Measure and Integration
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Professor B. Forrest

CHRIS ALMOST

Contents

1 Review of Riemann Integration

For this section let $f : [a, b] \to \mathbb{R}$ be a bounded function.

1.1 Definition. A *partition* of $[a, b]$ is a finite subset $\Pi = \{a = x_0 < x_1 < \cdots < x_n = b\}$. The *norm* of Π is $\|\Pi\| := \max_{i=1,\dots,n}\{\Delta x_i\}$. Let $M_i := \sup\{f(x)|x \in [x_{i-1}, x_i]\}$ and $m_i := \inf\{f(x)|x \in [x_{i-1}, x_i]\}$. The *lower Riemann sum* is $L(f, \Pi) := \sum_{i=1}^n m_i \Delta x_i$ and the *upper Riemann sum* is $U(f, \Pi) := \sum_{i=1}^n M_i \Delta x_i$.

It is clear that $L(f, \Pi) \le U(f, \Pi)$.

1.2 Definition. A partition Π_1 is a *refinement* of Π if $\Pi \subseteq \Pi_1$.

Clearly $L(f, \Pi) \le L(f, \Pi_1) \le L(f, \Pi_1) \le U(f, \Pi)$ and the collection of all partitions of $[a, b]$ is a directed set (directed by refinement).

1.3 Proposition. *If Π_1 and Π_2 are two partitions, then $L(f, \Pi_1) \le U(f, \Pi_2)$.*

1.4 Definition. The *upper Riemann integral* for f on $[a, b]$ is

$$\overline{\int_a^b} f(x)dx = \inf_{\Pi}\{U(f, \Pi)\}$$

The *lower Riemann integral* for f on $[a, b]$ is

$$\underline{\int_a^b} f(x)dx = \sup_{\Pi}\{L(f, \Pi)\}$$

f is said to be *Riemann integrable* if $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$. In this case the common value is denoted by $\int_a^b f(x)dx$, called the *Riemann integral*.

1.5 Proposition (Cauchy Criterion). *If f is bounded on $[a, b]$ then f is Riemann integrable if and only if for all $\varepsilon > 0$ there is a partition Π such that $U(f, \Pi) - L(f, \Pi) < \varepsilon$.*

For example, if f is continuous on $[a, b]$ then f is Riemann integrable on $[a, b]$. (Prove this as an exercise.) If f is a non-decreasing function on $[a, b]$ then f has at most countably many discontinuities, and they are all jump discontinuities. Let $P^{(n)}$ be the uniform partition of $[a, b]$. Then $M_i = f(x_i)$ for each i, and $m_i = f(x_{i-1})$ for all i. Hence

$$U(f, P^{(n)}) - L(f, P^{(n)}) = \sum_{i=1}^{n}(f(x_i) - f(x_{i-1}))\frac{b-a}{n} = (f(b) - f(a))\frac{b-a}{n} \longrightarrow 0$$

as n tends to infinity. Thus f is Riemann integrable.

1.6 Theorem. *Assume that f is bounded on $[a, b]$. Then the following are equivalent*

1. *f is Riemann integrable on $[a, b]$.*
2. *For every $\varepsilon > 0$ there is a partition Π such that $U(f, \Pi) - L(f, \Pi) < \varepsilon$.*
3. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\Pi\| < \delta$ implies that $U(f, \Pi) - L(f, \Pi) < \varepsilon$.*

Furthermore if any of these hold then for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|\Pi\| < \delta$ then $\left|S(f, \Pi) - \int_a^b f(x)dx\right| < \varepsilon$, where $S(f, \Pi) = \sum_{i=1}^n f(c_i)\Delta x_i$ with $c_i \in [x_{i-1}, x_i]$.

Note that if $s_k = S(f, P^{(k)})$ is any sequence of Riemann sums with respect to the uniform paritions $P^{(k)}$, then $\lim_{k\to\infty} s_k = \int_a^b f(x)dx$. This explains the usual definition given in first year calculus.

But there are some problems with the Riemann integral:

1. Many "nice" functions are not Riemann integrable.
2. The Riemann integral does not behave well with respect to limits of sequence of functions. That is, we cannot "interchange the limit and the integral". For example, let $\mathbb{Q} \cap [0,1] = \{r_1, r_2, \dots\}$, and let $f_n(x) = \chi_{\{r_1,\dots,r_n\}}(x)$. Then $\int_0^1 f_n(x)dx = 0$ for all n, but $\lim_{n\to\infty} f_n = \chi_{\mathbb{Q}}$ is not even Riemann integrable.

Assume that f is Riemann integrable on $[a,b]$, so that $\int_a^b f(x)dx = \sup_\Pi\{L(f,\Pi)\}$. Let Π be a parition of $[a,b]$ and $\varphi_\Pi := \sum_{i=1}^n m_i \chi_{[x_{i-1},x_i)}$. Then $\varphi_\Pi \le f$ for all $x \in [a,b]$, and $\int_a^b \varphi_\Pi(x)dx = L(f,\Pi)$. Hence

$$\int_a^b f(x)dx = \sup_\Pi \left\{ \int_a^b \varphi_\Pi(x)dx \right\}$$

The Riemann integral is essentially the weighted average of the function obtained by partitioning the domain of the function. An alternative approach is to partition the range of the function. Assume $\{y_0, y_1, \dots, y_n\}$ is partition of the range of f (whatever that means). Let $S_i = \{x \in [a,b] \mid y_{i-1} \le x < y_i\}$. Then $\int_a^b f(x)dx \approx \sum_{i=1}^n y_{i-1}\text{length}(S_i)$. But what do we mean by "length" of an arbitrary set of real numbers?

2 Measures and Measure Spaces

2.1 Definition. Let X be any non-empty set. An *algebra* of subsets of X is a collection $\mathscr{A} \subseteq \mathscr{P}(X)$ such that

1. $\varnothing \in \mathscr{A}$
2. $E_1, E_2 \in \mathscr{A}$ implies $E_1 \cup E_2 \in \mathscr{A}$
3. $E \in \mathscr{A}$ implies $E^c = X \setminus E \in \mathscr{A}$

$\mathscr{A}$ is said to be a σ-algebra if

1. $\varnothing \in \mathscr{A}$
2. $\{E_n\}_{n=1}^\infty \subseteq \mathscr{A}$ implies $\bigcup_{n=1}^\infty E_n \in \mathscr{A}$
3. $E \in \mathscr{A}$ implies $E^c = X \setminus E \in \mathscr{A}$

Remark.
1. Every σ-algebra is an algebra
2. $E_1 \cap E_2 = (E_1^c \cup E_2^c)^c$, so algebras are closed under intersection as well

2.2 Example. $\mathscr{P}(X)$ is a σ-algebra.

2.3 Proposition. *If $\{\mathscr{A}_\alpha\}_{\alpha\in I}$ is a collection of algebras (resp. σ-algebras) then $\bigcap_{\alpha\in I}\mathscr{A}_\alpha$ is an algebra (resp. σ-algebra).*

Proof: Exercise. □

2.4 Corollary. *Given any set $S \subseteq \mathscr{P}(X)$ then there exists a smallest algebra (resp. σ-algebra) that contains S.*

Notation. Given $S \subseteq \mathscr{P}(X)$, let $\mathscr{A}(S)$, the algebra generated by S, be the smallest algebra containing S, and $\sigma(S)$, the σ-algebra generated by S, be the smallest σ-algebra containing S.

2.5 Definition. Let $S = \{U \subseteq \mathbb{R} \mid U \text{ is open}\}$. The σ-algebra generated by S is called the Borel σ-algebra of $\mathbb{R}$ and is denoted $\mathscr{B}(\mathbb{R})$. More generally we may take the Borel σ-algebra of any topological space. A set $A \subseteq \mathbb{R}$ is called G_δ if $A = \bigcap_{n=1}^\infty U_n$ where each U_n is open. A set $A \subseteq \mathbb{R}$ is called F_σ if $A = \bigcup_{n=1}^\infty F_n$ where each F_n is closed. Additional subscripts may be appended in the most obvious way.

Remark. The G_δ sets are exactly the complements of the F_σ sets, and *visa versa*. Notice that the closed sets are G_δ.

Fact: $|\mathscr{B}(\mathbb{R})| = c$, the cardinality of $\mathbb{R}$.

2.6 Definition. Let (x, τ) be a topological space. We say that $A \subseteq X$ is *nowhere dense* if $\overline{A}$ has no interior. $A \subseteq X$ is of *first category* in (X, τ) if it is a countable union of nowhere dense sets. $A \subseteq X$ is of *second category* in (X, τ) if it is not first category in (X, τ). A is *residual* if A^c is of first category.

2.7 Theorem (Baire's Category Theorem). *Let X be a complete metric space or (X, τ) a locally compact, Hausdorff space. Then X is of second category in itself.*

2.8 Theorem. *Let X be a complete metric space or (X, τ) a locally compact, Hausdorff space. If $\{U_n\}_{n=1}^\infty$ is a collection of dense open sets then $\bigcap_{n=1}^\infty U_n$ is dense.*

2.9 Corollary. $\mathbb{Q}$ *is not G_δ in* $\mathbb{R}$.

PROOF: Assume that $\mathbb{Q} = \bigcap_{n=1}^\infty U_n$, where U_n is open for each n. Then $\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n=1}^\infty F_n$, where F_n is closed and nowhere dense. Let $\mathbb{Q} = \{r_1, r_2, \dots\}$. Then $F_n' = F_n \cup \{r_n\}$ is closed and nowhere dense and $\mathbb{R} = \bigcup_{n=1}^\infty F_n'$. This is a contradiction since $\mathbb{R}$ is not of first category. □

2.10 Example. Consider $[0, 1]$. Define an equivalence relation on $[0, 1]$ by $x \sim y$ if $x - y \in \mathbb{Q}$. Use the Axiom of Choice to choose one element from each equivalence class and denote this set by S. S is not Borel.

There is nothing special about using the open sets of $\mathbb{R}$ to define $\mathscr{B}(\mathbb{R})$.

$$\mathscr{B}(\mathbb{R}) = \sigma\{(a, b) \mid a, b \in \mathbb{R}\} = \sigma\{(a, b] \mid a, b \in \mathbb{R}\} = \sigma\{[a, b) \mid a, b \in \mathbb{R}\} = \sigma\{[a, b] \mid a, b \in \mathbb{R}\}$$

2.11 Example. The collection of all finite unions of sets of the form $\{\mathbb{R}, (-\infty, b], (a, b], (a, \infty)\}$ is an algebra.

2.12 Definition. A set together with a σ-algebra $(X, \mathscr{A})$ is called a *measurable space*. A *(countably additive) measure* on $\mathscr{A}$ is a function $\mu : \mathscr{A} \to \mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$ such that

1. $\mu(\varnothing) = 0$
2. $\mu(E) \geq 0$ for all $E \in \mathscr{A}$
3. If $\{E_n\}_{n=1}^\infty \subseteq \mathscr{A}$ is a sequence of pairwise disjoint set then $\mu(\bigcup_{n=1}^\infty E_n) = \sum_{n=1}^\infty \mu(E_n)$.

Condition (3) is known as *countable additivity*. If we replace (3) by

(3') If $\{E_n\}_{n=1}^N \subseteq \mathscr{A}$ is a sequence of pairwise disjoint set then $\mu(\bigcup_{n=1}^N E_n) = \sum_{n=1}^N \mu(E_n)$.

then μ is called a *finitely additive* measure on $\mathscr{A}$.

Countable additivity for measures give powerful convergence results for integration.

2.13 Example. Given X any set and $\mathscr{A} = \mathscr{P}(X)$, let

$$\mu(E) := \begin{cases} |E| & \text{if } |E| \text{ is finite} \\ \infty & \text{otherwise} \end{cases}$$

This is called the *counting measure*. If $|X|$ is finite then

$$\mu_*(E) := \frac{|E|}{|X|}$$

is called the *normalized counting measure*.

2.14 Definition. We call a measure μ *finite* if $\mu(X) < \infty$. We call μ *σ-finite* if there exists $\{E_n\}_{n=1}^{\infty} \subseteq \mathscr{A}$ such that $X = \bigcup_{n=1}^{\infty} E_n$ and $\mu(E_n) < \infty$ for all n.

2.15 Example. Let $X = \mathbb{N}$ and $\mathscr{A} = \mathscr{P}(X)$. Let $f : \mathbb{N} \to \mathbb{R}_+^*$ and define

$$\mu_f(E) := \sum_{n \in E} f(n) = \sum_{n=1}^{\infty} f(n)\chi_E(n)$$

Then μ_f is a measure on $\mathscr{A}$. μ_f is finite if and only if $\sum_{n=1}^{\infty} f(n) < \infty$. Which is to say, $f \in \ell^1(\mathbb{N})^+ = \{f \in \ell^1(\mathbb{N}) \mid f(n) \geq 0 \ \forall\, n \in \mathbb{N}\}$. μ_f is σ-finite if and only if $\infty \notin f(\mathbb{N})$. Conversely, if μ is any measure on $\mathscr{A}$ then $f_\mu(n) := \mu(\{n\})$ is a function $\mathbb{N} \to \mathbb{R}_+^*$ since $\mathbb{N}$ is countable and μ is countably additive. Then $\mu_{f_\mu} = \mu$, so we know all about the measures on $(\mathbb{N}, \mathscr{P}(\mathbb{N}))$.

In general, suppose that X is any set. Given $f : X \to \mathbb{R}_+^*$, we can define

$$\mu_f(E) = \sum_{x \in E} f(x) := \sup\{f(x_1) + \dots + f(x_n) \mid x_1, \dots, x_n \in E\}$$

Then μ_f is a measure on $\mathscr{P}(X)$. If $f \in \ell^1(X)^+$ then μ_f is finite. μ_f is σ-finite if it zero except on a countable subset of $\mathscr{P}(X)$ and $\infty \notin f(X)$. Are all measures on $(X, \mathscr{P}(X))$ of the form μ_f for some f? What about all finite measures? Consider

$$\mu(E) = \begin{cases} 0 & \text{if } E \text{ is countable} \\ \infty & \text{if } E \text{ is uncountable} \end{cases}$$

Then $\mu \neq \mu_f$ for any f. Can μ be made into a finite measure?

2.16 Definition. A triple $(X, \mathscr{A}, \mu)$ is called a *measure space*, where X is a set, $\mathscr{A}$ is a σ-algebra, and μ is a measure on $\mathscr{A}$. $(X, \mathscr{A}, \mu)$ is *complete* if $\mu(E) = 0$ and $S \subseteq E$ implies that $S \in \mathscr{A}$. (If $E \in \mathscr{A}$ then E is called *measurable*.) If $\mu(X) = 1$ then μ is called a probability measure and $(X, \mathscr{A}, \mu)$ is called a probability space. The elements of $\mathscr{A}$ in this case are called *events*.

2.17 Proposition (Monotonicity). *Let $(X, \mathscr{A}, \mu)$ be a measure space. If $E \subseteq F \in \mathscr{A}$ then $\mu(E) \leq \mu(F)$. If $\mu(E) < \infty$ then $\mu(F \setminus E) = \mu(F) - \mu(E)$.*

PROOF: This follows immediately since $F = E \cup (F \setminus E)$. □

2.18 Lemma. *Let $\mathscr{A} \subseteq \mathscr{P}(X)$ be an algebra. Assume that $\{E_n\}_{n=1}^{\infty} \subseteq \mathscr{A}$. Then there is $\{F_n\}_{n=1}^{\infty} \subseteq \mathscr{A}$ such that $\bigcup_{n=1}^{k} E_n = \bigcup_{n=1}^{k} F_n$ for all k and $\{F_n\}_{n=1}^{\infty}$ is pairwise disjoint. Moreover, $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n$*

PROOF: Let $F_1 := E_1 \in \mathscr{A}$ and let $F_n := E_n \setminus \left(\bigcup_{i=1}^{n-1} F_i\right) \in \mathscr{A}$. Notice that $F_n \subseteq E_n$ for each $n \geq 1$. □

2.19 Proposition (Countable Subadditivity). *Let $(X, \mathscr{A}, \mu)$ be a measure space. Let $\{E_n\}_{n=1}^{\infty} \subseteq \mathscr{A}$. Then*

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n)$$

PROOF: Let $\{F_n\}_{n=1}^{\infty} \subseteq \mathscr{A}$ be as in Lemma 2.18. Then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} \mu(F_n) \leq \sum_{n=1}^{\infty} \mu(E_n)$$

by monotonicity. □

2.20 Theorem (Continuity from Above). *Let $(X, \mathscr{A}, \mu)$ be a measure space and $\{E_n\}_{n=1}^{\infty} \subseteq \mathscr{A}$. If $\mu(E_1) < \infty$ and $E_{i+1} \subseteq E_i$ for all $i \geq 1$ then*

$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n)$$

2.21 Theorem (Continuity from Below). *Let $(X, \mathscr{A}, \mu)$ be a measure space and $\{E_n\}_{n=1}^{\infty} \subseteq \mathscr{A}$. If $E_i \subseteq E_{i+1}$ for all $i \geq 1$ then*

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n)$$

The theorem above is also known as the Monotone Convergence Theorem for Measures.

3 Constructing Measures: The Caratheodory Method

We are looking for a measure m on $\mathscr{P}(\mathbb{R})$ with the following nice properties:

1. $m(I) =$ length of I for all intervals I
2. m is countably additive
3. $m(x + A) = m(A)$ for all $A \in \mathscr{P}(\mathbb{R})$ and $x \in \mathbb{R}$

The problem is that no such measure exists. If we weaken our conditions and only look for a measure on some sufficiently large sub-σ-algebra $\mathscr{A}$ of $\mathscr{P}(\mathbb{R})$ then we may succeed. More specifically, we would like $\mathscr{A}$ to contain $\mathscr{B}(\mathbb{R})$ and be complete with respect to the measure.

3.1 Definition. A function $\mu^* : \mathscr{P}(X) \to \mathbb{R}^*$ is called an *outer measure* if

1. $\mu^*(\emptyset) = 0$
2. $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B$
3. If $\{E_n\}_{n=1}^{\infty} \subseteq \mathscr{A}$ then $\mu(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu(E_n)$.

