathrm{AV@R}_\lambda(X) = \mathrm{WCE}_\lambda(X) = \mathbb{E}[-X \mid -X \ge \mathrm{V@R}_\lambda(X)].$$

We conclude with a numerical example. Assume that the risk-free interest rate is zero and corporate bonds have a return of 2% with a 1% probability of default over a certain time period. Suppose we borrow an amount of \$1,000,000 and invest it in bonds. Let X be the final value if we invest it all in one bond, and let Y be the final value if we invest \$10,000 in each of 100 different bonds. Then

$$X = \begin{cases} -\$1,000,000 & \text{if the company defaults} \\ \$20,000 & \text{otherwise} \end{cases}$$

and $Y = \$200(100-d) - \$10,000d$, where d is the number of companies out of 100 that default. Clearly $\mathrm{V@R}_{0.05}(X) = -\$20,000$, so according to V@R position X is completely acceptable at level 0.05, regardless of the possible loss of the entire investment. However, $\mathrm{V@R}_{0.05}(Y) = \$10,600$, so in particular diversification of the portfolio has increased its measure of risk, contrary to intuition. For comparison,

$$\begin{aligned}
\mathrm{AV@R}_{0.05}(X) &= \frac{1}{0.05}(0.05-0.01)(-\$20,000) + (0.01-0)(\$1,000,000) \\
&= \$184,000
\end{aligned}$$

while $\mathrm{AV@R}_{0.05}(Y) = \$15,173$. Thus both positions are seen to be risky, but the risk of the diversified position is considerably less. See Appendix A for a detailed comparison.

3 Risk measurement over multiple periods

In the real world new information is revealed over time and financial positions change. Information is represented by a *filtration* $(\mathscr{F}_t)_{t\in I}$ on Ω, where I is usually an interval or a discrete set of times. A financial position is represented by a *bounded adapted process*, and we take $\mathscr{X}$ to be the set of all such processes. Whether an adapted process $(X_t)_{t\in I}$ is interpreted as a *payoff stream* (so X_t is the payoff at time t, as in [13] and [10]) or as a *cumulative cash flow* (so X_t is the sum of the payoffs received up to time t or the (market, book, liquidation, etc.) value of a portfolio at time t, as in [4] and [12]) is largely a matter of preference. In either case, the axioms (M), (C), and (H) have obvious analogs in the multi-period setting, but the extension of (TI) depends heavily on the interpretation. Several possible extensions will be discussed.

3.1 Initial period valuations

First we consider *initial period* valuations, where the risk assessment is made at the initial time. Initial period valuations are the simplest extension of concave valuations to the multi-period setting.

Example 24 (Standard capital requirements, [10]).
5 extends immediately to the multi-period setting. We interpret elements of $\mathscr{X}$ as payoff streams and take I to be finite. Let $\mathscr{L} \subseteq \mathscr{X}$ be those processes $(Y_n)_{n\in\{1,\dots,N\}}$ such that $\sum_{n=1}^N Y_n$ is constant. $\mathscr{L}$ is the set of *disinvestment strategies*, where (Y_n) is interpreted as the strategy of adding an amount Y_n of cash to a position at time n (where negative values correspond to removing cash). Given an acceptance set $\mathscr{A} \subseteq \mathscr{X}$ and a set of acceptable strategies $\mathscr{C} \subseteq \mathscr{L}$, the associated valuation is

$$\pi^0_{\mathscr{A},\mathscr{C}}(X) = -\inf\{\textstyle\sum_{n=1}^N Y_n \mid Y \in \mathscr{C}, X + Y \in \mathscr{A}\}.$$

Here translation invariance becomes (for $Y \in \mathscr{C}$),

$$\pi^0_{\mathscr{A},\mathscr{C}}(X+Y) = \pi^0_{\mathscr{A},\mathscr{C}}(X) + \textstyle\sum_{i=n}^N Y_n.$$

Frittelli and Scandolo discuss some economically feasible choices for $\mathscr{C}$, including

(i) $\mathscr{C} = \mathbb{R}^N$ or $(\mathbb{R}_+)^N$, the collection of deterministic strategies or those deterministic strategies which only involve adding cash to a position.

(ii) $\{Y \in \mathscr{L} \mid Y_n$ is $\mathscr{F}_m$-measurable for all $n \geq m\}$ for some time m. In this case all moves in an acceptable disinvestment strategy must be decided by date m.

