A Model for the Limit Order Book in Heavy Traffic

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Model

Our LOB model follows Cont et al. [1]:
- Discrete price levels.
- Orders of equal size.
- Arrivals are independent and Markovian.
- Rates depend on LOB state only via current bid and ask prices.

We investigate a sequence of such models:
- Order arrival rates are scaled by \( n \)
- Order size is scaled by \( 1/\sqrt{n} \).
- Simplifying assumptions for this investigation:
  - Combined market/limit order rate is \( \lambda > 1 \), arriving at opposite best price.
  - Limit order rates of 1 at next two prices.
  - Interesting but tractable behavior.

Heuristic

Assume there are “large” queues of buy and sell orders between which the action takes place.
- Simulation suggests this is often the case.
- With simplified arrival rates, there are only two intermediate prices, \( p \) and \( q \).
- In this case the signed order quantities \( P \) and \( Q \) form a CTMC on \( \mathbb{Z}^2 \).

Theorem

If \( \lambda = (1 + \sqrt{5})/2 \) then \( (\hat{P}_n, \hat{Q}_n) \Rightarrow (-B, B) \) in \( \mathbb{R}^2 \), where \( B \) is a one dimensional Brownian motion with variance parameter \( 4\lambda^2 \). Here \( \hat{P}_n(t) := P(nt)/\sqrt{n} \) and \( \hat{Q}_n(t) := Q(nt)/\sqrt{n} \) are the diffusion scaled queue length processes.

Evidence

- Simulation of \( (\hat{P}_n, \hat{Q}_n) \) with \( n = 100 \):
  - Black dots are final points of 1000 independent trials run for 2 scaled time units.
  - Coloured lines are independent paths 10 scaled time units long, sampled every 0.1 scaled time units.

- Simulation of \( (\hat{P}_n, \hat{Q}_n) \) with \( n = 1000 \) (same legend):
  - Distance to “backwards L” is crushed because it behaves like the CTMC:

Sketch of Proof

- Transform coordinates to “straighten out” the problem.

References