μ^* is *finite* if $\mu^*(X) < \infty$. μ^* is *σ-finite* if $X = \bigcup_{n=1}^{\infty} E_n$ and $\mu^*(E_n) < \infty$ for all n.

3.2 Definition. A set $E \in \mathscr{P}(X)$ is said to be *μ^*-measurable* or *measurable* if for every $A \in \mathscr{P}(X)$

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Notice that $\mu^*(A) \le \mu^*(A\cap E) + \mu^*(A\cap E^c)$ for every $A \in \mathscr{P}(X)$. Thus we need only show that $\mu^*(A) \ge \mu^*(A\cap E)+\mu^*(A\cap E^c)$ for every $A\in\mathscr{P}(X)$ to show that E is measurable. Furthermore, we need only assume that $\mu^*(A) < \infty$. In the case that $\mu^*(A) < \mu^*(A\cap E) + \mu^*(A\cap E^c)$ then $A = (A\cap E)\cup(A\cap E^c)$ is called a *paradoxical decomposition*.

3.3 Theorem. *The set $\mathscr{B}$ of μ^*-measurable sets in $\mathscr{P}(X)$ is a σ-algebra and if $\mu = \mu^*|_{\mathscr{B}}$, then μ is a complete measure on $\mathscr{B}$.*

PROOF: It is clear that $\varnothing\in\mathscr{B}$ since for any $A\in\mathscr{P}(X)$

$$\mu^*(A) = \mu^*(A\cap X) = \mu^*(A\cap\varnothing) + \mu^*(A\cap\varnothing^c)$$

It is also clear that if $E\in\mathscr{B}$ then $E^c\in\mathscr{B}$ by symmetry of the definition of μ^*-measurable.

Let $E_1, E_2\in\mathscr{B}$ and let $A\in\mathscr{P}(X)$. Since E_2 is measurable,

$$\mu^*(A) = \mu^*(A\cap E_2) + \mu^*(A\cap E_2^c)$$

Since E_1 is measurable,

$$\mu^*(A\cap E_2^c) = \mu^*(A\cap E_2^c\cap E_1) + \mu^*(A\cap E_2^c\cap E_1^c)$$

These together imply that

$$\mu^*(A) = \mu^*(A\cap E_2) + \mu^*(A\cap E_2^c\cap E_1) + \mu^*(A\cap E_2^c\cap E_1^c)$$

Notice that $A\cap(E_1\cup E_2) = (A\cap E_2)\cup(A\cap E_1\cap E_2^c)$. Thus

$$\mu^*(A\cap(E_1\cup E_2)) \le \mu^*(A\cap E_2) + \mu^*(A\cap E_1\cap E_2^c)$$

by subadditivity, so

$$\mu^*(A) \ge \mu^*(A\cap(E_1\cup E_2)) + \mu^*(A\cap E_2^c\cap E_1^c) = \mu^*(A\cap(E_1\cup E_2)) + \mu^*(A\cap(E_2\cup E_1)^c)$$

which implies $E_1\cup E_2\in\mathscr{B}$. Therefore $\mathscr{B}$ is an algebra.

Now let $\{E_i\}_{i=1}^\infty\subseteq\mathscr{B}$ be a sequence of pairwise disjoint μ^*-measurable sets and let $E = \bigcup_{i=1}^\infty E_i$. Let $G_n = \bigcup_{i=1}^n E_i$. Then $G_n\in\mathscr{B}$ and for any $A\in\mathscr{P}(X)$

$$\mu^*(A) = \mu^*(A\cap G_n) + \mu^*(A\cap G_n^c) \ge \mu^*(A\cap G_n) + \mu^*(A\cap E^c)$$

since $E^c\subseteq G_n^c$. Now $G_n\cap E_n = E_n$ and $G_n\cap E_n^c = G_{n-1}$. Since E_n is measurable

$$\mu^*(A\cap G_n) = \mu^*(A\cap G_n\cap E) + \mu^*(A\cap G_n\cap E^c) = \mu^*(A\cap E_n) + \mu^*(A\cap G_{n-1})$$

An inductive argument shows that $\mu^*(A\cap G_n) = \sum_{i=1}^n \mu^*(A\cap E_i)$. Therefore

$$\mu^*(A) \ge \mu^*(A\cap E^c) + \sum_{i=1}^n \mu^*(A\cap E_i)$$

for all n. Hence

$$\mu^*(A) \ge \mu^*(A\cap E^c) + \sum_{i=1}^\infty \mu^*(A\cap E_i) \ge \mu^*(A\cap E^c) + \mu^*\left(\bigcup_{i=1}^\infty (A\cap E_i)\right) = \mu^*(A\cap E^c) + \mu^*(A\cap E)$$

Thus E is measurable. Given any sequence $\{E_i\}_{i=1}^{\infty} \subseteq \mathscr{B}$ with $E = \bigcup_{i=1}^{\infty} E_i$, we can fine a pairwise disjoint sequence $\{F_i\}_{i=1}^{\infty} \subseteq \mathscr{B}$ with $E = \bigcup_{i=1}^{\infty} F_i$. Hence $E \in \mathscr{B}$ and $\mathscr{B}$ is a σ-algebra.

Let $E_1, E_2 \in \mathscr{B}$ be disjoint. Then since E_2 is measurable,

$$\mu(E_1 \cup E_2) = \mu^*(E_1 \cup E_2) = \mu^*((E_1 \cup E_2) \cap E_2) + \mu^*((E_1 \cup E_2) \cap E_2^c) = \mu^*(E_2) + \mu^*(E_1) = \mu(E_2) + \mu(E_1)$$

Hence μ is finitely additive. Let $\{E_i\}_{i=1}^{\infty} \subseteq \mathscr{B}$ be a sequence of pairwise disjoint μ^*-measurable sets and let $E = \bigcup_{i=1}^{\infty} E_i$. Then

$$\mu(E) \geq \mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i)$$

for every n. Taking the limit we have $\mu(E) \geq \sum_{i=1}^{\infty} \mu(E_i)$. On the other hand,

$$\mu(E) = \mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i) = \sum_{i=1}^{\infty} \mu(E_i)$$

Thus $\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$ and μ is countably additive. Clearly $\mu(\varnothing) = 0$ and $\mu(E) \geq 0$ for all $E \in \mathscr{B}$, so μ is a measure on $\mathscr{B}$.

Let $\mu(E) = 0$ and $F \subseteq E$. Then

$$\mu(A) \geq \mu^*(A \cap F^c) = \mu^*(A \cap F^c) + \mu^*(A \cap E) \geq \mu^*(A \cap F^c) + \mu^*(A \cap F)$$

so $F \in \mathscr{B}$. Therefore $(X, \mathscr{B}, \mu = \mu^*|_{\mathscr{B}})$ is complete. □

4 Example: Lebesgue Measure on $\mathbb{R}$

Let I be an interval in $\mathbb{R}$. Let $\ell(I)$ denote its length. For any $E \subseteq \mathbb{R}$ define

$$m^*(E) = \inf\left\{\sum_{n=1}^{\infty} \ell(I_n) \mid E \subseteq \bigcup_{n=1}^{\infty} I_n,\ I_n\text{'s are open intevals}\right\}$$

From this definition it is clear that:

1. $m^*(\varnothing) = 0$
2. $m^*(E) \geq 0$
3. If $F \subseteq E$, then $m^*(F) \leq m^*(E)$.

4.1 Theorem. *m^* is an outer measure on $\mathscr{P}(\mathbb{R})$.*

PROOF: We need only show that m^* is coutably subadditive. Let $\{E_n\}_{n=1}^{\infty} \subseteq \mathscr{P}(\mathbb{R})$. We may assume without loss of generality that $m^*(E_n) < \infty$ for all n. Let $\varepsilon > 0$. For each n choose a countable collection $\{I_{i,n}\}$ of open intevals with $E_n \subseteq \bigcup_{i=1}^{\infty} I_{i,n}$ such that $\sum_{i=1}^{\infty} \ell(I_{i,n}) \leq m^*(E_n) + \frac{\varepsilon}{2^n}$. Note that $\bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{i,n=1}^{\infty} I_{i,n}$, so

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{i,n=1}^{\infty} \ell(I_{i,n}) = \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} \ell(I_{i,n}) \leq \sum_{n=1}^{\infty}\left(m^*(E_n) + \frac{\varepsilon}{2^n}\right) \leq \left(\sum_{n=1}^{\infty} m^*(E_n)\right) + \varepsilon$$

Since this is true for all $\varepsilon > 0$ we get $m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m^*(E_n)$. □

4.2 Definition. m^* is called the *Lebesgue outer measure* on $\mathbb{R}$. We denote the σ-algebra of m^*-measurable sets by $\mathcal{M}(\mathbb{R})$. Elements of $\mathcal{M}(\mathbb{R})$ are said to be *Lebesgue measurable*. $m = m^*|_{\mathcal{M}(\mathbb{R})}$ is called the *Lebesgue measure* on $\mathbb{R}$.

4.3 Proposition. *If I is an interval then $m^*(I) = \ell(I)$.*

PROOF: Assume that $I = [a, b]$ and let $\varepsilon > 0$. Then if $I_1 = (a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2})$ we have $I \subseteq I_1$, so

$$m^*(I) \leq \ell(I_1) = b - a + \varepsilon = \ell(I) + \varepsilon$$

Since ε was arbitrary, $m^*(I) \leq \ell(I)$. Assume that $I \subseteq \bigcup_{i=1}^{\infty} I_i$, where each I_i is an open interval. Since $[a, b]$ is compact there is a finite subcover. Out of these finitely many elements, choose one that contains a. Without loss of generality suppose it is I_1. If $b \in I_1$ then we are done. Otherwise choose an interval that contains b_1 and call it I_2. Continue this process to get $I_1, \dots, I_k$ such that $I \subseteq \bigcup_{i=1}^{k} I_i$. Furthermore, if $I_j = (a_j, b_j)$ then $a_1 < a < b_1 < \cdots < b < b_k$ and $b_i - a_i \geq b_i - b_{i-1}$. Hence

$$\sum_{i=1}^{k} b_i - a_i \geq b_1 - a_1 + \sum_{i=2}^{k} b_i - b_{i-1} = b_k - a_1 > b - a$$

Therefore

$$\sum_{i=1}^{\infty} \ell(I_i) \geq \sum_{i=1}^{k} \ell(I_i) = \sum_{i=1}^{k} b_i - a_i > b - a = \ell(I)$$

Hence $m^*(I) \geq \ell(I)$, so $m^*(I) = \ell(I)$.

Assume that I is a finite interval. For any $\varepsilon > 0$ we can find a closed interval $J \subseteq I$ such that $\ell(I) < \ell(J) + \varepsilon$. Hence

$$\ell(I) - \varepsilon < \ell(J) = m^*(J) \leq m^*(I) \leq m^*(\bar{I}) = \ell(\bar{I}) = \ell(I)$$

so $m^*(I) = \ell(I)$.

Finally, if $\ell(I) = \infty$ then for any $M > 0$ we can find a finite interval $J \subseteq I$ with $\ell(J) > M$. Then $m^*(I) \geq m^*(J) = \ell(J) > M$, so $m^*(I) = \infty = \ell(I)$. □

4.4 Lemma. *The intervals (a, ∞) and $(-\infty, a]$ are m^*-measurable.*

PROOF: Let $A \subseteq \mathbb{R}$. We need to show that

$$m^*(A) \geq m^*(A \cap (a, \infty)) + m^*(A \cap (-\infty, a])$$

and moreover, we may assume that $m^*(A) < \infty$. Let $\varepsilon > 0$. We can find a collection of open intervals with $A \subseteq \bigcup_{i=1}^{\infty} I_i$ and $\sum_{i=1}^{\infty} \ell(I_i) < m^*(A) + \varepsilon$. Let $I_n' = I_n \cap (a, \infty)$ and $I_n'' = I_n \cap (-\infty, a]$. Then I_n' and I_n'' are intervals and $\ell(I_n) = \ell(I_n') + \ell(I_n'') = m^*(I_n') + m^*(I_n'')$. We have

$$m^*(A \cap (a, \infty)) \leq m^*\left(\bigcup_{i=1}^{\infty} I_i'\right) \leq \sum_{i=1}^{\infty} m^*(I_i') \quad \text{and} \quad m^*(A \cap (-\infty, a]) \leq m^*\left(\bigcup_{i=1}^{\infty} I_i''\right) \leq \sum_{i=1}^{\infty} m^*(I_i'')$$

Hence

$$\begin{aligned} m^*(A\cap(a,\infty))+m^*(A\cap(-\infty,a]) &\leq \sum_{i=1}^{\infty} m^*(I_i') + \sum_{i=1}^{\infty} m^*(I_i'') \\ &= \sum_{i=1}^{\infty} m^*(I_i') + m^*(I_i'') \\ &= \sum_{i=1}^{\infty} \ell(I_i') + \ell(I_i'') \\ &= \sum_{i=1}^{\infty} \ell(I_i) \leq m^*(A)+\varepsilon \end{aligned}$$

Since ε was arbitrary, $m^*(A) \geq m^*(A\cap(a,\infty))+m^*(A\cap(-\infty,a])$. □

4.5 Theorem. $\mathscr{B}(\mathbb{R}) \subseteq \mathcal{M}(\mathbb{R})$.

PROOF: Trivial, given the last lemma. □

Clearly every countable or finite set has Lebesgue measure 0. m^* is translation invariant, so Lebesgue measure is also translation invariant. Indeed, let $E \in \mathcal{M}(\mathbb{R})$, $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Then

$$\begin{aligned} m^*(x+A) &= m^*(A) \\ &= m^*(A\cap E) + m^*(A \cap E^c) \\ &= m^*((x+A)\cap(x+E)) + m^*((x+A)\cap(x+E^c)) \\ &= m^*((x+A)\cap(x+E)) + m^*((x+A)\cap(x+E)^c) \end{aligned}$$

If A is arbitrary then so is B, so $x+A \in \mathcal{M}(\mathbb{R})$.

4.6 Theorem. *Let $E \subseteq \mathbb{R}$. The following are equivalent:*

1. $E \in \mathcal{M}(\mathbb{R})$
2. *Given $\varepsilon > 0$ there is an open set $U \subseteq \mathbb{R}$ with $E \subseteq U$ and $m^*(U \setminus E) < \varepsilon$.*
3. *Given $\varepsilon > 0$ there is a closed set $F \subseteq \mathbb{R}$ with $F \subseteq E$ and $m^*(E \setminus F) < \varepsilon$.*
4. *There is a G_δ set $G \subseteq \mathbb{R}$ with $E \subseteq G$ and $m^*(G \setminus E) = 0$.*
5. *There is an F_σ set $F \subseteq \mathbb{R}$ with $F \subseteq E$ and $m^*(E \setminus F) = 0$.*

If $m^*(E) < \infty$ then these are all equivalent to:

(6) Given $\varepsilon > 0$ there is a set $U \subseteq \mathbb{R}$ which is the union of finitely many open intervals such that $m^*(U \Delta E) < \varepsilon$.

PROOF: Exercise. □

4.7 Example. Let $x, y \in [0,1)$ and let $x \oplus y := x + y \pmod 1$. If $E \subseteq [0,1)$ is measurable then $x \oplus E$ is measurable and $m(E) = m(x \oplus E)$. Indeed, let $E_1 = \{y \in E \mid x + y < 1\} = E \cap (-\infty, 1-x)$ and $E_2 = \{y \in E \mid x + y \geq 1\} = E \cap [1-x, \infty)$. Then $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \varnothing$, and E_1, E_2 are both measurable. It follows that

$$x \oplus E = (x \oplus E_1) \cup (x \oplus E_2) = (x \oplus E_1) \cup ((x-1) \oplus E_2)$$

is measurable and $m(x \oplus E) = m(E)$.

Define an equivalence relation on $[0,1)$ by $x \sim y \iff x - y \in \mathbb{Q}$. Using the Axiom of Choice, construct a set $E \subseteq [0,1)$ consisting of one element from each equivalence class. Let $\mathbb{Q} \cap [0,1) = \{r_1, r_2, \ldots\}$ be an enumeration of the rationals in $[0,1)$. Let $E_n := r_n \oplus E$. Is E measurable? If E is measurable then E_n is measurable and $m(E) = m(E_n)$ for each n. But $[0,1) = \bigcup_{n=1}^{\infty} E_n$, and the E_n's are pairwise disjoint, so

$$1 = m([0,1)) = \left(\bigcup_{n=1}^{\infty} E_n \right) = \sum_{n=1}^{\infty} m(E_n) = \sum_{n=1}^{\infty} m(E)$$

which is impossible. As an exercise, show that, assuming the Axiom of Choice, if $m(E) > 0$, then E contains a non-measurable set.

The set E in the previous example is not Borel, since the Borel sets are measurable. Unfortunately E is not a particularily nice non-Borel set, as its existence depends upon the Axiom of Choice. Given a Borel set A in $\mathbb{R}^2$, is it necessarily true that the projection of A onto the real line is Borel? The answer is no, but this is not obvious.

4.8 Example. The Cantor set C is compact, nowhere dense, and has cardinality c. It turns out that it also has measure zero. It follows from this that the cardinality of $\mathcal{M}(\mathbb{R})$ is 2^c.

5 Extending Measures

5.1 Definition. Let $\mathscr{A} \subseteq \mathscr{P}(X)$ be an algebra. A *measure* on $\mathscr{A}$ is a function $\mu : \mathscr{A} \to \mathbb{R}^*$ such that

1. $\mu(\emptyset) = 0$
2. $\mu(E) \geq 0$ for all $E \in \mathscr{A}$
3. If $\{E_n\}_{n=1}^{\infty} \subseteq \mathscr{A}$ is are pairwise disjoint and $\bigcup_{n=1}^{\infty} E_n \in \mathscr{A}$, then $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$.

Given a measure μ on $\mathscr{A}$ define $\mu^* : \mathscr{A} \to \mathbb{R}^*$ by

$$\mu^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \mu(E_n) \mid \{E_n\} \subseteq \mathscr{A}, A \subseteq \bigcup_{n=1}^{\infty} E_n \right\}$$

5.2 Proposition. *Using the notation defined above:*

1. $\mu^*(\emptyset) = 0$
2. $\mu^*(B) \geq 0$ *for any* $B \subseteq X$
3. *If* $A \subseteq B$ *then* $\mu^*(A) \leq \mu^*(B)$.
4. *If* $B \in \mathscr{A}$ *then* $\mu^*(B) = \mu(B)$
5. *If* $\{B_n\}_{n=1}^{\infty} \subseteq \mathscr{P}(X)$ *then* $\mu^*\left(\bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty} \mu(B_n)$.