(iii) $\{Y \in \mathscr{L} \mid \sum_{n\geq m} Y_n \leq b, m = 1,\dots,N\}$, where $b \in [0,\infty]$. Here b is a *credit limit* and puts a bound on the amount of cash that may be borrowed to bail out a risky position.

The rest of this Section follows [4]. From now on we interpret elements of $\mathscr{X}$ as cumulative cash flows and take $I = \mathbb{Z}_+$. We work in a filtered probability space with discrete time steps $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_n)_{n\geq 0})$, where $\mathscr{F} = \bigvee_{n\geq 0} \mathscr{F}_n$. We would like to reduce the multi-period case to the single period case. To this end, let $\Omega' := \Omega \times \mathbb{Z}_+$ and $\mathscr{F}'$ be the σ-algebra consisting of sets of the form $\bigcup_{n\geq 0} E_n \times \{n\}$, where $E_n \in \mathscr{F}_n$ for each n. $\mathbb{P}$ gives rise to a probability $\mathbb{P}'$ on Ω' via

$$\mathbb{P}'\left[\bigcup_{n\geq 0} E_n \times \{n\}\right] = \sum_{n\geq 0} \frac{1}{2^{n+1}} \mathbb{P}[E_n].$$

Lemma 25. *The set $\mathscr{X}$ is canonically identified with $L^\infty(\Omega', \mathscr{F}', \mathbb{P}')$.*

Proof: Let $X = (X_n)_{n\geq 0} \in \mathscr{X}$ be an adapted process. Then the corresponding element of L^∞ is $X' : (\omega, n) \mapsto X_n(\omega)$ (it is clearly bounded and

$$(X')^{-1}(B) = \bigcup_{n\geq 0} X_n^{-1}(B) \times \{n\} \in \mathscr{F}'.$$

for any Borel set B, so X' is $\mathscr{F}'$-measurable). Conversely, if $X' \in L^\infty$ then the corresponding process is $X_n : \omega \mapsto X'(\omega, n)$. This process is adapted since

$$X_n^{-1}(B) = \{\omega \in \Omega \mid X(\omega, n) \in B\} = p_n((X')^{-1})(B) \in \mathscr{F}_n$$

for any Borel set B. □

25 allows us to reduce the study of the multi-period case to the single period case. Given a convex cone $\mathscr{A}$ of acceptable processes, the associated initial period valuation $\pi^0(X) = \sup\{m \in \mathbb{R} \mid X - m \in \mathscr{A}\}$ can be interpreted as a concave valuation on $L^\infty(\Omega', \mathscr{F}', \mathbb{P}')$ associated with $\mathscr{A}$. Note that under this identification the translation for processes is required to be given by, for $m \in \mathbb{R}$,

$$(X_n)_{n\geq 0} + m = (X_n + m)_{n\geq 0}.$$

Theorem 26. *For each initial period coherent valuation π^0 on the collection of bounded adapted processes with the Fatou property, there is a set $\mathcal{A}$ of positive adapted non-decreasing processes A with $\mathbb{E}[A_\infty] = 1$ and such that for each process X,*

$$\pi^0(X) = \inf_{A\in\mathcal{A}} \mathbb{E}\left[\sum_{n=0}^{\infty} X_n \cdot (A_n - A_{n-1})\right].$$

Proof: From the discussion above, π^0 may be interpreted as a valuation on $L^\infty(\Omega', \mathscr{F}', \mathbb{P}')$, so by the representation result 15 there is a set $\mathscr{Q}$ of test probabilities, each absolutely continuous with respect to $\mathbb{P}'$, such that for each $X \in \mathscr{X}$,

$$\pi^0(X) = \inf_{Q\in\mathscr{Q}} \mathbb{E}^Q[X].$$

Now each $Q \in \mathscr{Q}$ can be described by its *density* with respect to $\mathbb{P}'$, a non-negative $\mathscr{F}'$-measurable function f' with expected value 1. But we may think of f' as an adapted process $(f_n)_{n\geq 0}$, where each f_n is a non-negative $\mathscr{F}_n$-measurable function on Ω such that

$$\sum_{n=0}^{\infty} \frac{1}{2^{n+1}}\,\mathbb{E}[f_n] = 1,$$

and further, for each $X = (X_n)_{n\geq 0} \in \mathscr{X}$, we have

$$\mathbb{E}^Q[X] = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}}\,\mathbb{E}[f_n X_n].$$