PROOF: (1), (2), (3), and (5) are trivial. For (4), notice that $\mu^*(B) \leq \mu(B)$ for all $B \in \mathscr{A}$. On the other hand, let $\{E_n\}_{n=1}^{\infty} \subseteq \mathscr{A}$ be pairwise disjoint such that $B \subseteq \bigcup_{n=1}^{\infty} E_n$. Then $B = \bigcup_{n=1}^{\infty} B \cap E_n$, so since μ is a measure on $\mathscr{A}$, $\mu(B) \leq \sum_{n=1}^{\infty} \mu(B \cap E_n) \leq \sum_{n=1}^{\infty} \mu(E_n)$. Therefore $\mu(B) = \mu^*(B)$. □

It follows that μ^* is an outer measure on $\mathscr{P}(X)$ that extends μ. We call μ^* the outer measure generated by μ.

5.3 Theorem (Caratheodory Extension Theorem). *Let μ be a measure on an algebra $\mathscr{A} \subseteq \mathscr{P}(X)$. Let μ^* be the measure generated by μ. Let $\mathscr{A}^*$ be the σ-algebra of μ^*-measurable sets. Then $\mathscr{A} \subseteq \mathscr{A}^*$. In particular, μ extends to a measure $\overline{\mu}$ on $\mathscr{A}^*$.*

PROOF: We need only show that $\mathscr{A} \subseteq \mathscr{A}^*$. Let $E \in \mathscr{A}$ and let $A \subseteq X$. As before, we can assume that $\mu^*(A) < \infty$ and we need only show that $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$. Let $\varepsilon > 0$. Let $\{F_n\}_{n=1}^\infty \subseteq \mathscr{A}$ such that $A \subseteq \bigcup_{n=1}^\infty F_n$ and $\sum_{n=1}^\infty \mu(F_n) < \mu^*(A) + \varepsilon$. Note that

$$A \cap E \subseteq \bigcup_{n=1}^\infty E \cap F_n \quad \text{and} \quad A \cap E^c \subseteq \bigcup_{n=1}^\infty E^c \cap F_n$$

It follows that

$$\mu^*(A \cap E) \leq \sum_{n=1}^\infty \mu(E \cap F_n) \quad \text{and} \quad \mu^*(A \cap E^c) \leq \sum_{n=1}^\infty \mu(E^c \cap F_n)$$

Hence

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) \leq \sum_{n=1}^\infty \mu(E \cap F_n) + \sum_{n=1}^\infty \mu(E^c \cap F_n) = \sum_{n=1}^\infty \mu(F_n) < \mu^*(A) + \varepsilon$$

Since ε was arbitrary, $\mu^*(A \cap E) + \mu^*(A \cap E^c) \leq \mu^*(A)$. □

Given a measure μ on an algebra $\mathscr{A}$, is the extension of μ to $\mathscr{A}^*$ unique?

5.4 Example. Let $X = (0,1] \cap \mathbb{Q}$. Let $\mathscr{A}$ be the finite union of sets of the form $(a,b] \cap \mathbb{Q}$ where $a, b \in X$. It is easy to see that $\mathscr{P}(X)$ is the smallest (and only) σ-algebra that contains $\mathscr{A}$. Let μ be the counting measure on $\mathscr{A}$. Then

$$\mu(E) = \begin{cases} 0 & \text{if } E = \varnothing \\ \infty & \text{otherwise} \end{cases}$$

This implies for all $A \in \mathscr{P}(X)$ that

$$\mu^*(A) = \begin{cases} 0 & \text{if } A = \varnothing \\ \infty & \text{otherwise} \end{cases}$$

The extension theorem gives that $\mathscr{A}^* = \mathscr{P}(X)$ and $\overline{\mu} = \mu^*$. On the other hand, the counting measure on $\mathscr{P}(X) = \mathscr{A}^*$ also extends μ, but is not equal to $\overline{\mu}$.

5.5 Theorem (Hahn Extension Theorem). *Suppose that μ is a σ-finite measure on an algebra $\mathscr{A}$. Then there exists a unique extension of μ to a measure $\overline{\mu}$ on $\mathscr{A}^*$, the σ-algebra of all μ^*-measurable sets.*

PROOF: Let γ be a measure on $\mathscr{A}^*$ that agrees with μ on $\mathscr{A}$. Let $\overline{\mu}$ be the Caratheodory extension. First assume that μ is finite (so that $\mu(X) < \infty$). Then $\overline{\mu}$ and γ are both finite. Let $E \in \mathscr{A}^*$ and let $\{E_n\}_{n=1}^\infty \subseteq \mathscr{A}$ with $E \subseteq \bigcup_{n=1}^\infty E_n$. Then

$$\gamma(E) \leq \gamma\left(\bigcup_{n=1}^\infty E_n\right) \leq \sum_{n=1}^\infty \gamma(E_n) = \sum_{n=1}^\infty \mu(E_n)$$

Therefore $\gamma(E) \leq \mu^*(E) = \overline{\mu}(E)$. But

$$\gamma(E) + \gamma(E^c) = \gamma(X) = \mu(X) = \overline{\mu}(X) = \overline{\mu}(E) + \overline{\mu}(E^c)$$

and since $\gamma(E) \leq \overline{\mu}(E)$ and $\gamma(E^c) \leq \overline{\mu}(E^c)$ we must have $\gamma(E) = \overline{\mu}(E)$ for all $E \in \mathscr{A}^*$.

Now assume that μ is σ-finite. Let $\{F_n\}_{n=1}^\infty$ be an increasing sequence in $\mathscr{A}$, with $\mu(F_n) < \infty$ and $X = \bigcup_{n=1}^\infty F_n$. We would then have that $\overline{\mu}(E \cap F_n) = \gamma(E \cap F_n)$ for all $n \geq 1$ and for all $E \in \mathscr{A}^*$. However,

$$\overline{\mu}(E) = \lim_{n\to\infty} \overline{\mu}(E \cap F_n) = \lim_{n\to\infty} \gamma(E \cap F_n) = \gamma(E)$$

and the result is proven. □

5.6 Example. (Lebesgue-Stieltjes Measures)

Let $\mathscr{A}$ be the collection of all finite unions of sets of the form $(-\infty, b]$, (a, ∞), or $(a, b]$. Then $\mathscr{A}$ is an algebra. Let $F(x)$ be an nondecreasing function on $\mathbb{R}$, with $F(c) = \lim_{x \to c^+} F(x)$ for every $c \in \mathbb{R}$. Since F is monotonic, $\lim_{x\to\infty} F(x)$ and $\lim_{x\to-\infty} F(x)$ both exist as extended real numbers. Define

1. $\mu_F((a, b]) = F(b) - F(a)$
2. $\mu_F((a, \infty)) = \lim_{x\to\infty} F(x) - F(a)$
3. $\mu_F((-\infty, b]) = F(b) - \lim_{x\to-\infty} F(x)$
4. $\mu_F((-\infty, \infty)) = \lim_{x\to\infty} F(x) - \lim_{x\to-\infty} F(x)$

We extend μ_F to $\mathscr{A}$ in the obvious way. Then μ_F is a σ-finite measure on $\mathscr{A}$ (check this claim). The Hahn Extension theorem tells us that μ_F extends to a unique measure on $\mathscr{A}^*$, which we shall also denote by μ_F. $\mathscr{B}(\mathbb{R}) \subseteq \mathscr{A}^*$ for each F, and for $F(x) = x$ we get $\mu_F = m$, the Lebesgue measure.

5.7 Definition. μ_F is called the *Lebesgue-Stieltjes measure* generated by F. $\mu_F|_{\mathscr{B}(\mathbb{R})}$ is often called the *Borel-Stieltjes measure* generated by F.

5.8 Definition. Let $(X, \mathscr{A})$ be a measurable space and let μ, ν be measures on $\mathscr{A}$. A property holds *μ-almosts everywhere*, or μ a.e., if the property holds everywhere except on a set E with $\mu(E) = 0$. We say that μ is *absolutely continuous* with respect to ν if for every $E \in \mathscr{A}$ with $\nu(E) = 0$ then $\mu(E) = 0$. We write $\mu \ll \nu$. We say that μ and ν are *mutually singular* if there exist $A, B \in \mathscr{A}$ that partition X such that $\mu(A) = 0 = \nu(B)$. We write $\mu \perp \nu$.

6 Measurable Functions

6.1 Definition. Let $(X, \mathscr{A})$ be a measurable space. We say that $f : X \to \mathbb{R}$ is measurable if $f^{-1}((\alpha, \infty)) \in \mathscr{A}$ for every $\alpha \in \mathbb{R}$. Let $\mathcal{M}(X, \mathscr{A})$ denote the set of measurable functions.

6.2 Lemma. *The following are equivalent:*

1. *f is measurable.*
2. *$f^{-1}((-\infty, \alpha]) \in \mathscr{A}$ for all $\alpha \in \mathbb{R}$.*
3. *$f^{-1}([\alpha, \infty)) \in \mathscr{A}$ for all $\alpha \in \mathbb{R}$.*
4. *$f^{-1}((-\infty, \alpha)) \in \mathscr{A}$ for all $\alpha \in \mathbb{R}$.*

6.3 Example. 1. Given any $(X, \mathscr{A})$, $f(x) = c \in \mathbb{R}$ for all $x \in \mathbb{R}$ is measurable.

2. If $A \subseteq X$, the characteristic function of A, $\chi_A(x)$, is measurable if and only if $A \in \mathscr{A}$.
3. If $X = \mathbb{R}$ and $\mathscr{A} = \mathscr{B}(\mathbb{R})$ then the continuous functions are measurable since $f^{-1}((\alpha, \infty))$ is open and hence measurable.

6.4 Lemma. *Assume that $f, g \in \mathcal{M}(X, \mathscr{A})$.*

1. *$cf \in \mathcal{M}(X, \mathscr{A})$ for every $c \in \mathbb{R}$*
2. *$f^2 \in \mathcal{M}(X, \mathscr{A})$*
3. *$f + g \in \mathcal{M}(X, \mathscr{A})$*
4. *$fg \in \mathcal{M}(X, \mathscr{A})$*
5. *$|f| \in \mathcal{M}(X, \mathscr{A})$*

PROOF: Exercise. □

6.5 Definition. Let $(X, \mathscr{A})$ be a measure space and let $f : X \to \mathbb{R}$. Let $f^+(x) := \sup\{f(x), 0\}$ and $f^-(x) := \sup\{-f(x), 0\}$.

It follows that $f = f^+ - f^-$ and $|f| = f^+ + f^-$, so $f^+ = \frac{1}{2}(|f| + f)$ and $f^- = \frac{1}{2}(|f| - f)$. Hence f^+ and f^- are measureable if and only if f is measurable.

6.6 Definition. Let $(X, \mathscr{A})$ be a measure space. An extended real-valued function is a function $f : X \to \mathbb{R}^*$ is *measurable* if $f^{-1}((\alpha, \infty]) \in \mathscr{A}$.

Notice that both $f^{-1}(\{\infty\}) = \bigcap_{n=1}^{\infty} f^{-1}((n, \infty])$ and $f^{-1}(\{-\infty\}) = \left(\bigcup_{n=1}^{\infty} f^{-1}((-n, \infty])\right)^c$ and measurable.

6.7 Proposition. *An extended real-valued function $f : X \to \mathbb{R}^*$ is measurable if and only if $f^{-1}(\{\infty\})$ and $f^{-1}(\{-\infty\})$ are measurable and the real-valued function defined by*

$$f_1(x) = \begin{cases} 0 & x \in f^{-1}(\{\infty\}) \cup f^{-1}(\{-\infty\}) \\ f(x) & \textit{otherwise} \end{cases}$$

The collection of all extended real-valued measurable functions will be denoted by $\mathscr{M}(X, \mathscr{A})$. With the convention that $0(\pm\infty) = 0$, $cf, f^2, f^+, f^- \in \mathscr{M}(X, \mathscr{A})$ if and only if $f \in \mathscr{M}(X, \mathscr{A})$. For $f + g$, we use the convention that $\infty - \infty = 0$. Then $f + g \in \mathscr{M}(X, \mathscr{A})$ if $f, g \in \mathscr{M}(X, \mathscr{A})$.

6.8 Definition. Given $\{a_n\} \subseteq \mathbb{R}$, define

$$\limsup_{n\to\infty} a_n := \inf_n \sup_{k \geq n}\{a_k\} \quad \text{and} \quad \liminf_{n\to\infty} a_n := \sup_n \inf_{k \geq n}\{a_k\}$$

In general, $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$ for any sequence $\{a_n\}$.

6.9 Proposition. *If $\{f_n\} \in \mathscr{M}(X, \mathscr{A})$ then let*

$$f(x) = \inf_n\{f_n(x)\} \qquad F(x) = \sup_n\{f_n(x)\}$$
$$f^*(x) = \liminf_n\{f_n(x)\} \qquad F^*(x) = \limsup_n\{f_n(x)\}$$

All of these functions are measurable.

PROOF: Exercise. □

6.10 Corollary. *Assume that $\{f_n\} \in \mathscr{M}(X, \mathscr{A})$ and $f(x) = \lim_{n\to\infty} f_n(x)$ exists for each $x \in X$. Then $f \in \mathscr{M}(X, \mathscr{A})$.*

6.11 Definition. $\mathscr{M}^+ := \{f \in \mathscr{M}(X, \mathscr{A}) \mid f(x) \geq 0 \ \forall\, x \in X\}$.

6.12 Definition. A *simple function* is a function $\varphi : X \to \mathbb{R}$ such that $\varphi(X)$ is finite. If the range of φ is $\{a_1, \dots, a_n\}$ and $E_i = \varphi^{-1}(\{a_i\})$ then $\varphi = \sum_{i=1}^n a_i \chi_{E_i}$. We say that φ is in *standard form* if the a_i's are distinct and the E_i's partition X.

Up to order, the standard form of a simple function is unique. $\varphi = \sum_{i=1}^n a_i \chi_{E_i}$ (in standard form) is mearsurable if and only if each E_i is measurable.

6.13 Theorem. *Let $f \in \mathscr{M}^+(X, \mathscr{A})$. Then there exists $\{\varphi_n\} \subseteq \mathscr{M}^+(X, \mathscr{A})$ such that*

1. *φ_n is simple for each n.*

2. $0 \le \varphi_n(x) \le \varphi_{n+1}(x)$ *for all* $x \in X$ *and each* n.
3. $f(x) = \lim_{n\to\infty} \varphi_n(x)$ *for all* $x \in X$.

PROOF: Let $n \in \mathbb{N}$. For $k = 0, 1, \ldots, n2^n - 1$, let $E_{n,k} := \{x \in X \mid \frac{k}{2^n} \le f(x) < \frac{k+1}{2^n}\}$ and if $k = n2^n$ then let $E_{n,k} := \{x \in X \mid f(x) \ge n\}$. Define $\varphi_n = \sum_{k=0}^{n2^n} \frac{k}{2^n} \chi_{E_{n,k}}$. It is easy to see that $\{\varphi_n\}$ is the desired sequence. (Fill in the details.) □

Let $f : X \to \mathbb{C}$. Then there are real valued functions $f_1, f_1 : X \to \mathbb{R}$ such that $f = f_1 + if_2$. We say that f is measurable if f_1 and f_2 are measurable.

7 Integration

7.1 Definition. Assume that $\varphi \in \mathscr{M}^+(X, \mathscr{A})$ is simple, and suppose that $\varphi = \sum_{i=1}^n a_i \chi_{E_i}$ is the standard form for φ. For any measure μ on $(X, \mathscr{A})$, we define $\int \varphi \, d\mu := \sum_{i=1}^n a_i \mu(E_i)$.

7.2 Exercise. Assume that $\varphi = \sum_{i=1}^n a_i \chi_{E_i}$, where $E_i \in \mathscr{A}$, but this is not the standard form for φ. Prove that we still have $\int \varphi \, d\mu := \sum_{i=1}^n a_i \mu(E_i)$.

7.3 Lemma. *Let* $\varphi, \psi \in \mathscr{M}^+(X, \mathscr{A})$ *be simple functions and* $c \ge 0$. *Then*

1. $\int c\varphi \, d\mu = c \int \varphi \, d\mu$
2. $\int (\varphi + \psi) d\mu = \int \varphi \, d\mu + \int \psi \, d\mu$

Furthermore, if we define $\lambda : \mathscr{A} \to \mathbb{R}^*$ *by* $\lambda(E) = \int \varphi \chi_E \, d\mu$ *then* λ *is a measure on* $\mathscr{A}$.

PROOF: Exercise. □

7.4 Definition. If $f \in \mathscr{M}^+(X, \mathscr{A})$ we define

$$\int f \, d\mu := \sup_{\varphi} \int \varphi \, d\mu$$

where the supremum is taken over all simple functions $\varphi \in \mathscr{M}(X, \mathscr{A})$ with $0 \le \varphi \le f$. If $E \in \mathscr{A}$, then $\int_E f \, d\mu := \int f \chi_E \, d\mu$.

7.5 Lemma. 1. *If* $f, g \in \mathscr{M}^+(X, \mathscr{A})$ *and* $f \le g$ *then* $\int f \, d\mu \le \int g \, d\mu$.

2. *If* $E \subseteq F \in \mathscr{A}$ *then* $\int_E f \, d\mu \le \int_F f \, d\mu$ *for all* $f \in \mathscr{M}^+(X, \mathscr{A})$.