Define the increasing adapted process A by $A_n := A_{n-1} + \frac{1}{2^{n+1}} f_n$ and $A_{-1} := 0$. Then A_∞ exists and $\mathbb{E}[A_\infty] = 1$, and the representation result follows by taking the infimum over the set $\mathcal{A}$ of processes corresponding to $\mathscr{Q}$. □

Example 27. Let τ be a stopping time and $\pi^0(X) = \mathbb{E}[X_\tau]$. Then π^0 is a coherent valuation describing how the process fares when τ happens. Assuming $\mathbb{P}[\tau < \infty] > 0$, take

$$A_n^\tau := \frac{1}{\mathbb{P}[\tau < \infty]}\mathbf{1}_{\{\tau \leq n\}}$$

for each n, so A^τ is a non-decreasing process and

$$\pi^0(X) = \mathbb{E}\left[\sum_{n\geq 0} X_n (A_n^\tau - A_{n-1}^\tau)\right].$$

Example 28. Let $(c_n)_{n\geq 0}$ be non-negative constants such that $\sum_{n\geq 0} c_n = 1$ and

$$\pi^0(X) = \mathbb{E}\left[\sum_{n\geq 0} c_n X_n\right].$$

Then π^0 is the weighted average of future values with weights $(c_n)_{n\geq 0}$. The process A that gives π^0 is clear.

Example 29. Let $\pi^0(X) = \mathbb{E}[\inf_{0\leq n\leq N} X_n]$, a coherent valuation. Recall that times of extrema are not generally stopping times, so it is non-trivial that we are able to represent π^0 with adapted processes. To this end, for any random time τ let $A^\tau := A_{n-1}^\tau + \mathbb{E}[\mathbf{1}_{\{\tau=n\}} \mid \mathscr{F}_n]$ and $A_{-1} = 0$. Then

$$\mathbb{E}\left[\sum_{n\geq 0} X_n (A_n^\tau - A_{n-1}^\tau)\right] = \mathbb{E}\left[\sum_{n\geq 0} X_n \mathbf{1}_{\{\tau=n\}}\right].$$

For each X let $\tau_X(\omega)$ be the time n such that $X_n(\omega)$ is minimal for the set $\{X_n(\omega) \mid n \geq 0\}$, and let $\mathcal{A}$ be the convex hull of $\{A^{\tau_X} \mid X \in \mathscr{X}\}$. Then

$$\pi^0(X) = \inf_{A\in\mathcal{A}} \mathbb{E}\left[\sum_{n=0}^{\infty} X_n \cdot (A_n - A_{n-1})\right].$$

3.2 Dynamic valuations, and further reading

Finally, we come to true multi-period valuations, or *dynamic valuations*. This section outlines current results on this topic and suggests further reading. For dynamic valuations, at each (future) stopping time τ a cumulative cash flow X is associated with its *time-τ risk-adjusted value*, $\pi_\tau(X)$, which is a bounded $\mathscr{F}_\tau$-measurable function that represents the amount of cash that may be removed from X at time τ and have it remain acceptable.

Example 30. In [4], Artzner et al. suggest one method of extending an initial period valuation to a dynamic valuation. Supposing I is finite and a coherent initial period valuation π^0 is given. For each $n \in \{1, \dots, N\}$ consider the set

$$\{f \ \mathscr{F}_n\text{-measurable} \mid \pi^0((X-f)\mathbf{1}_A\mathbf{1}_{[[n,N]]}) \geq 0 \text{ for all } A \in \mathscr{F}_n\},$$

where $(X_n) - f = (X_n - f)$, $\mathbf{1}_{[[n,N]]}$ is the process that is zero to time n and one from n to N, and the product of processes is defined via $(X_n)(Y_n) = (X_nY_n)$. They claim that this set has a maximal element, and define $\pi_t^0(X)$ to be it. For a general stopping time τ, they define

$$\pi_\tau^0(X) := \sum_{n=1}^{N} \pi_n^0(X)\mathbf{1}_{\{\tau=n\}}.$$

This family of valuations has the following properties.

(i) Each mapping $\pi_\tau^0 : \mathscr{X} \to L^\infty(\Omega, \mathscr{F}_\tau, \mathbb{P})$ is positive, concave, positively homogeneous, and satisfies the Fatou property.

(ii) For each $Y \in L^\infty(\Omega, \mathscr{F}_\tau, \mathbb{P})$ we have $\pi_\tau^0(X + Y\mathbf{1}_{[[\tau,N]]}) = \pi_\tau^0(X) + Y$.

In some cases a position is "locked in" for a certain period of time, possibly due to the absence of intermediate markets or the nature of the financial instrument under consideration. In this simpler case only the *final value* $X_N \in L^\infty(\Omega, \mathscr{F}, \mathbb{P})$ matters, and its risk-adjusted valuation reflects how its acceptability changes as information is revealed over time. Given a set $\mathscr{Q} \subseteq \mathscr{M}_1(\mathbb{P})$ of test probabilities, one construction of a coherent risk-adjusted valuation is obtained by mimicking the representation of a single period valuation, but using conditional expectation at each time, namely

$$\overline{\pi}_n(X_N) := \inf_{Q \in \mathscr{Q}} \mathbb{E}^Q[X \mid \mathscr{F}_n]$$

for each n. Another construction from the same set of test probabilities is by backwards induction, defining $\widetilde{\pi}_N(X_N) := X_N$ and

$$\widetilde{\pi}_n(X_N) := \inf_{Q \in \mathscr{Q}} \mathbb{E}[\widetilde{\pi}_{n+1}(X_N) \mid \mathscr{F}_n]$$

for each n. It can be shown that these valuations are equal when the collection of test probabilities $\mathscr{Q}$ has a property known as *m-stability* (see [7]), which is also known as *stable under pasting* in [4]. With some work these dynamic valuations can be extended to all of $\mathscr{X}$ (not just final values) and the generalizations have the same property. It should be noted that the set of test probabilities $\mathscr{Q}_\lambda$ associated with AV@R$_\lambda$ is not m-stable.

A forthcoming paper from Jobert and Rogers, [12] suggests a collection of axioms for families of concave dynamic valuations

$$\{\pi_\tau : \mathscr{X} \to L^\infty(\mathscr{F}_\tau) \mid \tau \text{ a stopping time}\},$$

namely,

(C) π_τ is concave for all τ.

(M) If $X_n \geq X'_n$ for all n then $\pi_\tau(X) \geq \pi(X')$ for all τ.

(TI) $\pi_\tau(X + Y\mathbf{1}_{[[\tau,N]]}) = \pi_\tau(X) + Y$ for $Y \in L^\infty(\mathscr{F}_\tau)$.

(DC) For stopping times $\tau \leq \sigma$, $\pi_\tau(X) = \pi_\tau(X\mathbf{1}_{[[\tau,\sigma[[} + \pi_\sigma(X)\mathbf{1}_{[[\sigma,T]]})$.

(L) $\pi_\tau(\mathbf{1}_A\mathbf{1}_{[[\tau,N]]}X) = \mathbf{1}_A\pi_\tau(X)$ for all τ, $X \in \mathscr{X}$, and $A \in \mathscr{F}_\tau$.

(CL) If τ and τ' are two stopping times then $\pi_\tau(X) = \pi_{\tau'}(X)$ on $\{\tau = \tau'\}$.

(Z) $\pi_\tau(0) = 0$ for all τ.

Axioms (C) and (M) are obvious extensions of the axioms from the single period case, and (TI) is a natural extension, especially given 30. Axiom (L), (for *local*) expresses the obvious economic fact that, at time τ, the only thing that matters is that we have the cash, not the precise details of how it was obtained. It goes further by stating that, at time τ, if event A has not happened then the cash balance $\mathbf{1}_A\mathbf{1}_{[\tau,N]}K$ is worthless, and if A has happened then it is worth the same as K. Axiom (CL) (for *constant localization*, apparently) is natural, as is axiom (Z). Finally, axiom (DC), *dynamic consistency*, is what ties the family together. It says that, at time τ, we may remove as much cash from X as we could remove from the cash balance process which is X up to time σ and then requires us to hand in X_σ in exchange for the amount of cash we may remove from X at time σ. This axiom is referred to as *Bellman's principle* in [4] and is satisfied by dynamic valuations arising from m-stable collections of probabilities.