7.6 Theorem (Monotone Convergence Theorem). *If* $\{f_n\}_{n=1}^\infty \subseteq \mathscr{M}^+(X, \mathscr{A})$ *is such that* $f_n \le f_{n+1}$ *for all* $n \ge 1$ *and* $\lim_{n\to\infty} f_n(x) = f(x)$ *then*

$$\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu$$

PROOF: We know that $f \in \mathscr{M}^+(X, \mathscr{A})$ and

$$\int f_n \, d\mu \le \int f_{n+1} \, d\mu \le \int f \, d\mu$$

Hence $\lim_{n\to\infty}\int f_n d\mu \le \int f\,d\mu$. Let $0<\alpha<1$ and let $\varphi\in\mathscr{M}^+(X,\mathscr{A})$ be simple with $0\le\varphi\le f$. For each n, let $A_n=\{x\in X \mid f_n(x)\ge\alpha\varphi(x)\}$. Then $A_n\in\mathscr{A}$, $A_n\subseteq A_{n+1}$, and $X=\bigcup_{n=1}^{\infty}A_n$. We have

$$\int_{A_n}\alpha\varphi\,d\mu \le \int_{A_n} f_n d\mu \le \int f_n d\mu$$

By the Monotone Convergence Theorem for measures, $\int\varphi\,d\mu=\lim_{n\to\infty}\int_{A_n}\varphi\,d\mu$ (since $\lambda(E)=\int_E\varphi\,d\mu$ is a measure). Therefore

$$\alpha\int\varphi\,d\mu=\lim_{n\to\infty}\int_{A_n}\alpha\varphi\,d\mu\le\lim_{n\to\infty}\int f_n d\mu$$

Since $0\le\alpha<1$ is arbitrary, $\int\varphi\,d\mu\le\lim_{n\to\infty}\int f_n d\mu$. Thus $\int f\,d\mu=\sup_{0\le\varphi\le f}\int\varphi\,d\mu\le\lim_{n\to\infty}\int f_n d\mu$, and the result is proved. □

7.7 Corollary. *1. If $f\in\mathscr{M}^+(X,\mathscr{A})$ and $c\ge 0$ then $\int cf\,d\mu=c\int f\,d\mu$.*

2. If $f,g\in\mathscr{M}^+(X,\mathscr{A})$ then $\int(f+g)d\mu=\int f\,d\mu+\int f\,d\mu$.

7.8 Corollary (Fatou's Lemma). *If $\{f_n\}_{n=1}^{\infty}\subseteq\mathscr{M}^+(X,\mathscr{A})$ then*

$$\int\liminf_{n\to\infty}f_n d\mu\le\liminf_{n\to\infty}\int f_n d\mu$$

PROOF: For $m\in\mathbb{N}$, let $g_m=\inf_{n\ge m}\{f_n\}$. Then $g_m\le f_n$ for all $n\ge m$. Hence $\int g_m d\mu\le\int f_n d\mu$ for all $n\ge m$. This implies that $\int g_m d\mu\le\liminf_{n\to\infty}\int f_n d\mu$. Note that $g_m\nearrow\liminf_{n\to\infty}f_n$. By the Monotone Converge Theorem,

$$\int\liminf_{n\to\infty}f_n d\mu=\lim_{m\to\infty}\int g_m d\mu\le\liminf_{n\to\infty}\int f_n d\mu$$

□

7.9 Corollary. *Let $f\in\mathscr{M}^+(X,\mathscr{A})$ and let $\lambda(E)=\int_E f\,d\mu$ for all $E\in\mathscr{A}$. Then λ is a measure.*

PROOF: Since $f\ge 0$, $\lambda(E)\ge 0$ for all $E\in\mathscr{A}$, and clearly $\lambda(\varnothing)=0$. Let $\{E_n\}_{n=1}^{\infty}\subseteq\mathscr{A}$ be pairwise disjoint and $E=\bigcup_{n=1}^{\infty}E_n$. Let $f_n=\sum_{k=1}^{n}f\chi_{E_k}$. Then

$$\int f_n d\mu=\sum_{k=1}^{\infty}\int_{E_k}f\,d\mu=\sum_{k=1}^{n}\lambda(E_k)$$

Notice that $f_n\nearrow f\chi_E$, so by the Monotone Convergence Theorem,

$$\lambda(E)=\int_E f\,d\mu=\lim_{n\to\infty}f_n d\mu=\lim_{n\to\infty}\sum_{k=1}^{n}\lambda(E_k)=\sum_{n=1}^{\infty}\lambda(E_n)$$

□

Question: If we have a measure space $(X,\mathscr{A},\mu)$ and another measure λ on $\mathscr{A}$, if $\lambda(E)=\int_E f\,d\mu$ for some $f\in\mathscr{M}^+(X,\mathscr{A})$? That is, does a converse to the above corollary hold?

7.10 Corollary. *Suppose that $f\in\mathscr{M}^+(X,\mathscr{A})$. Then $f(x)=0$ μ-a.e. if and only if $\int f\,d\mu=0$.*

PROOF: Assume that $\int f\,d\mu = 0$. For each $n \in \mathbb{N}$ let $E_n = \{x \in X \mid f(x) > \frac{1}{n}\}$ Then $f \geq \frac{1}{n}\chi_{E_n}$ for each $n \in \mathbb{N}$, which implies that

$$0 = \int f\,d\mu \geq \int_{E_n} \frac{1}{n}\,d\mu = \frac{1}{n}\mu(E_n)$$

so $\mu(E_n) = 0$. If $E = \{x \in X \mid f(x) \neq 0\}$ then $E = \bigcup_{n=1}^{\infty} E_n$, so $\mu(E) \leq \sum_{n=1}^{\infty} \mu(E_n) = 0$.

Now suppose that $f(x) = 0$ a.e. Let $E = \{x \in X \mid f(x) > 0\}$ and let $f_n = n\chi_E$. Since $f \leq \liminf_{n\to\infty} f_n$, by Fatou's Lemma

$$0 \leq \int f\,d\mu \leq \liminf_{n\to\infty} \int f_n\,d\mu = \liminf_{n\to\infty} n\mu(E) = 0$$

Therefore $\int f\,d\mu = 0$. □

7.11 Proposition. *If $f \in \mathcal{M}^+(X, \mathscr{A})$ and $\lambda(E) = \int_E f\,d\mu$ then λ is absolutely continuous with respect to μ.*

PROOF: If $\mu(E) = 0$ then $\chi_E f = 0$, so μ-a.e. Hence $\lambda(E) = \int_E f\,d\mu = 0$. □

Question: Given measure μ on $(X, \mathscr{A})$, if λ is any measure on $\mathscr{A}$ with $\lambda \ll \mu$, does there exist $f \in \mathscr{M}^+(X, \mathscr{A})$ such that $\lambda(E) = \int_E f\,d\mu$ for all $E \in \mathscr{A}$. That is, does the converse of the last proposition hold?

7.12 Theorem (Monotone Convergence Theorem, II). *If $\{f_n\}_{n=1}^{\infty}$ is a monotonically increasing sequence in $\mathscr{M}^+(X, \mathscr{A})$ which converges μ-a.e. to $f \in \mathscr{M}^+(X, \mathscr{A})$ then*

$$\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu$$

PROOF: Let $N \in \mathscr{A}$ be such that $\mu(N) = 0$ and $f_n \to f$ on $M = X \setminus N$. Then $f_n\chi_M \nearrow f\chi_M$. The Monotone Convergence Theorem shows that $\int f\chi_M\,d\mu = \lim_{n\to\infty} \int f_n\chi_M\,d\mu$. But

$$\begin{aligned}
\int f\,d\mu &= \int f\chi_N\,d\mu + \int f\chi_M\,d\mu \\
&= 0 + \lim_{n\to\infty} \int f_n\chi_M\,d\mu \\
&= \lim_{n\to\infty} 0 + \int f_n\chi_M\,d\mu \\
&= \lim_{n\to\infty} \int f_n\chi_N\,d\mu + \int f_n\chi_M\,d\mu \\
&= \lim_{n\to\infty} \int f_n\,d\mu
\end{aligned}$$

and the result is proved. □

7.13 Corollary. *Let $\{g_n\}_{n=1}^{\infty} \subseteq \mathscr{M}^+(X, \mathscr{A})$. Then*

$$\int \sum_{n=1}^{\infty} g_n\,d\mu = \sum_{n=1}^{\infty} \int g_n\,d\mu$$

8 Integrable Functions

8.1 Definition. Given $(X, \mathscr{A}, \mu)$, we say that an *extended real-valued function* f is integrable if $\int f^+ d\mu < \infty$ and $\int f^- d\mu < \infty$. In this case we write $\int f\,d\mu := \int f^+ d\mu - \int f^- d\mu$. If $E \in \mathscr{A}$ we define (unsurprisingly) $\int_E f\,d\mu := \int f\chi_E d\mu$.

If $f = f_1 - f_2$, where $f_1, f_2 \in \mathscr{M}^+(X, \mathscr{A})$ and $\int f_1 d\mu < \infty$ and $\int f_2 d\mu < \infty$ then we would like $\int f\,d\mu = \int f_1 d\mu - \int f_2 d\mu$. Indeed, $f^+ - f^- = f_1 - f_2$, so $f^+ + f_2 = f_1 + f^-$. Integrating and using additivity gives us the result.

8.2 Definition. Let $(X, \mathscr{A})$ be a measurable space. A *signed measure* on $(X, \mathscr{A})$ is a function $\mu : \mathscr{A} \to \mathbb{R}^*$ such that

1. μ assumes at most one of ∞ and $-\infty$.
2. $\mu(\varnothing) = 0$.
3. If $\{E_n\} \subseteq \mathscr{A}$ is pairwise disjoint then $\mu(\bigcup_{n=1}^\infty E_n) = \sum_{n=1}^\infty \mu(E_n)$.

8.3 Example. If $(X, \mathscr{A})$ is a measurable space and μ_1 and μ_2 are measures on $(X, \mathscr{A})$ with at least one begin finite, then $\lambda = \mu_1 - \mu_2$ is a signed measure on $(X, \mathscr{A})$.

8.4 Theorem. *If f is integrable with respect to μ then $\lambda(E) = \int_E f\,d\mu$ is a signed measure.*

Proof: $\lambda = \lambda_1 - \lambda_2$ where $\lambda_1(E) = \int_E f^+ d\mu$ and $\lambda_2(E) = \int_E f^- d\mu$. □

8.5 Definition. Let $\mathscr{L}(X, \mathscr{A}, \mu)$ be the set of all integrable functions on $(X, \mathscr{A}, \mu)$.

8.6 Theorem. *If f measurable then $f \in \mathscr{L}(X, \mathscr{A}, \mu)$ if and only if $|f| \in \mathscr{L}(X, \mathscr{A}, \mu)$.*

Proof: We know $f \in \mathscr{L}(X, \mathscr{A}, \mu)$ if and only if $f^+, f^- \in \mathscr{L}(X, \mathscr{A}, \mu)$. Since $|f| = f^+ + f^-$ we get that $|f| \in \mathscr{L}(X, \mathscr{A}, \mu)$. Conversely, if $|f| \in \mathscr{L}(X, \mathscr{A}, \mu)$ then $\int |f| d\mu < \infty$, so both $\int f^+ d\mu^\infty$ and $\int f^- d\mu < \infty$, so $f \in \mathscr{L}(X, \mathscr{A}, \mu)$. □

Notice that if $f \in \mathscr{L}(X, \mathscr{A}, \mu)$ then

$$\left|\int f\,d\mu\right| = \left|\int f^+ d\mu - \int f^- d\mu\right| \le \int f^+ d\mu - \int f^- d\mu = \int |f| d\mu$$

8.7 Proposition. *If $f, g \in \mathscr{L}(X, \mathscr{A}, \mu)$ and $\alpha, \beta \in \mathbb{R}$ then $\alpha f + \beta g \in \mathscr{L}(X, \mathscr{A}, \mu)$ and $\int (\alpha f + \beta g) d\mu = \alpha \int f\,d\mu + \beta \int g\,d\mu$.*

Proof: Exercise. First show that $\alpha f \in \mathscr{L}(X, \mathscr{A}, \mu)$ and that $\int \alpha f\,d\mu = \alpha \int f\,d\mu$ Then show that $f + g \in \mathscr{L}(X, \mathscr{A}, \mu)$ and $\int (f+g) d\mu = \int f\,d\mu + \int g\,d\mu$. Use the fact that $|f + g| \le |f| + |g|$. □

8.8 Theorem (Lebesgue Dominated Convergence Theorem). *Let $\{f_n\}_{n=1}^\infty \subseteq \mathscr{L}(X, \mathscr{A}, \mu)$. Assume that $f = \lim_{n\to\infty} f_n$ μ-a.e. If there exists an integrable function $g \in \mathscr{L}(X, \mathscr{A}, \mu)$ such that $|f_n| \le g$ for all n then f is integrable and $\int f\,d\mu = \lim_{n\to\infty} \int f_n d\mu$.*

PROOF: By redefining f_n, f if necessary, we may assume that $f = \lim_{n\to\infty} f_n$ everywhere. This shows that f is measurable. We have $|f| \le g$, so $|f|$ is integrable and so f is integrable. Notice that $g + f_n \ge 0$. By Fatou's Lemma

$$\begin{aligned}\int g\,d\mu + \int f\,d\mu &= \int (g+f)d\mu \\ &\le \liminf_{n\to\infty} \int (g+f_n)d\mu \\ &= \liminf_{n\to\infty} \int g\,d\mu + \int f_n d\mu \\ &= \int g\,d\mu + \liminf_{n\to\infty} \int f_n d\mu\end{aligned}$$

so $\int f\,d\mu \le \liminf_{n\to\infty} \int f_n d\mu$. On the other hand, $g - f_n \ge 0$, so

$$-\int f\,d\mu \le \liminf_{n\to\infty} \int -f_n d\mu = -\limsup_{n\to\infty} \int f_n d\mu$$

Therefore $\int f\,d\mu = \lim_{n\to\infty} \int f_n d\mu$. □

9 L^p-spaces

This section is in need of repairs.

We have seen that $\mathscr{L}(X, \mathscr{A}, \mu)$ is a vector space over $\mathbb{R}$. For $f \in \mathscr{L}(X, \mathscr{A}, \mu)$, define $\|f\|_1 = \int |f| d\mu$. Then $\|\cdot\|_1$ defines a seminorm on $\mathscr{L}(X, \mathscr{A}, \mu)$. Let $\sim$ be the equivalence relation defined on $\mathscr{L}(X, \mathscr{A}, \mu)$ by $f \sim g \iff \|f - g\|_1 = 0$. Let $S = \{f \in \mathscr{L}(X, \mathscr{A}, \mu) \mid \|f\|_1 = 0\}$, the subspace of all integrable functions whose seminorm is zero.

9.1 Definition. Let $L^1(X, \mathscr{A}, \mu) = L^1(X, \mu) := \mathscr{L}(X, \mathscr{A}, \mu)/\sim$.

L^1 is a normed vector space with respect to the norm $\|[f]\|_1 := \|f\|_1$. In an abuse of notation we will normally write f to represent $[f]$.

9.2 Example. Take $X = \mathbb{N}$ and μ the counting measure on $\mathbb{N}$. Then $\mathscr{L}(\mathbb{N}, \mathscr{P}(\mathbb{N}), \mu) = L^1(\mathbb{N}, \mu) = \ell_1(\mathbb{N})$.

9.3 Definition. Let $1 \le p < \infty$. Define

$$L^1(X, \mathscr{A}, \mu) = L^1(X, \mu) := \{[f] \mid f \text{ is measurable and } |f|^p \text{ is integrable}\}$$

Let $\|f\|_p := \left(\int |f|^p d\mu\right)^{\frac{1}{p}}$.

9.4 Definition. Let $f \in \mathscr{M}(X, A)$. We say that f is essentially bounded if there exists M such that $\mu(\{x \in X \mid |f(x)| > M\}) = 0$. Let $\|f\|_\infty = \int\{M \mid \mu(\{x \in X \mid |f(x)| > M\}) = 0\}$. Then $\|\cdot\|_\infty$ is a seminorm on $\mathscr{L}^\infty(X, \mathscr{A}, \mu) = \{f \in \mathscr{M}(X, \mathscr{A}) \mid f \text{ is essentially bounded}\}$. Let $L^\infty(X, \mathscr{A}, \mu) = L^\infty(X, \mu) = \mathscr{L}^\infty(X, \mathscr{A}, \mu)/\sim$.

9.5 Theorem (Hölder's Inequality). *Let* $1 \le p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Let* $f \in L^p(X, \mu)$ *and* $g \in L^q(X, \mu)$. *Then* $fg \in L^1(X, \mu)$ *and*

$$\|fg\|_1 \le \|f\|_p \|g\|_q$$

9.6 Theorem (Minkowski's Inequality). *If $f, g \in L^p(X,\mu)$ then $f+g \in L^p(X,\mu)$ and*

$$\|f+g\|_p \le \|f\|_p + \|g\|_p$$

9.7 Corollary. *$(L^p(X,\mu), \|\cdot\|_p)$ is a normed linear space for all $1 \le p < \infty$.*

9.8 Theorem (Completeness of L^p). *Let $1 \le p \le \infty$. Then $(L^p(X,\mu), \|\cdot\|_p)$ is a Banach space.*

PROOF: Assume that $1 \le p < \infty$. Let $\{f_n\}_{n=1}^{\infty} \subseteq L^p(X,\mu)$ be a Cauchy sequence in $(L^p(X,\mu), \|\cdot\|_p)$. We can find a subsequence $\{g_k\}$ of $\{f_n\}$ such that $\|g_{k+1} - g_k\|_p < \frac{1}{2^k}$. Define, for all $x \in X$,

$$g(x) = |g_1(x)| + \sum_{k=1}^{\infty} |g_{k+1}(x) - g_k(x)|$$

Then $g \in \mathscr{M}^+(X, \mathscr{A})$, and by Fatou's Lemma,

$$\begin{aligned}\int |g|^p d\mu &\le \liminf_{n\to\infty} \int \left(|g_1(x)| + \sum_{k=1}^{n} |g_{k+1}(x) - g_k(x)| \right)^p d\mu \\ &\le \liminf_{n\to\infty} \|g_1(x)\|_p + \sum_{k=1}^{n} \|g_{k+1}(x) - g_k(x)\|_p \\ &\le \|g_1\|_p + 1 < \infty\end{aligned}$$

Let $E = \{x \in X \mid g(x) < \infty\}$, so that $\mu(X \setminus E) = 0$. Hence $|g_1(x)| + \sum_{k=1}^{\infty} |g_{k+1}(x) - g_k(x)|$ converges to a finite number μ-a.e. Let

$$f(x) = \begin{cases} g_1(x) + \sum_{k=1}^{\infty} g_{k+1}(x) - g_k(x) & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Note that $g_k \to f$ μ-a.e. We also that that $|g_k(x)| \le g(x)$ for all $x \in X$. The Lebesgue Dominated Convergence Theorem shows us that

$$\int |f|^p d\mu = \lim_{k\to\infty} \int |g_k|^p d\mu \le \int |g|^p d\mu < \infty$$

Therefore $f \in L^p(X,\mu)$. Since $|f| \le g$, $|f - g_k|^p \le 2^p |g|$ and again by LDCT,

$$0 = \lim_{k\to\infty} \int |f - g_k|^p d\mu$$

since $g_k \to f$ μ-a.e. Therefore the subsequence $\{g_k\}$ converges to f in $L^p(X,\mu)$. It follows that $\{f_n\}$ converges to f in $L^p(X,\mu)$.