A Numerical AV@R example

The following table extends the example discussed in Section 2.5. $V@R_\lambda$ and $AV@R_\lambda$ values are given for various levels λ for the portfolios consisting of $\$1,000,000$ divided evenly between 1, 10, 100, and 1000 corporate bonds, assuming the risk-free rate is zero, a return of 2% on bonds, a default rate of 1%, and that companies default independently.

level	1 bond		10 bonds		100 bonds		1000 bonds	
λ	$V@R_\lambda$ (\$)	$AV@R_\lambda$ (\$)	$V@R_\lambda$	$AV@R_\lambda$	$V@R_\lambda$	$AV@R_\lambda$	$V@R_\lambda$	$AV@R_\lambda$
0.001	1000000.00	1000000.00	184000.00	195819.23	31000.00	37270.55	1420.00	2541.09
0.002	1000000.00	1000000.00	184000.00	189909.61	31000.00	34135.28	400.00	1723.75
0.003	1000000.00	1000000.00	184000.00	187939.74	31000.00	33090.18	400.00	1282.50
0.004	1000000.00	1000000.00	184000.00	186954.81	20800.00	31120.06	-620.00	880.41
0.005	1000000.00	1000000.00	82000.00	171394.33	20800.00	29056.05	-620.00	580.33
0.006	1000000.00	1000000.00	82000.00	156495.28	20800.00	27680.04	-620.00	380.27
0.007	1000000.00	1000000.00	82000.00	145853.09	20800.00	26697.18	-1640.00	223.53
0.008	1000000.00	1000000.00	82000.00	137871.46	20800.00	25960.03	-1640.00	-9.41
0.009	1000000.00	1000000.00	82000.00	131663.52	20800.00	25386.69	-1640.00	-190.59
0.01	-20000.00	1000000.00	82000.00	126697.17	20800.00	24928.02	-1640.00	-335.53
0.02	-20000.00	490000.00	82000.00	104348.58	10600.00	22034.77	-2660.00	-1302.30
0.03	-20000.00	320000.00	82000.00	96899.06	10600.00	18223.18	-3680.00	-1877.57
0.04	-20000.00	235000.00	82000.00	93174.29	10600.00	16317.39	-3680.00	-2328.18
0.05	-20000.00	184000.00	82000.00	90939.43	10600.00	15173.91	-4700.00	-2641.98
0.06	-20000.00	150000.00	82000.00	89449.53	10600.00	14411.59	-4700.00	-2984.99
0.07	-20000.00	125714.29	82000.00	88385.31	10600.00	13867.08	-4700.00	-3229.99
0.08	-20000.00	107500.00	82000.00	87587.15	400.00	13378.78	-4700.00	-3413.74
0.09	-20000.00	93333.33	82000.00	86966.35	400.00	11936.69	-5720.00	-3642.65
0.10	-20000.00	82000.00	-20000.00	82000.00	400.00	10783.02	-5720.00	-3850.39
0.11	-20000.00	72727.27	-20000.00	72727.27	400.00	9839.11	-5720.00	-4020.35
0.12	-20000.00	65000.00	-20000.00	65000.00	400.00	9052.52	-5720.00	-4161.99
0.13	-20000.00	58461.54	-20000.00	58461.54	400.00	8386.94	-5720.00	-4281.84
0.14	-20000.00	52857.14	-20000.00	52857.14	400.00	7816.44	-6740.00	-4425.11
0.15	-20000.00	48000.00	-20000.00	48000.00	400.00	7322.01	-6740.00	-4579.43
0.16	-20000.00	43750.00	-20000.00	43750.00	400.00	6889.39	-6740.00	-4714.47
0.17	-20000.00	40000.00	-20000.00	40000.00	400.00	6507.66	-6740.00	-4833.62
0.18	-20000.00	36666.67	-20000.00	36666.67	400.00	6168.34	-6740.00	-4939.53
0.19	-20000.00	33684.21	-20000.00	33684.21	400.00	5864.75	-6740.00	-5034.29
0.20	-20000.00	31000.00	-20000.00	31000.00	400.00	5591.51	-6740.00	-5119.57
0.21	-20000.00	28571.43	-20000.00	28571.43	400.00	5344.30	-7760.00	-5208.94
0.22	-20000.00	26363.64	-20000.00	26363.64	400.00	5119.55	-7760.00	-5324.89
0.23	-20000.00	24347.83	-20000.00	24347.83	400.00	4914.36	-7760.00	-5430.77
0.24	-20000.00	22500.00	-20000.00	22500.00	400.00	4726.26	-7760.00	-5527.82
0.25	-20000.00	20800.00	-20000.00	20800.00	400.00	4553.21	-7760.00	-5617.11

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