Now let $\{f_n\}_{n=1}^{\infty} \subseteq L^{\infty}(X,\mu)$ be a Cauchy sequence in $(L^{\infty}(X,\mu), \|\cdot\|_{\infty})$. Let $E \subseteq X$ be such that $\mu(E) = 0$ and if $x \notin E$ then $|f_n(x)| \le \|f_n\|_{\infty}$ and $|f_n(x) - f_m(x)| \le \|f_n - f_m\|_{\infty}$. Then $\{f_n(x)\}$ converges uniformly on $X \setminus E$ to some function $f(x)$. It follows that f is measurable and $\|f_n - f\|_{\infty} \to 0$. Hence $f \in L^{\infty}(X,\mu)$. ☐

9.9 Corollary. *If $1 \le p < \infty$ and if $\{f_n\}_{n=1}^{\infty} \subseteq L^p(X,\mu)$ is such that $f_n \to f$ in $L^p(X,\mu)$ then there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}$ such that $f_{n_k}(x) \to f(x)$ μ-a.e.*

Observe that it is possible to $f_n \to f$ in $L^p(X,\mu)$ but $\{f_n(x)\}$ diverges everywhere. Let $f_1 = \chi_{[0,1)}$, $f_2 = \chi_{[0,\frac{1}{2})}$, $f_3 = \chi_{[\frac{1}{2},1)}$, and so on, so that f_n is the characteristic function of exceedingly small intervals. Then $f_n \to 0$ but $\limsup_{n\to\infty} f_n(x) = 1$ and $\liminf_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$.

9.10 Example. Let $X=[0,1]$ and $\mu=m|_{[0,1]}$. Hölder's Inequality implies that $L^1[0,1]\supseteq L^p[0,1]\supseteq L^\infty[0,1]$ for all $1<p<\infty$. If $L^2[0,1]\subseteq L^1[0,1]$, is it closed with respect to $\|\cdot\|_1$?

9.11 Theorem. *$C[0,1]$ is dense in $L^p[0,1]$ for all $1\le p<\infty$ and it is closed in $L^\infty[0,1]$.*

PROOF: Exercise. □

10 Signed Measures

10.1 Definition. Let μ be signed measure on $(X,\mathscr{A})$ and let $P,N,M\in\mathscr{A}$. Then

1. P is *positive* if $\mu(E\cap P)\ge 0$ for all $E\in\mathscr{A}$.
2. N is *negative* if $\mu(E\cap N)\le 0$ for all $E\in\mathscr{A}$.
3. M is *null* if $\mu(E\cap M)=0$ for all $E\in\mathscr{A}$.

10.2 Lemma. *1. Every subset of a positive (resp. negative, null) set is postive (resp. negative, null).*

2. A countable union of positive (resp. negative, null) sets is postive (resp. negative, null).

PROOF: Trivial. □

10.3 Lemma. *Let $E\in\mathscr{A}$ be such that $\mu(E)>0$. Then there is a postive set $A\subseteq E$ such that $\mu(A)>0$.*

PROOF: Recursively define E_k as follows. If $E\setminus\bigcup_{j=1}^{k-1}E_j$ is not positive then let n_k be the least positive integer such that there exists $E_k\subseteq E\setminus\bigcup_{j=1}^{k-1}E_j$ with $\mu(E_k)<-\frac{1}{n_k}$, otherwise let $E_k=\varnothing$. Let $A=E\setminus\bigcup_{j=1}^{\infty}E_j=\bigcap_{k=1}^{\infty}(E\setminus\bigcup_{j=1}^{k-1}E_j)$. Since $E=A\cup\bigcup_{j=1}^{\infty}E_j$ and the union is disjoint, $\mu(E)=\mu(A)+\sum_{j=1}^{\infty}\mu(E_j)$. It follows that $\mu(A)>0$ since $\sum_{j=1}^{\infty}\mu(E_j)<0$, and hence $\sum_{j=1}^{\infty}\mu(E_j)>-\infty$. In particular, $\lim_{n\to\infty}n_k=\infty$ or the sequence terminates. Now for $F\subseteq A$, if $\mu(F)=r<0$ then let $k\ge 1$ such that $n_k>1-\frac{1}{r}$. $F\subseteq E\setminus\bigcup_{j=1}^{k-1}E_j$, but by choice of n_k, $E\setminus\bigcup_{j=1}^{k-1}E_j$ has no subset of measure less than or equal to $\frac{1}{n_k-1}<r$. This contradiction shows that A is positive. □

10.4 Theorem (Hahn Decomposition Theorem). *Let μ be a signed measure on $(X,\mathscr{A})$. Then there exists a positive set P and a negative set N that partition X. This is called a Hahn Decomposition.*

PROOF: Without loss of generality we may assume that $\mu(E)<\infty$ for all $E\in\mathscr{A}$, for otherwise $-\mu$ is a measure with this property and postive with respect to μ is the same as negative with respect to $-\mu$ and *visa versa*. Let $\lambda=\sup\{\mu(A)\mid A\text{ is positive}\}<\infty$. Let $\{A_n\}_{n=1}^{\infty}\subseteq\mathscr{A}$ be a sequence of positive sets such that $\lim_{n\to\infty}\mu(A_n)=\lambda$, and let $P=\bigcup_{n=1}\infty A_n$. Then P is positive so $\mu(P)\le\lambda$, and for all $n\ge 1$ $\mu(P)=\mu(P\cap A_n)+\mu(P\cap A_n^c)\ge\mu(A_n)$, so $\mu(P)=\lambda$. We need only check that $N=P^c$ is negative. If $A\subseteq N$ is such that $\mu(A)>0$ then by the previous lemma there is $B\subseteq A$ such that $\mu(B)>0$ and B is positive. But then $P\cup B$ is positive and $\mu(P\cup B)=\mu(P)+\mu(B)>\lambda$, a contradiction. □

10.5 Lemma. *If (P_1,N_1) and (P_2,N_2) are Hahn Decompositions of $(X,\mathscr{A},\mu)$ then*

1. *$P_1\Delta P_2=N_1\Delta N_2$ is null.*
2. *$\mu(P_1\cap E)=\mu(P_2\cap E)$ and $\mu(N_1\cap E)=\mu(N_2\cap E)$ for all $E\in\mathscr{A}$.*

PROOF: Notice first that $N_i=P_i^c$, so it's redundant to represent an Hahn Decomposition by an ordered pair when a singlton will suffice. $P_1\Delta P_2=(P_1\cap P_2^c)\cup(P_1^c\cap P_2)$ is both positive and negative, so it is null. Similarily for $N_1\Delta N_2$. For the second part, $\mu(E\cap P_1)=\mu(E\cap P_1\cap P_2)$ since $E\cap P_1\setminus P_2\subseteq P_1\Delta P_2$ is null. By symmetry the lemma is proved. □

10.6 Theorem (Jordan Decomposition Theorem). *Let μ be a signed measure on $(X, \mathscr{A})$. Then there exists two positive measures μ^+ and μ^- such that $\mu = \mu^+ - \mu^-$. Moreover, there is only one such pair for which $\mu^+ \perp \mu^-$.*

PROOF: Let (P, N) be a Hahn Decomposition. Let $\mu^+(E) = \mu(E \cap P)$ and $\mu^- = -\mu(E \cap N)$. Then clearly μ^+ and μ^- are positive measures and $\mu(E) = \mu^+(E) - \mu^-(E)$ for all $E \in \mathscr{A}$. Moreover, since $N \cap P = \varnothing$, $\mu^+ \perp \mu^-$. Assume that $\mu = \mu_1 - \mu_2$ where $\mu_1 \perp \mu_2$. Let A, B be a partition of X such that $\mu_1(B) = 0$ and $\mu_2(A) = 0$. For every $E \in \mathscr{A}$,

$$\mu(E \cap A) = \mu_1(E \cap A) - \mu_2(E \cap A) = \mu_1(E \cap A) \geq 0$$

so A is a positive set. Similarily, B is negative with respect to μ, so (A, B) is a Hahn Decomposition. But then for any $E \in \mathscr{A}$,

$$\mu^+(E) = \mu(E \cap P) = \mu(E \cap A) = \mu_1(E) \quad \text{and} \quad \mu^-(E) = -\mu(E \cap N) = -\mu(E \cap B) = \mu_2(E)$$

Hence $\mu_1 = \mu^+$ and $\mu_2 = \mu^-$. □

Note that if $\mu = \mu_1 - \mu_2$ with μ_1 and μ_2 positive measures then

$$\mu(E) = \mu(E \cap P) = \mu_1(E \cap P) - \mu_2(E \cap P) \leq \mu_1(E \cap P) \leq \mu_1(E)$$

and similarly $\mu^-(E) \leq \mu_2(E)$, for all $E \in \mathscr{A}$.

10.7 Example. If μ is a measure on $(X, \mathscr{A})$ and f is integrable, then $\lambda(E) = \int_E f \, d\mu$, $\lambda^+(E) = \int f^+ d\mu$, and $\lambda^-(E) = \int f^- d\mu$. Notice that $\lambda^+(X) + \lambda^-(X) = \int |f| d\mu = \|f\|_1$.

10.8 Definition. Given a postive measure μ, the positive measure $|\mu| = \mu^+ + \mu^-$ is called the *total variation* of μ.

Note that $\mu^+, \mu^- \ll |\mu|$. Assume that μ is finite. Then $|\mu(E)| \leq |\mu|(E) < \infty$. Given $(X, \mathscr{A})$, let

$$\text{Meas}(X, \mathscr{A}) = \{\mu \mid \mu \text{ is a finite signed measure on } (X, \mathscr{A})\}$$

Notice that $\text{Meas}(X, \mathscr{A})$ is a vector space over $\mathbb{R}$. Define $\|\mu\|_{\text{Meas}} = |\mu|(X)$. It is clear that $\|\mu\| = 0$ if and only if $\mu = 0$. It is also easy to see that $\|\alpha\mu\| = |\alpha\mu|(X) = |\alpha||\mu|(X) = |\alpha|\|\mu\|$. Assume that $\mu, \gamma \in \text{Meas}(X, \mathscr{A})$. Then

$$\mu + \gamma = (\mu^+ + \gamma^+) - (\mu^- + \gamma^-)$$

and so $(\mu+\gamma)^+ \leq \mu^+ + \gamma^+$ and $(\mu+\gamma)^- \leq \mu^- + \gamma^-$. It follows that $\|\mu+\gamma\| \leq \|\mu\| + \|\gamma\|$. Therefore $(\text{Meas}(X, \mathscr{A}), \|\cdot\|_{\text{Meas}})$ is a normed linear space.

Problem: Is $\text{Meas}(X, \mathscr{A})$ a Banach space with this norm?

Suppose $\{\mu_n\}_{n=1}^{\infty}$ is Cauchy with respect to $\|\cdot\|_{\text{Meas}})$. Then for all $\varepsilon > 0$ there is N such that if $n, m \geq N$ then $|\mu_m - \mu_n|(X) \leq \|\mu_m - \mu_n\| < \varepsilon$. Since $|\mu(E)| \leq |\mu|(E)$ for all $E \in \mathscr{A}$, we get that

$$|\mu_m(E) - \mu_n(E)| = |(\mu_m - \mu_n)(E)| \leq |\mu_m - \mu_n|(E) \leq |\mu_m - \mu_n|(X) < \varepsilon$$

if $n, m \geq N$. Therefore $\{\mu_n(E)\}$ is Cauchy in $\mathbb{R}$ for all $E \in \mathscr{A}$. Define $\mu(E) = \lim_{n\to\infty} \mu_n(E)$ for all $E \in \mathscr{A}$. Then

1. $\mu(\varnothing) = \lim_{n\to\infty} \mu_n(\varnothing) = 0$.
2. Since $\{\mu_n\}$ is Cauchy, if is bounded. Let M be such that $\|\mu_n\| < M$ for all n. It follows that $|\mu_n(E)| \leq M$. For all $E \in \mathscr{A}$.
3. We must show that μ is countably additive. Prove this as an exercise.

We have proven the following theorem:

10.9 Theorem. *$(Meas(X, \mathscr{A}), \|\cdot\|_{Meas})$ is a Banach space.*

Problem: What does $\text{Meas}(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ “look like”?

11 The Radon-Nikodym Theorem

11.1 Lemma. *Let μ and ν be finite measures on $(X, \mathscr{A})$. Then $\nu \ll \mu$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $E \in \mathscr{A}$ with $\mu(E) < \delta$ then $\nu(E) < \varepsilon$.*

PROOF: Assume that $\nu \ll \mu$ and let $\varepsilon > 0$. Assume that no such δ exists for this ε. For each $n \in \mathbb{N}$, let $E_n \in \mathscr{A}$ be such that $\mu(E_n) < \frac{1}{2^{-n}}$ but $\nu(E_n) \geq \varepsilon$. Let $F_n = \bigcup_{k=n}^{\infty} E_k$, so that $\mu(F_n) < \frac{1}{2^{n-1}}$ and $\mu(F_n) \geq \varepsilon$. Let $F = \bigcap_{n=1}^{\infty} F_n$, so that $F_n \searrow F$. Hence $\mu(F) = \lim_{n\to\infty} \mu(F_n) = 0$ but $\nu(F) = \lim_{n\to\infty} \nu(F) \geq \varepsilon$, contradicting that $\nu \ll \mu$.

Conversely, let $\varepsilon > 0$ and $E \in \mathscr{A}$. If $\mu(E) = 0$ then $\mu(E) < \delta(\varepsilon)$, so $\nu(E) < \varepsilon$. But ε was arbitrary, so $\nu(E) = 0$. □

11.2 Theorem (Radon-Nikodym). *Let λ and μ be σ-finite measures on $(X, \mathscr{A})$. Assume that $\lambda \ll \mu$. Then there exists $f \in \mathscr{M}^+(X, \mathscr{A})$ such that*

$$\lambda(E) = \int_E f \, d\mu$$

for every $E \in \mathscr{A}$. Moreover, this f is uniquely determined μ-almost everywhere.

PROOF: We assume that λ, μ are finite. For each $c > 0$ let $(P(c), N(c))$ be an Hahn decomposition for the signed measure $\lambda - c\mu$. For each $k \in \mathbb{N}$ let $A_1 = N(c)$ and $A_{k+1} = N((k+1)c) \setminus \bigcup_{i=1}^{k} A_i$. Then $\{A_i\}_{i=1}^{\infty}$ is pairwise disjoint and $\bigcup_{i=1}^{k} N(ic) = \bigcup_{i=1}^{k} A_i$. Therefore $A_k = N(kc) \cap \bigcap_{i=1}^{k-1} P(ic)$. If $E \in \mathscr{A}$, $E \subseteq A_k$, then $E \subseteq N(ck)$ and $E \subseteq P((k-1c)$. This implies that

$$(k-1)c\mu(E) \leq \lambda(E) \leq kc\mu(E) \qquad (*)$$

Let $B = X \setminus \bigcup_{i=1}^{\infty} A_i = \bigcap_{i=1} \infty P(jc)$. Hence $B \subseteq P(kc)$ for all $k \in \mathbb{N}$. It follows that

$$0 \leq kc\mu(B) \leq \lambda(B) \leq \lambda(X) < \infty$$

for every $k \in N$, so $\mu(B) = 0$ implies that $\lambda(B) = 0$. For each c, define

$$f_c(x) = \begin{cases} (k-1)c & \text{for } x \in A_k \\ 0 & \text{for } x \in B \end{cases}$$

Then if E is measurable we have $E = (E \cap B) \cup (\bigcup_{i=1}^{\infty}(E \cap A_k))$, and so $(*)$ implies

$$\int_E f_c d\mu \leq \lambda(E) \leq \int_E (f_c + c) d\mu \leq \int_E f_c d\mu + c\mu(X)$$

For each $n \in \mathbb{N}$, let $g_n = f_{\frac{1}{2^n}}$. Then

$$\int_E g_n d\mu \leq \lambda(E) \leq \int_E g_n d\mu + \frac{\mu(X)}{2^n} \qquad (**)$$

If $m \geq n$ then $(**)$ implies

$$\int_E g_n d\mu \leq \lambda(E) \leq \int_E g_m d\mu + \frac{\mu(X)}{2^m} \quad \text{and} \quad \int_E g_m d\mu \leq \lambda(E) \leq \int_E g_n d\mu + \frac{\mu(x)}{2^n}$$

Adding gives

$$\left| \int_E g_n d\mu - \int_E g_m d\mu \right| \leq \frac{\mu(X)}{2^n}$$

for all $E \in \mathscr{A}$. Let $E_1 = \{x \in X \mid g_n - g_m \geq 0\}$ and $E_2 = \{x \in X \mid g_n - g_m < 0\}$. This gives us that

$$\int_E |g_n - g_m| d\mu \leq \frac{2\mu(X)}{2^n} \leq \frac{\mu(X)}{2^{n-1}}$$

so $\{g_n\}_{n=1}^{\infty}$ is Cauchy in $L^1(X, \mathscr{A})$. Suppose that $g_n \to f \in L^1(X, \mathscr{A})$. Since $g_n \in \mathscr{M}^+(X, \mathscr{A})$, we can take $f \in \mathscr{M}^+(X, \mathscr{A})$. Now

$$\left| \int_E g_n d\mu - \int_E f d\mu \right| \leq \int_E |g_n - f| d\mu \leq \|g_n - f\|_1 \to 0$$

Hence $(**)$ implies that $\lambda(E) = \lim_{n\to\infty} \int_E g_n d\mu = \int_E f d\mu$.

Suppose $f, h \in \mathscr{M}^+(X, \mathscr{A})$ are such that $\int_E f d\mu = \lambda(E) = \int_E h d\mu$ for all $E \in \mathscr{A}$. Let $E_1 = \{x \in X \mid f(x) > h(x)\}$ and $E_2 = \{x \in X \mid f(x) < h(x)\}$. Then

$$\int_{E_1} (f - h) d\mu = \int_{E_1} f d\mu - \int_{E_1} h d\mu = \lambda(E_1) - \lambda(E_1) = 0$$

so $\mu(E_1) = 0$. Similarily, $\mu(E_2) = 0$, so $f = h$ μ-a.e.

Now assume that μ, λ are σ-finite. Let $\{X_n\} \subseteq \mathscr{A}$ be an increasing sequence such that $X = \bigcup_{n=1}^{\infty} X_n$ and $\mu(X_n) < \infty$, $\lambda(X_n) < \infty$. We get $f_n \in \mathscr{M}^+(X, \mathscr{A})$ with $f_n|_{X_n^c} \equiv 0$ and if $E \subseteq X_n$ then $\lambda(E) = \int_E f_n d\mu$. If $n \leq m$ then $X_n \subseteq X_m$, so $\int_E f_n d\mu = \int_E f_m d\mu$ for all $E \subseteq X_n$. It follows that $f_n = f_m$ μ-a.e. Let $F_n = \sup\{f_1, \dots, f_n\}$, so that $\{F_n\}_{n=1}^{\infty}$ is an increasing sequence of positive measurable functions. Let $f = \lim_{n\to\infty} F_n$. If $E \in \mathscr{A}$ then $\lambda(E \cap X_n) = \int_E F_n d\mu$ since $F_n = f_n$ μ-a.e. on X_n. But $E \cap X_n \nearrow E$, so

$$\lambda(E) = \lim_{n\to\infty} \lambda(E \cap X_n) = \lim_{n\to\infty} \int_E F_n d\mu = \int_E f d\mu$$

The uniqueness of f is determined as in the finite case. □

From the proof, we note that is λ is finite then we can choose f to be integrable, and hence finite μ-a.e. If λ is σ-finite then we can choose the f_n's to be finite on X_n. From this we can deduce that f can be chosen to be integrable in this case as well.

Remark. Let $\lambda \ll \mu$, where λ is a finite signed measure and μ is a positive σ-finite measure. Then $|\lambda| \ll \mu$ and so $\lambda^+ \ll \mu$ and $\lambda^- \ll \mu$. Hence there are $f^+, f^- \in \mathscr{M}^+(X, \mathscr{A})$ that represent λ^+ and λ^-. But then

$$\lambda(E) = \lambda^+(E) - \lambda^-(E) = \int_E f^+ d\mu - \int_E f^- d\mu = \int_E f^+ - f^- d\mu$$

so $f = f^+ - f^-$ represents λ. Notice that $\int |f| d\mu = |\lambda|(X) < \infty$, so $f \in L^1(X, \mathscr{A})$ and $\|f\|_1 = \|\lambda\|_{\text{Meas}}$. Given μ positive and σ-finite, let $AC(X, \mathscr{A}, \mu) = \{\lambda \in \text{Meas}(X, \mathscr{A}) \mid \lambda \ll \mu\}$. Then $AC(X, \mathscr{A}, \mu)$ is a closed subspace of $(\text{Meas}(X, \mathscr{A}), \|\cdot\|_{\text{Meas}})$. If $\nu \in AC(X, \mathscr{A}, \mu)$ and $f \in L^1(X, \mathscr{A}, \mu)$ is such that $\nu(E) = \int_E f d\mu$ then $\nu \mapsto f$ is an isometric isomorphism of AC onto L^1.

11.3 Example. Let $\mathscr{A} = \mathscr{P}(X)$ and $\mu_f(E) = \sum_{x \in E} f(x) = \int_E f d\lambda$, where λ is the counting measure. Is every finite measure on $(X, \mathscr{P}(X))$ of this form? How about every finite measure? Is this measure absolutely continuous with respect to the counting measure? The Radon-Nikodym Theorem says yes.

11.4 Definition. Assume that λ, μ are σ-finite measures and $\lambda \ll \mu$. The function $f \in \mathscr{M}^+(X, \mathscr{A})$ such that $\lambda(E) = \int_E f d\mu$ is called the *Radon-Nikodym derivative* of λ with respect to μ, and is denoted by $\frac{d\lambda}{d\mu}$.

11.5 Example. Assume that $F:\mathbb{R}\to\mathbb{R}$ is continuously differentiable with $F'(x)\geq 0$. Then F is monotonic. Let μ_F be the Lebesgue-Stieltjes measure on $(\mathbb{R},\mathscr{B}(\mathbb{R}))$ generated by F. Then μ_F is σ-finite and $\mu_F \ll m$. Moreover,

$$\mu_F((a,b]) = F(b)-F(a) = \int_a^b F'(x)dx = \int_{[a,b]} F'dm = \int_{(a,b]} F'dm$$

From here we get that if $E\in\mathscr{B}(\mathbb{R})$ then $\mu_F(E)=\int_E F'dm$, so $F'=\frac{d\mu_F}{dm}$.

11.6 Theorem (Lebesgue Decomposition Theorem). *Let λ,μ be σ-finite measurs on $(X,\mathscr{A})$. Then there exist two measures λ_1 and λ_2 on $(X,\mathscr{A})$ such that $\lambda=\lambda_1+\lambda_2$ and $\lambda_1\perp\mu$ and $\lambda_2\ll\mu$. Moreover, λ_1 and λ_2 are unique with these properties.*

PROOF: Let $\gamma=\lambda+\mu$, so that γ is σ-finite and $\lambda\ll\gamma$ and $\mu\ll\gamma$. By the Radon-Nikodym Theorem, there are functions $f,g\in\mathscr{M}^+(X,\mathscr{A})$ such that

$$\lambda(E)=\int_E f\,d\gamma \quad \text{and} \quad \mu(E)=\int_E g\,d\gamma$$

for all $E\in\mathscr{A}$. Let $A=\{x\in X\mid g(x)=0\}$ and $B=\{x\in X\mid g(x)>0\}$, so that A and B partition X. Let $\lambda_1(E)=\lambda(E\cap A)$ and $\lambda_2(E)=\lambda(E\cap B)$ for all $E\in\mathscr{A}$. It is clear that $\lambda=\lambda_1+\lambda_2$. Since $\mu(A)=0$, it follows that $\lambda_1\perp\mu$. If $\mu(E)=0$ then $\int_E g\,d\gamma=0$, so $g(x)=0$ for γ-a.e. $x\in E$. But then $\gamma(E\cap B)=0$, and so $\lambda_2(E)=\lambda(E\cap B)=0$ since $\lambda\ll\gamma$. Therefore $\lambda_2\ll\mu$. To see that the decomposition is unique, observe that for any σ-finite signed measure ν, if $\nu\perp\mu$ and $\nu\ll\mu$ then $\nu\equiv 0$. If $\lambda=\nu_1+\nu_2$ is another decomposition then $\lambda_1-\nu_1=\nu_2-\lambda_2$. This is not quite correct, as we cannot be certain that $\lambda_i-\nu_i$ is a signed measure. Complete this proof as an exercise. □

12 Differentiation of Monotone Functions

12.1 Definition. Let $\mathfrak{J}$ be a collection of non-degenerate intervals. We say that $\mathfrak{J}$ is a *Vitali cover* of $E\subseteq\mathbb{R}$ if for all $\varepsilon>0$ and any $x\in E$ there exists $I\in\mathfrak{J}$ such that $x\in I$ and $\ell(I)<\varepsilon$.

12.2 Lemma (Vitali's Lemma). *Let $E\subseteq\mathbb{R}$ be such that $m^*(E)<\infty$ and let $\mathfrak{J}$ be a Vitali cover of E. Then there exists, for each $\varepsilon>0$, a finite pairwise disjoint collection $\{I_1,\dots,I_n\}\subseteq\mathfrak{J}$ such that $m^*(E\setminus\bigcup_{i=1}^n I_i)<\varepsilon$.*

PROOF: We may assume that the intervals in $\mathfrak{J}$ are closed since finitely many endpoints will contribute only a set of measure zero. Let $E\subseteq U\subseteq\mathbb{R}$ be an open set with $m(U)<\infty$. Since $\mathfrak{J}$ is a Vitali cover, we may assume that if $I\in\mathfrak{J}$ then $I\subseteq U$. Let $I_1\in\mathfrak{J}$. Suppose that $I_1,\dots,I_n$ have been chosen to be pairwise disjoint. If $E\subseteq\bigcup_{i=1}^n I_i$ then stop. Otherwise, let k_n be the supremum of the lengths of the intervals in $\mathfrak{J}$ that are disjoint from $\bigcup_{i=1}^n I_i$. Since $E\not\subseteq\bigcup_{i=1}^n I_i$, we can find $I_{n+1}\in\mathfrak{J}$ such that $\ell(I_{n+1})>\frac{1}{2}k_n$ and I_{n+1} is disjoint from $\bigcup_{i=1}^n I_i$. This gives us a, possibly finite, sequence $\{I_n\}_{n=1}^\infty$ of disjoint intervals in $\mathfrak{J}$ with $\bigcup_{n=1}^\infty I_n\subseteq U$. Hence $m(\bigcup_{n=1}^\infty I_n)\leq m(U)<\infty$. We can find $N\in\mathbb{N}$ such that $\sum_{n=N+1}^\infty \ell(I_n)<\frac{\varepsilon}{5}$. Let $R=E\setminus\bigcup_{i=n}^N I_n$. Let $x\in R$. Since $x\in(\bigcup_{i=n}^N I_n)^c$, an open set, there is an interval $x\in I\in\mathfrak{J}$ such that I is disjoint from $\bigcup_{i=n}^N I_n$. It follows that $\ell(I)\leq k_n<2\ell(I_{n+1})$ if $I\cap I_n=\emptyset$. Since $\lim_{n\to\infty}\ell(I_n)=0$, I must meet at least one of the I_n's. Let n_0 be the smallest n for which $I\cap I_n\neq\emptyset$. Then $n_0>N$ and $\ell(I)\leq K_{n_0-1}<2\ell(I_{n_0})$. Since $I\cap I_{n_0}\neq\emptyset$ and $x\in I$, the distance from x to the midpoint of I_{n_0} is at most $\ell(I)+\frac{1}{2}\ell(I_{n_0})\leq\frac{5}{2}\ell(I_{n_0})$. Therefore $x\in J_{n_0}$, where J_{n_0} is the interval with the same center as I_{n_0} and five times the length. For each $n\geq N+1$, let J_n be the interval with the same center as I_n but five times the length. Then $R\subseteq\bigcup_{n=N+1}^\infty J_n$, and hence $m^*(R)\leq 5\sum_{n=N+1}^\infty \ell(I_n)<\varepsilon$. □

12.3 Definition. Let $f : \mathbb{R} \to \mathbb{R}$. For any $x \in \mathbb{R}$, let

$$\begin{aligned} D^+f(x) &= \limsup_{h\to 0^+} \frac{f(x+h)-f(x)}{h} \\ D_+f(x) &= \liminf_{h\to 0^+} \frac{f(x+h)-f(x)}{h} \\ D^-f(x) &= \limsup_{h\to 0^+} \frac{f(x)-f(x-h)}{h} \\ D_-f(x) &= \liminf_{h\to 0^+} \frac{f(x)-f(x-h)}{h} \end{aligned}$$

These four quantities are called the *Dini derviatives* of $f(x)$ at x. f is differentiable at x if and only if $D^+f(x) = D^-f(x) = D_+f(x) = D_-f(x) < \infty$. In this case $f'(x) = D^+f(x)$. If $D^+f(x) = D_+f(x)$, we denote the common value by $f(x^+)$. If both are finite then $f(x^+)$ is called the *right hand dervative*. $f(x^-)$ is defined in a similar fashion and is called the *left hand derviative* if it exists.

In general, $D^+f(x) \geq D_+f(x)$ and $D^-f(x) \geq D_-f(x)$.

12.4 Proposition. *If $f \in C[a,b]$ and one of its Dini dervatives is everywhere non-negative then f is non-decreasing on $[a,b]$*

PROOF: Exercise. □

12.5 Theorem. *Let $f : [a,b] \to \mathbb{R}$ be increasing. Then f is differentiable almost everywhere on $[a,b]$, f' is measureable, and*

$$\int_{[a,b]} f' dm \leq f(b) - f(a)$$

PROOF: Consider $E = \{x \in [a,b] \mid D^+f(x) > D_-f(x)\}$. Let $E_{r,s} = \{x \in [a,b] \mid D^+f(x) > r > s > D_-f(x)\}$, for $r,s \in \mathbb{Q}$. Then $E = \bigcup_{r,s\in\mathbb{Q}} E_{r,s}$. Let $\alpha = m^*(E_{r,s})$ and $\varepsilon > 0$. Choose U open such that $m(U) < \alpha + \varepsilon$ and $E_{r,s} \subseteq U$. For each $x \in E_{r,s}$, there is an arbitrarily small interval $[x-h,x] \subseteq U$ so that $\frac{f(x)-f(x-h)}{h} < s$. By Vitali's Lemma, there are finitely many disjoint intervals $I_1,\dots,I_N$ of this type such that the interiors of the I_n's cover a subset A of $E_{r,s}$ with $m^*(A) > \alpha - \varepsilon$. We have

$$\sum_{n=1}^{N} [f(x_n) - f(x_n - h_n)] \leq s \sum_{n=1}^{N} h_n < sm(U) < s(\alpha + \varepsilon)$$

Now for each point $y \in A$, there is an arbitrarily small interval $[y, y+k]$ that is contained in some I_n and is such that $\frac{f(y+k)-f(y)}{k} > r$. Therefore $f(y+k) - f(y) > rk$. Applying Vitali's Lemma again, we get intervals $J_1,\dots,J_M$ disjoint and of the type $[y,y+k] \subseteq I_n$ such that $\bigcup_{j=1}^{M} J_j$ contains a subset of A with outer measure at least $\alpha - 2\varepsilon$. Hence

$$\sum_{j=1}^{M} [f(y_j + k_j) - f(y_j)] > r \sum_{j=1}^{M} k_j \geq r(\alpha - 2\varepsilon)$$

We know $J_j \subseteq I_n$. We have $\sum_{J_j \subseteq I_n} [f(y_j + k_j) - f(y_j)] \leq f(x_n) - f(x_n - h_n)$ since f is increasing. It follows that

$$r(\alpha - 2\varepsilon) \leq \sum_{j=1}^{M} [f(y_j + k_j) - f(y_j)] \leq \sum_{n=1}^{N} [f(x_n) - f(x_n - h_n)] \leq s(\alpha - \varepsilon)$$

Since $\varepsilon > 0$ is arbitrary, $r\alpha \le s\alpha$. Since $r > s$, this imples that $\alpha = 0$. Therefore $m^*(E_{r,s}) = 0$, so $m(E_{r,s}) = 0$ and so $m(E) = 0$.

Similarily, if $E_1 = \{x \in [a,b] \mid D^- f(x) > D_+ f(x)\}$ then $m(E_1) = 0$. From this we can deduce that $D^+ f(x) = D^- f(x) = D_+ f(x) = D_- f(x)$ a.e. (Verify this.) This means that $g(x) = \lim_{h\to 0} \frac{f(x)-f(x+h)}{h}$ is defined as an extended real number a.e. on $[a,b]$. Let $g_n(x) = n[f(x+\frac{1}{n}) - f(x)]$, where $f(x) := f(b)$ for $x \ge b$. Then $g_n \to g$ a.e. on $[a,b]$. Since f is increasing, $g_n > 0$. By Fatou's Lemma,

$$\begin{aligned}
\int_{[a,b]} g\,dm &\le \liminf_n \int_{[a,b]} g_n\,dm \\
&= \liminf_n n \int_{[a,b]} [f(x+\frac{1}{n}) - f(x)]\,dx \\
&= \liminf_n \left[n \int_{[b,b-\frac{1}{n}]} f\,dm - \int_{[a,a+\frac{1}{n}]} f\,dm \right] \\
&= \liminf_n \left[f(b) - \int_{[a,a+\frac{1}{n}]} f\,dm \right] \\
&\le f(b) - f(a)
\end{aligned}$$

Therefore g is integrable. Hence $g(x)$ is finite a.e. This shows that $f(x)$ is differentiable a.e. and $f'(x) = g(x)$ is finite. It follows that $f' = g$ a.e. on $[a,b]$ and finally,

$$\int_{[a,b]} f'\,dm = \int_{[a,b]} g\,dm \le f(b) - f(a)$$

□

13 Bounded Variation

13.1 Definition. Let $f : [a,b] \to \mathbb{R}$. For a partition $\Pi = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a,b]$, define

$$V_a^b(f,\Pi) = \sum_{i=1}^n |f(x_i) - f(x_{i-1})|$$

Notice that if $\Pi_1 \subseteq \Pi_2$ then $V_a^b(f,\Pi_1) \le V_a^b(f,\Pi_2)$. The *variation* of f on $[a,b]$ is

$$V_a^b(f) = \sup_\Pi V_a^b(f,\Pi)$$

f is of *bounded variation* on $[a,b]$ if $V_a^b(f) < \infty$. Define

$$BV[a,b] = \{f : [a,b] \to \mathbb{R} \mid f \text{ is of bounded variation}\}$$

13.2 Proposition. *$BV[a,b]$ is a vector space and $V_a^b(\cdot)$ is a seminorm on it.*

PROOF: Trivial. □

13.3 Definition. For $f : [a,b] \to \mathbb{R}$ and $\Pi = \{a = x_0 < x_1 < \cdots < x_n = b\}$ a partition of $[a,b]$, let

$$V_a^{b+}(f,\Pi) = \sum_{i=1}^n (f(x_i) - f(x_{i-1}))^+ \qquad \text{and} \qquad V_a^{b-}(f,\Pi) = \sum_{i=1}^n (f(x_i) - f(x_{i-1}))^-$$

where $\alpha^+ = \max\{\alpha, 0\}$ and $\alpha^- = (-\alpha)^+$. The *postive variation* and *negative variation* of f are

$$V_a^{b+}(f) = \sup_\Pi V_a^{b+}(f,\Pi) \qquad \text{and} \qquad V_a^{b-}(f) = \sup_\Pi V_a^{b-}(f,\Pi)$$

13.4 Proposition. *1.* $V_a^b(f) = V_a^{b+}(f) + V_a^{b-}(f)$

2. $f(b) - f(a) = V_a^{b+}(f) - V_a^{b-}(f)$

PROOF: 1. Trivial.

2. For $\varepsilon > 0$ let Π be a partition of $[a,b]$ such that $V_a^{b+}(f) - V_a^{b+}(f,\Pi) < \varepsilon$ and $V_a^{b-}(f) - V_a^{b-}(f,\Pi) < \varepsilon$. Then

$$\begin{aligned}&|V_a^{b+}(f) - V_a^{b-}(f) - (f(b) - f(a))| \\ &\qquad \le |V_a^{b+}(f) - V_a^{b+}(f,\Pi)| + |V_a^{b-}(f) - V_a^{b-}(f,\Pi)| + |V_a^{b+}(f,\Pi) - V_a^{b-}(f,\Pi) - (f(b) - f(a))|\end{aligned}$$

But $V_a^{b+}(f,\Pi) - V_a^{b-}(f,\Pi) = \sum_{i=1}^n f(x_i) - f(x_{i-1}) = f(b) - f(a)$, so the result is proved. □

13.5 Theorem (Jordan Decomposition Theorem).

$$BV[a,b] = \{f - g \mid f, g : [a,b] \to \mathbb{R} \text{ is non-decreasing}\}$$

PROOF: If f is non-decreasing then $V_a^b(f) = f(b) - f(a) < \infty$, so $f \in BV[a,b]$. Since $BV[a,b]$ is a vector space, one containment is proved.

Conversely, given $f \in BV[a,b]$, $f(x) = V_a^{x+}(f) - V_a^{x-}(f) - f(a)$. But $V_a^{x+}(f)$ and $V_a^{x-}(f)$ are non-decreasing, so f is a difference of non-decreasing functions. □

13.6 Corollary. *If $f \in BV[a,b]$ then f is differentiable almost everywhere.*

13.7 Proposition. *Suppose that f is integrable on $[a,b]$. Let $F(x) = \int_{[a,x]} f\,dm$. Then F is continuous and of bounded variation on $[a,b]$.*

PROOF: To see that F is continuous, let $\lambda(E) = \int_E |f|dm$ for $E \in \mathscr{B}([a,b])$. λ is a (positive) measure and $\lambda \ll m$. Thus, given $\varepsilon > 0$ there is $\delta > 0$ such that $m(E) < \delta$ implies that $\lambda(E) < \varepsilon$. Hence if $|x - y| < \delta$ then

$$|f(x) - f(y)| = \left| \int_{[x,y]} f\,dm \right| \le \int_{[x,y]} |f|dm < \varepsilon$$

To see that F is of bounded variation, let Π be a partition of $[a,b]$. Then

$$\begin{aligned}\sum_{i=1}^n |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^n \left| \int_{[x_{i-1},x_i]} f\,dm \right| \\ &\le \sum_{i=1}^n \int_{[x_{i-1},x_i]} |f|dm \\ &= \int_{[a,b]} |f|dm\end{aligned}$$

Therefore $V_a^b(f) \le \int_{[a,b]} |f|dm < \infty$. □

13.8 Proposition. *If $f \in C[a,b] \cap BV[a,b]$ then $V_a^x(f) \in C[a,b]$.*

PROOF: Let $f \in C[a,b] \cap BV[a,b]$ and suppose for contradiction that $V_a^x(f)$ is not continous at $x_0 \in [a,b]$. Since $V_a^x(f)$ is monotone, this must be a jump discontinuity. By symmetry and without loss of generality, assume that $\lim_{x\to x_0} V_a^x(f) = a > 0$. Let $\delta > 0$ be such that if $x \in [x_0, x_0+\delta)$ then $|f(x) - f(x_0)| < \frac{a}{3}$.

Claim. For all $y \in (x_0, x_0+\delta)$ there exists $z \in (x_0, y)$ such that $V_z^y(f) > \frac{a}{3}$.

Surely, $V_{x_0}^y(f) \geq a$, so let Π be a partition of $[a,b]$ such that $V_{x_0}^y(f, \Pi) > \frac{2a}{3}$. Then $x_1 \in [x_0, x_0+\delta)$, so $|f(x_1) - f(x_0)| < \frac{a}{3}$. Let $z = x_1$ and notice that

$$V_z^y(f) \geq \sum_{i=2}^{n} |f(x_i) - f(x_{i-1})| > \frac{a}{3}$$

Consequently, there is an infinite series $x_0 + \delta > y_0 > y_1 > \cdots > a$ such that $V_{y_n}^{y_0}(f) = \sum_{i=1}^{n} V_{y_i}^{y_{i-1}}(f) > n\frac{a}{3}$. This implies that $V_a^b(f) > n\frac{a}{3}$ for all n, contradicting that $f \in BV[a,b]$. □

13.9 Lemma. *If $f : [a.b] \to \mathbb{R}$ is integrable and $\int_a^x f(t)dt = 0$ for every $a \in [a,b]$ then $f(x) = 0$ a.e.*

PROOF: Let $E = \{x \in [a,b] \mid f(x) > 0\}$. Assume that $m(E) > 0$. Then there exists a closed set $F \subseteq E$ with $m(F) > 0$. Let $U = (a,b) \setminus F$, an open set. Since U is open, $U = \bigcup_{i=1}^{\infty}(a_i, b_i)$, where $\{(a_i,b_i)\}_{i=1}^{\infty}$ is pairwise disjoint. We also have that $0 = \int_a^b f(t)dt = \int_U f\,dm + \int_F f\,dm$, so $\int_U f\,dm = -\int_F f\,dm \neq 0$. Since $0 \neq \int_U f\,dm = \sum_{i=1}^{\infty} \int_{a_i}^{b_i} f(t)dt0$, there is some n such that $\int_{a_n}^{b_n} f(t)dt \neq 0$. But then either $\int_a^{a_n} f(t)dt \neq 0$ or $\int_a^{b_n} f(t)dt \neq 0$. Since this is impossible, $m(E) = 0$. Similarily, $m(\{x \in [a,b] \mid f(x) < 0\}) = 0$. □

13.10 Lemma. *If $f : [a,b] \to \mathbb{R}$ is bounded and measurable and $F(x) = \int_a^x f(t)dt + F(a)$ then $F'(x) = f(x)$ a.e. on $[a,b]$.*

PROOF: We know that $F \in BV[a,b]$, and so F' exists a.e. on $[a,b]$. Suppose that $|f| \leq k$. Let $f_n(x) = n(F(x + \frac{1}{n}) - F(x))$, so that $f_n(x) = n\int_x^{x+\frac{1}{n}} f(t)dt$. Hence $|f_n(x)| \leq k$ for all $n \in \mathbb{N}$ and $x \in [a,b]$. Since $f_n \to F'$ a.e., the Lebegue Dominated Convergence Theorem shows that

$$\begin{aligned}
\int_a^c F'(x)dx &= \lim_{n\to\infty} \int_a^c f_n(x)dx \\
&= \lim_{n\to\infty} n \int_a^c (F(x + \frac{1}{n}) - F(x))dx \\
&= \lim_{n\to\infty} \left[n \int_a^{c+\frac{1}{n}} F(x)dx - n \int_a^{a+\frac{1}{n}} F(x)dx \right] \\
&= F(c) - F(a) && \text{by the FTC since } F \text{ is continuous} \\
&= \int_a^c f(t)dt
\end{aligned}$$

Hence $\int_a^c (F'(t) - f(t))dt = 0$ for all $c \in [a,b]$. Therefore $F'(x) = f(x)$ a.e. on $[a,b]$. □

13.11 Theorem. *Let $f : [a,b] \to \mathbb{R}$ be integrable and $F(x) = F(a) + \int_a^x f(t)dt$. Then $F'(x) = f(x)$ a.e.*

PROOF: Assume without loss of generality that $f(x) \geq 0$ on $[a,b]$. Let

$$f_n(x) = \begin{cases} f(x) & \text{if } f(x) \leq n \\ n & \text{if } f(x) \geq n \end{cases}$$

Then $f - f_n \geq 0$. Hence $G_n(x) = \int_a^x (f(t) - f_n(t))dt$ is increasing on $[a,b]$. Therefore G_n is differentiable and $G_n'(x) \geq 0$ a.e. Moreover, $\frac{d}{dx}\int_a^x f_n(t)dt = f_n(x)$ a.e. Hence $F'(x) = \frac{d}{dx}G_n + \frac{d}{dx}\int_a^x f_n(t)dt$, so $F'(x) \geq f_n(x)$ a.e. for all $n \in \mathbb{N}$. Therefore $F'(x) \geq f(x)$ a.e. It follows that $\int_a^b F'(x)dx \geq \int_a^b f(x)dx = F(b) - F(a)$. However, $F(b) - F(a) \geq \int_a^b F'(x)dx$, so $\int_a^b F'(x)dx = F(b) - F(a) = \int_a^b f(x)dx$. Thus $0 = \int_a^b (F'(x) - f(x))dx$, so $F'(x) = f(x)$ a.e. on $[a,b]$ since $F'(x) \geq f(x)$ a.e. on $[a,b]$. □

14 Absolutely Continuous Functions

14.1 Definition. A function $f : [a,b] \to \mathbb{R}$ is said to be *absolutely continuous* on $[a,b]$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $\{(a_i, b_i)\}_{i=1}^n$ is any finite collection of pairwise disjoint open subintervals in $[a,b]$ with $\sum_{i=1}^n b_i - a_i < \delta$ then $\sum_{i=1}^n |F(b_i) - F(a_i)| < \varepsilon$.

14.2 Lemma. *If f is absolutely continuous on $[a,b]$ then it is of bounded variation.*

PROOF: Assignment. □

14.3 Corollary. *If f is absolutely continuous on $[a,b]$ then it is differentiable a.e.*

14.4 Lemma. *If f is absolutely continuous on $[a,b]$ and $f'(x) = 0$ a.e. then f is constant on $[a,b]$.*

PROOF: Let $c \in [a,b]$ with $c > a$. Let $E \subseteq [a,c]$ be such that $m(E) = c - a$ and $f'(x) = 0$ for all $x \in E$. Let $\varepsilon, \eta > 0$ be arbitrary. For each $x \in E$ we can find arbitrarily small h's such that $[x, x+h] \subseteq [a,c]$ and $|f(x+h) - f(x)| < \eta h$. Let $\delta > 0$ be chosen from ε in the definition of absolute continuity. By the Vitali Covering Lemma there is a finite disjoint collection $\{[x_k, y_k]\}$ of such intervals such that the union covers E, except for a set of measure at most δ. Without loss of generality, we may assume that $x_k < x_{k+1}$. We have $y_0 := a \leq x_1 < y_1 \leq x_2 < y_2 \leq \cdots \leq x_n < y_n \leq c =: x_{n+1}$. Then $\sum_{k=0}^n |x_{k+1} - y_k| < \delta$. Hence $\sum_{k=0}^n |f(x_{k+1}) - f(y_k)| < \varepsilon$. We also have $\sum_{k=1}^n |f(y_k) - f(x_k)| \leq \eta \sum_{k=0}^n |y_k - x_k| \leq \eta(c-a)$. But we can write all of this as a telescoping sum

$$|f(c) - f(a)| = \left| \sum_{k=0}^n [f(x_{k+1}) - f(y_k)] + \sum_{k=0}^n [f(y_k) - f(x_k)] \right| < \varepsilon + \eta(c-a)$$

Since ε and η were arbitrary, $f(c) = f(a)$. Therefore f is constant on $[a,b]$. □

14.5 Theorem. *A function $F : [a,b] \to \mathbb{R}$ is of the form $F(x) = F(a) + \int_a^x f(t)dt$ for some integrable function f if and only if F is absolutely continuous.*

PROOF: If F has this form then we've seen that F is absolutely continuous. Assume that F is absolutely continuous on $[a,b]$. Then F is of bounded variation, so $F(x) = F_1(x) - F_2(x)$, where each F_i is increasing on $[a,b]$. Therefore F is differentiable a.e. on $[a,b]$ and $|F'(x)| \leq F_1'(x) + F_2'(x)$. It follows that

$$\int_a^b |F'(x)|dx \leq \int_a^b F_1'(x)dx + \int_a^b F_2'(x)dx \leq F_1(b) - F_1(a) + F_2(b) - F_2(a)$$

This shows that F' is integrable on $[a,b]$. Let $G(x) = \int_a^x F'(t)dt$. Then G is absolutely continuous and hence so is $f := F - G$. However, $f' = F' - G' = 0$ a.e. on $[a,b]$, so f is constant on $[a,b]$. But $f(a) = F(a) - G(a) = F(a)$. This shows that $F(x) = F(a) + G(x) = F(a) + \int_a^x F'(t)dt$. □

14.1 Measures on $[a,b]$

Let $\text{Meas}[a,b] = \{\mu \mid \mu \text{ is a finite Borel sign-measure}\}$ and define $\|\mu\|_{\text{Meas}} = |\mu|([a,b])$. Then we've seen that $(\text{Meas}[a,b], \|\cdot\|_{\text{Meas}})$ is a Banach space.

Consider $BV[a,b] = \{f : [a,b] \to \mathbb{R} \mid V_a^b(f) < \infty\}$. Define $\|f\|_{BV} = V_a^b(f) + |f(a)|$. Clearly this is non-negative, and in fact $\|f\|_{BV} = 0$ if and only if $f \equiv 0$. Variation is a seminorm, so $\|\cdot\|_{BV}$ is a norm on $BV[a,b]$. We would like the consider the right continuous functions in this space. Let $BV_r[a,b]$ denote the right continuous functions of bounded variation on $[a,b]$. Then $BV_r[a,b]$ is a closed subspace of $BV[a,b]$. It is clearly a subspace. From the assignment, variation dominates the uniform norm, so a sequence that converges in variation converges unformly, and uniform convergence preserves (right) continuity.

14.6 Theorem. $(BV_r[a,b], \|\cdot\|_{BV}) \cong (\mathit{Meas}[a,b], \|\cdot\|_{\mathit{Meas}})$.

15 Bounded Linear Functionals on $L^p(X,\mu)$ and $C[a,b]$

15.1 Definition. A *linear functional* on a normed linear space $(X, \|\cdot\|)$ is a linear map $\varphi : X \to \mathbb{R}$. Denote the set of all linear functionals on X by $X^\#$. We say that a linear functional φ is *bounded* if $\|\varphi\| := \sup\{|\varphi(x)| : \|x\| \leq 1\} < \infty$.

If φ is bounded then $|\varphi(x)| \leq \|\varphi\|\|x\|$ for all $x \in X$. Let X^* be the set of all bounded linear functionals on X. Then $\|\cdot\|$ is a norm on X^* and $(X^*, \|\cdot\|)$ is called the *dual space* of $(X, \|\cdot\|)$.

15.2 Theorem. *The following are equivalent for $\varphi \in X^\#$*

1. *φ is bounded.*
2. *φ is continuous.*
3. *φ is continuous at a point.*

15.3 Theorem. *Let $(X, \|\cdot\|)$ be a normed linear space. Then $(X^*, \|\cdot\|)$ is a Banach space.*

Problem: What is $L^p(X,\mu)^*$?

15.4 Lemma. *If $(X, \mathscr{A}, \mu)$ is a measure space and if $1 \leq p \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ then for every $g \in L^q(X,\mu)$ the map $\Gamma_g : L^p(X,\mu) \to \mathbb{R}$ defined by $\Gamma_g(f) = \int_X f g \, d\mu$ is a continuous linear functional on $L^p(X,\mu)$. Further, $\|\Gamma_g\| \leq \|g\|_q$ and if $1 < p \leq \infty$ then $\|\Gamma_g\| = \|g\|_q$.*

PROOF: Assignment. □

If (X,μ) is σ-finite then equality holds for $p = 1$ as well.

15.5 Lemma. *Let $(X, \mathscr{A}, \mu)$ be a finite measure space and $1 \leq p < \infty$. Let g be an integrable function such that there exists a constant M with $|\int g\varphi \, d\mu \leq M\|\varphi\|_p$ for all simple functions φ. Then $g \in L^q(X,\mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

PROOF: Assume that $p > 1$. Let ψ_n be a sequence of simple functions with $\psi_n \nearrow |g|^q$. Let $\varphi_n = (\psi_n)^{\frac{1}{p}}\text{sgn}(g)$. Then φ_n is also simple and $\|\varphi_n\|_p = (\int \psi_n d\mu)^{\frac{1}{p}}$. Since $|\varphi_n g| \geq |\varphi_n||\psi_n|^{\frac{1}{q}} = |\psi_n|$, we have

$$\int \psi_n d\mu \leq \int \varphi_n g \, d\mu \leq M\|\varphi_n\|_p = M\left(\int \psi_n d\mu\right)^{\frac{1}{p}}$$

Therefore $\int \psi_n d\mu \leq M^q$. By the Monotone Convergence Theorem we get that $\|g\|_q \leq M$, so $g \in L^q(X,\mu)$.

If $p = 1$ then we need to show that g is bounded almost everywhere. Let $E = \{x \in X \mid |g(x)| > M\}$. Let $f = \frac{1}{\mu(E)}\chi_E \text{sgn}(g)$. Then f is a simple function and $\|f\|_1 = 1$. Get a contradiction. □

15.6 Lemma. *Let $1 \le p < \infty$. Let $\{E_n\}$ be a sequence of disjoint sets. Let $\{f_n\} \subseteq L^p(X,\mu)$ be such that $f_n(x) = 0$ if $x \notin E_n$ for each $n \ge 1$. Let $f = \sum_{n=1}^{\infty} f_n$. Then $f \in L^p(X,\mu)$ if and only if $\sum_{n=1}^{\infty} \|f\|_p^p < \infty$. In this case $\|f\|_p^p = \sum_{n=1}^{\infty} \|f\|_p^p$.*

PROOF: Exercise. □

15.7 Theorem (Riesz Representation Theorem, I). *Let $\Gamma \in L^p(X,\mu)^*$, where $1 \le p < \infty$ and μ is σ-finite. Then if $\frac{1}{p} + \frac{1}{q=1}$ then there exists a unique $g \in L^q(X,\mu)$ such that*

$$\Gamma(f) = \int_X f g \, d\mu =: \phi_g(f)$$

Moreover, $\|\Gamma\| = \|g\|_q$.

PROOF: Assume that μ is finite. Then every bounded measurable function is in $L^p(X,\mu)$. Define $\lambda : \mathscr{A} \to \mathbb{R} : E \mapsto \Gamma(\chi_E)$. Let $\{E_n\} \subseteq \mathscr{A}$ be a sequence of disjoint sets, and let $E = \bigcup_{n=1}^{\infty} E_n$. Let $\alpha_n = \operatorname{sgn}\Gamma(\chi_{E_n})$ and $f = \sum_{n=1}^{\infty} \alpha_n \chi_{E_n}$. Then $f \in L^p(X,\mu)$ and $\Gamma(f) = \sum_{n=1}^{\infty} |\lambda(E_n)| < \infty$ and so $\sum_{n=1}^{\infty} \lambda(E_n) = \Gamma(\chi_E) = \lambda(E)$. Therefore λ is a finite signed measure. Clearly, if $\mu(E) = 0$ then $\chi_E = 0$ a.e., so $\lambda(E) = \Gamma(0) = 0$. Therefore $|\lambda| \ll \mu$. By the Radon-Nikodym Theorem, there is an integrable function g such that $\lambda(E) = \int_E g \, d\mu$ for all $E \in \mathscr{A}$. If φ is simple then $\Gamma(\varphi) = \int \varphi g \, d\mu$ by linearity of the integral. But $|\Gamma(\varphi)| \le \|\Gamma\|\|\varphi\|_p$ for all simple functions φ, so $g \in L^q(X,\mu)$ by the lemma above. Now $\Gamma - \phi_g \in L^p(X,\mu)^*$ and $\Gamma - \phi_g = 0$ on the space of simple functions. Since the simple functions are dense in $L^p(X,\mu)$, $\Gamma - \phi_g = 0$ on $L^p(X,\mu)$, so $\Gamma = \phi_g$. We have that $\|\Gamma\| = \|\phi_g\| = \|g\|_q$ as before.

Now assume that μ is σ-finite. We can write $X = \bigcup_{n=1}^{\infty} X_n$, where $\mu(X_n) < \infty$ and $X_n \subseteq X_{n+1}$ for all $n \ge 1$. For each $n \ge 1$, the proof above gives us $g_n \in L^q(X,\mu)$, vanishing outside X_n, such that $\Gamma(f) = \int f g \, d\mu$ for all $f \in L^p(X,\mu)$ vanishing off of X_n. Moreover, $\|g_n\|_q \le \|\Gamma\|$. By the uniqueness of the g_n's, we can assume that $g_{n+1} = g_n$ on X_n. Let $g(x) = \lim_{n\to\infty} g_n(x)$. We have that $|g_n| \nearrow |g|$. By the Monotone Convergence Theorem

$$\int |g|^q d\mu = \lim_{n\to\infty} \int |g_n|^q d\mu \le \|\Gamma\|^q$$

Hence $g \in L^q(X,\mu)$. Let $f \in L^p(X,\mu)$ and $f_n = f\chi_{X_n}$. Then $f_n \to f$ pointwise and $f_n \in L^p(X,\mu)$ for all $n \ge 1$. Since $|fg| \in L^1(X,\mu)$ and $|f_n g| \le |fg|$, the Lebesgue Dominated Convergence Theorem shows

$$\int f g \, d\mu = \lim_{n\to\infty} \int f_n g \, d\mu = \lim_{n\to\infty} \int f_n g_n d\mu = \lim_{n\to\infty} \Gamma(f_n) = \Gamma(f)$$

□

If $p = 1$ then we cannot drop the assumption of σ-finiteness.

15.8 Theorem (RRT II). *Let $\Gamma \in L^p(X,\mu)^*$, where $1 < p < \infty$. Then if $\frac{1}{p} + \frac{1}{q} = 1$ then there exists a unique $g \in L^q(X,\mu)$ such that*

$$\Gamma(f) = \int f g \, d\mu$$

for all $f \in L^p(X,\mu)$. Moreover, $\|\Gamma\| = \|g\|_q$.

PROOF: Let $E \subseteq X$ be σ-finite. Then there exists a unique $g_E \in L^q(X,\mu)$, vanishing outside of E, such that $\Gamma(f) = \int f g_E d\mu$ for all $g \in L^p(X,\mu)$ vanishing outside of E. Moreover, if $A \subseteq E$ then $g_A = g_E$ a.e. on A. For each σ-finite set E let $\lambda(E) = \int |g_E|^q d\mu$. If $A \subseteq E$ then $\lambda(A) \le \lambda(E) \le \|\Gamma\|^q$. Let $M = \sup\{\lambda(E) \mid E \text{ is } \sigma\text{-finite}\}$.

Let $\{E_n\}$ be a sequence of σ-finite sets such that $\lim_{n\to\infty}\lambda(E_n) = M$. If $H = \bigcup_{n=1}^{\infty} E_n$ then H is σ-finite and $\lambda(H) = M$. If E is σ-finite with $H \subseteq E$ then $g_E = g_H$ a.e on H. But

$$\int |g_E|^q d\mu = \lambda(E) \leq \lambda(H) = \int |g_H|^q d\mu$$

so $g_E = 0$ a.e. on $E \setminus H$. Let $g = g_H \chi_H$. Then $g \in L^q(X,\mu)$ and if E is σ-finite with $H \subseteq E$ then $g_E = g$ a.e. If $f \in L^p(X,\mu)$ then let $E = \{x \in X \mid f(x) \neq 0\}$. E is σ-finite and hence $E_1 := E \cup H$ is σ-finite. Hence

$$\Gamma(f) = \int f g_{E_1} d\mu = \int f g \; d\mu = \phi_g(f)$$

Therefore $\Gamma = \phi_g$ and as before $\|\Gamma\| = \|g\|_q$. □

We have shown that if $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ then, for any measure space $(X, \mathscr{A}, \mu)$, $L^p(X,\mu)^* \cong L^q(X,\mu)$. If μ is σ-finite then $L^1(X,\mu)^* \cong L^\infty(X,\mu)$. What happens when $p = \infty$? $L^1(X,\mu) \hookrightarrow L^\infty(X,\mu)^*$, but this embedding is not usually surjective. There exists a compact Hausdorff space Ω such that $L^\infty(X,\mu) \cong C(\Omega)$. What is $C(\Omega)^*$?

Let $\varphi : [a,b] \to \mathbb{R}$ be defined by $\varphi(f) = f(x_0)$. Then $\varphi \in C[a,b]^*$, and $\|\varphi\| = 1$. Let μ_{x_0} the measure on $[a,b]$ of the point mass x_0. If $g \in L^1([a,b],m)$ then $\varphi_g(f) = \int_a^b f g \; dm$ is a linear functional on $C[a,b]$, and $\|\varphi_g\| \leq \|g\|_1$. g is the Radon-Nikodym derivative of an absolutely continuous measure μ on $[a,b]$, and $\varphi_g(f) = \int f \; d\mu$. If $\mu \in \text{Meas}[a,b]$ then $\varphi_\mu(f) = \int f \; d\mu$ is a bounded linear functional on $C[a,b]$, with $\|\varphi_\mu\| \leq \|\mu\|_{\text{Meas}}$.

15.9 Definition. A linear functional $\Gamma \in C[a,b]^*$ is *positive* if $\Gamma(f) \geq 0$ for all $f \in C[a,b]$ with $f \geq 0$. For any $\Gamma \in C[a,b]^*$ and $f \geq 0$, define $\Gamma^+(f) = \sup\{\Gamma(g) \mid 0 \leq g \leq f\}$.

Γ^+ is linear. If $f \in C[a,b]$ define

$$\Gamma^+(f) = \Gamma^+(f^+) - \Gamma^+(f^-)$$
$$\Gamma^-(f) = \Gamma^+(f) - \Gamma(f)$$

Then $\Gamma^+, \Gamma^- \in C[a,b]^*$ and $\Gamma = \Gamma^+ - \Gamma^-$.

15.10 Lemma (Jordon Decomposition). *Let $\Gamma \in C[a,b]^*$. Then there exists $\Gamma^+, \Gamma^- \in C[a,b]^*$ that are positive with $\Gamma = \Gamma^+ - \Gamma^-$ and $\|\Gamma\| = \|\Gamma^+\| + \|\Gamma^-\| = \Gamma^+(1) + \Gamma^-(1)$.*

15.11 Theorem (RRT III). *Let $\Gamma \in C[a,b]^*$. Then there exists a unique finite signed measure μ on $\mathscr{B}[a,b]$ such that*

$$\Gamma(f) = \int_a^b f \; d\mu$$

Moreover, $\|\Gamma\| = |\mu|([a,b])$.

PROOF: Find this in a book. □

16 Product Measures

16.1 Definition. Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be measurable spaces. A *measurable rectangle* is a set of the form $A \times B$, where $A \in \mathscr{A}$ and $B \in \mathscr{B}$. Let $Z = X \times Y$ and

$$\mathscr{Z}_0 = \{\coprod_{i=1}^{n} A_i \times B_i \mid A_i \in \mathscr{A}, B_i \in \mathscr{B}\}$$

16.2 Lemma. *$\mathscr{Z}_0$ is an algebra in $\mathscr{P}(Z)$.*

Let $\mathscr{Z}$ be the σ-algebra generated by $\mathscr{Z}_0$. We write $\mathscr{Z} = \mathscr{A} \times \mathscr{B}$. Assume that $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \lambda)$ are measure spaces. A measure π on $(Z, \mathscr{Z})$ is called a *product measure* if $\pi(A \times B) = \mu(A)\lambda(B)$.

16.3 Theorem (Product Measure Theorem). *Let $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \lambda)$ be measure spaces. Then there exists a measure π on $(X \times Y, \mathscr{A} \times \mathscr{B})$ such that $\pi(A \times B) = \mu(A)\lambda(B)$. Moreover, if μ and λ are σ-finite then π is unique and σ-finite.*

PROOF: Suppose that $A \times B$ can be written as $\sum_{i=1}^{\infty} A_i \times B_i$, where each of the measurable rectangles $A_i \times B_i$ are dijoint. Then

$$\chi_A(x)\chi_B(y) = \chi_{A\times B}(x,y) = \sum_{i=1}^{\infty} \chi_{A_i}(x)\chi_{B_i}(y)$$

for all $x \in X$ and $y \in Y$. Fix x and integrate with respect to λ.

$$\begin{aligned}
\int_Y \chi_{A\times B}(x,y)d\lambda(y) &= \int_Y \sum_{i=1}^{\infty} \chi_{A_i}(x)\chi_{B_i}(y)d\lambda(y) \\
\int_Y \chi_A(x)\chi_B(y)d\lambda(y) &= \sum_{i=1}^{\infty} \chi_{A_i}(x)\int_Y \chi_{B_i}(y)d\lambda(y) && \text{by MCT} \\
\chi_A(x)\lambda(B) &= \sum_{i=1}^{\infty} \chi_{A_i}(x)\lambda(B_i)
\end{aligned}$$

Further integrating with respect to μ yeilds (again by MCT) $\mu(A)\lambda(B) = \sum_{i=1}^{\infty} \mu(A_i)\lambda(B_i) \quad (*)$

Define π_0 on $\mathscr{Z}_0$ by $\pi_0(\cup_{i=1}^{n} A_i \times B_i) = \sum_{i=1}^{n} \mu(A_i)\lambda(B_i)$. Then π_0 is a measure on $\mathscr{Z}$ by $(*)$ (the only nontrivial thing to check was countable additivity). Caratheodory's Extension Theorem gives us a measure π defined on at least $\mathscr{Z}$ that extends π_0. If μ and λ are σ-finite then π_0 is σ-finite, so Hahn's Extension Theorem tells us that π is unique. □

16.4 Definition. Let $E \subseteq Z = X \times Y$. An *x-section* of E is the set $E_x = \{y \in Y \mid (x,y) \in E\}$. A *y-section* is $E^y = \{x \in X \mid (x,y) \in E\}$. Let $f : Z \to [-\infty, \infty]$ and $x \in X$. The *x-section of f* is $f_x(y) = f(x,y)$. For $y \in Y$ the *y-section of f* is $f^y(x) = f(x,y)$.

16.5 Lemma. *If $E \subseteq Z$ is measurable in the product measure then E_x, E^y are measurable in the factors for each $x \in X$ and $y \in Y$. Similarily, if $f : Z \to [-\infty, \infty]$ is measurable in the product measure then f_x, f^y is are measurable in each factor for every $x \in X$, $y \in Y$.*

PROOF: Exercise. □

16.6 Definition. A *monotone class* is a non-empty collection $M \subseteq \mathscr{P}(X)$ such that

1. If $\{E_n\}_{n=1}^{\infty} \subseteq M$ with $E_n \subseteq E_{n+1}$ then $\bigcup_{n=1}^{\infty} \in M$.

2. If $\{E_n\}_{n=1}^{\infty} \subseteq M$ with $E_n \supseteq E_{n+1}$ then $\bigcap_{n=1}^{\infty} \in M$.

Every σ-algebra is a monotone class. If $\mathscr{A} \subseteq \mathscr{P}(X)$ is any collection of subsets then there is a smallest monotone class $M(\mathscr{A})$ that contains $\mathscr{A}$. Simply take the intersection of all monotone classes that contain $\mathscr{A}$. With this in mind, it is clear that $M(\mathscr{A}) \subseteq \sigma(\mathscr{A})$, the smallest σ-algebra that contains $\mathscr{A}$. In fact, the reverse inclusion also holds when $\mathscr{A}$ is an algebra.

16.7 Lemma (Monotone Class Lemma). *If $\mathscr{A} \subseteq \mathscr{P}(X)$ is an algebra them $M(\mathscr{A}) = \sigma(A)$.*

PROOF: We need only show that $M := M(\mathscr{A})$ is an algebra, since this combined with the fact that M is closed under countable ascending unions implies that M is closed under countable unions. For $E \in M$ define $M(E) = \{F \in M \mid E \setminus F, E \cup F, F \setminus E \in M\}$. Then $\varnothing \in M(E)$ and $E \in M(E)$. Further, if $F \in M(E)$ then $E \in M(F)$ by the symmetry of the definition. $M(E)$ is a monotone class since complementation, union, and intersection play nicely together.

Suppose that $E \in \mathscr{A}$. Then since $\mathscr{A}$ is an algebra, $\mathscr{A} \subseteq M(E)$. But $M(E)$ is also a monotone class, so $M \subseteq M(E) \subseteq M$ and $M = M(E)$. It follows that $\mathscr{A} \subseteq M(F)$ for every $F \in M$, and again we have $M = M(F)$. But $\varnothing, X \in \mathscr{A} \subseteq M(E)$, so this implies that M is closed under intersections and finite unions. □

16.8 Lemma. *Let $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \lambda)$ be σ-finite. If $E \in \mathscr{Z} = \mathscr{A} \times \mathscr{B}$, then $f(x) = \lambda(E_x)$ and $g(y) = \mu(E^y)$ are measurable and*

$$\int_X f \, d\mu = \pi(E) = \int_Y g \, d\lambda$$

PROOF: Get this from Aaron. □

16.9 Theorem (Tonelli's Theorem). *Let $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \lambda)$ be σ-finite. Let $F : Z = X \times Y \to [0, \infty]$ be measurable. Then the functions defined by $f(x) = \int_Y F_x d\lambda$ and $g(y) = \int_X F^y d\mu$ are measurable and $\int_X f \, d\mu = \int_Z F \, d\pi = \int_Y g \, d\lambda$, where $\pi = \mu \times \lambda$. This is to say that*

$$\int_X \left(\int_Y F(x,y) d\lambda(y) \right) d\mu(x) = \int_Z F \, d\pi = \int_Y \left(\int_X F(x,y) d\mu(x) \right) d\lambda(y)$$

PROOF: If $F = \chi_E$ for some $E \in \mathscr{Z} = \mathscr{A} \times \mathscr{B}$ then the theorem is exactly the previous lemma. If follows immediately that the theorem holds for all non-negative measurable simple functions. If F is arbitrary, we can find a sequence $\{\Phi_n\}_{n=1}^{\infty}$ of non-negative measurable simple functions such that $\Phi_n \nearrow F$. Let $\varphi_n(x) = \int_Y (\Phi_n)_x d\lambda$ and $\psi_n(y) = \int_X (\Phi_n)^y d\mu$. Then φ_n and ψ_n are measurable and monotonic in n. By the Monotone convergence Theorem,

$$\lim_{n\to\infty} \varphi_n(x) = f(x) \qquad \text{and} \qquad \lim_{n\to\infty} \psi_n(y) = g(y)$$

By the Monotone Convergence Theorem again,

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_X \varphi_n d\mu = \lim_{n\to\infty} \int_Z \Phi_n d\pi = \int_Z F \, d\pi$$

and similarly $\int_Y g \, d\lambda = \int_Z F \, d\pi$. □

16.10 Theorem (Fubini's Theorem). *Let $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \lambda)$ be σ-finite and let $\pi = \mu \times \lambda$. If F is integrable with respect to π on $Z = X \times Y$ then the extended real valued functions defined almost everywhere by $f(x) = \int_Y F_x d\lambda$ and $g(y) = \int_X F^y d\mu$ have finite integrals and $\int_X f \, d\mu = \int_Z F \, d\pi = \int_Y g \, d\lambda$. That is to say,*

$$\int_X \left(\int_Y F(x,y) d\lambda(y) \right) d\mu(x) = \int_Z F \, d\pi = \int_Y \left(\int_X F(x,y) d\mu(x) \right) d\lambda(y)$$

PROOF: Since F is π-integrable, so are F^+ and F^-. Apply Tonelli's Theorem to establish that f^+ and f^- have finite integrals and hence are finite a.e. Therefore $f = f^+ - f^-$ is defined a.e. and $\int_X f \, d\mu = \int_Z F \, d\pi$. Similarly, we can show that $g = g^+ - g^-$ is defined a.e. and $\int_Y g \, d\lambda = \int_Z F \, d\pi$. □