Complex Analysis
Fall 2004
Professor L. Marcoux

Chris Almost

Contents

1 Introduction

1.1 Motivation

In real analysis, we observe that a distinct advantage of the real numbers over the rational numbers is that $\mathbb{R}$ is complete. This is the propositionerty that allows us to define things like suprema and infima, and in turn allows us to prove theorems such as the Mean/Intermediate value theorems. One way in which the real numbers are deficient is that they are not algebraically closed. (e.g. $x^2 + 1 = 0$ has no roots in $\mathbb{R}$.) Our goal is study a new number system, the set of complex numbers, which contains $\mathbb{R}$ and is algebraically closed. As we shall see, whereas notions of differentiability of real valued functions lead to examples which exhibit pathologies, the theory of "analytic" complex valued functions is much richer and the examples are much better behaved.

1.2 Definitions

Denote by $\mathbb{C}$ the set $\mathbb{R}^2$ equipped with the following operations.

$$(a,b)+(c,d)=(a+b,c+d) \text{ and } (a,b)(c,d)=(ac-bd,ad+bc)$$

It is not hard to see that $\mathbb{C}$ is a commutative ring. Here $(1,0)$ is the multiplicative identity. We shall identify the real numbers with the subset $\mathbb{R}\times\{0\}$ of $\mathbb{C}$. We write i for the ordered pair $(0,1)$, and refer to $i\mathbb{R}$ as the imaginary axis of $\mathbb{C}$, and $\mathbb{R}$ as the real axis of $\mathbb{C}$. Note that $i^2=(-1,0)=-1$. We tend to use a,b for real numbers and w,z for complex numbers.

1.1 Exercise. $\mathbb{C}$ is a field

1.3 Algebraic properties

Given $z=a+bi$, the number $\overline{z}=a-bi$ is called the complex conjugate of z. The map $z\to\overline{z}$ is an involution on an algebra.

$z\overline{z}=a^2+b^2=:|z|^2$. This is the Euclidean length of the vector z in $\mathbb{C}$. This is also called the norm or modulus of z. Note that if $0\neq z$ if and only if $|z|\neq 0$, in which case $\frac{z\overline{z}}{|z|^2}=1$, whence $z^{-1}=\frac{\overline{z}}{|z|^2}=\frac{a-bi}{a^2+b^2}$. We define the real and imaginary parts of z to be $\Re z=a$ and $\Im z=b$, so that $z=\Re z+i\Im z$. Then functions $z\to\Re z$ and $z\to\Im z$ are real-linear (but not complex linear).

1.4 Polar coordinates

$z=a+bi$. Define $r=|z|$ and θ by $r\cos\theta=a$ and $r\sin\theta=b$. We call θ the argument of z and write $\theta=\arg z$ (not defined if $z=0$. Of course, sin and cos are periodic, and so the argument is uniquely defined once we choose some interval $[\lambda,\lambda+2\pi)$ for some $\lambda\in\mathbb{R}$. This is called the branch of the argument. This representation is useful for multiplying complex numbers. The product of complex numbers is equal to the complex number who's modulus is the product of the moduli and who's argument is the sum of the arguments (modulo 2π). For example, $z\to iz$ is the map that takes a complex number and rotates it counterclockwise $\pi/2$. Using induction, we can prove:

1.5 DeMoivre's formula

If $z=r(\cos\theta+i\sin\theta)$ and if $n\in\mathbb{N}$ then $z^n=r^n(\cos(n\theta)+i\sin(n\theta))$.

Now consider the inverse problem, given $w\in\mathbb{C}, n\in\mathbb{N}$, find $z\in\mathbb{C}$ so that $z^n=w$. We see clearly that if $w=0$ then $z=0$ works, and if $w=\rho(\cos\phi+i\sin\phi)$ then $z=\phi^{\frac{1}{n}}(\cos(\phi/n)+i\sin(\phi/n))$ will work. But there are more solutions, in fact replacing ϕ/n with $(\phi+2k\pi)/n$ will give us a solution z_k for any $k\in\mathbb{Z}$, and the solutions are distinct for $0\leq k<n$.

1.6 n^{th} root functions

The situation becomes more delicate if we consider the behaviour of functions, $f(z) = z^n$ and their "inverses". Indeed, except at the point 0, the function $f(z) = z^n$ is an n to 1 function (i.e. it maps n different numbers to the same number), and maps each section S of the disk of angle $2\pi/n$ onto the whole disk. As such, we cannot talk of an inverse function. Can we at least find a continuous function g such that $g^n(z) = z$? Try $g(z) = z^{\frac{1}{n}}$. Fix any branch of the arg function, say $[0, 2\pi)$. Consider the behaviour of g as z traverses the unit circle $\mathbb{T}$. With $z_0 = 1$, we have $\arg(z_0) = 0$. Suppose that $g(z_0) = 1$. Then as $z \to z_0$ in the first quadrant, $g(z) \to g(z_0) = 1$, so $\arg g(z) \to \arg g(z_0) = 0$. In the forth quadrant, as $z \to z_0$, the argument of $g(z)$ is approaching $2\pi/n \neq 0$. Thus g is not continuous at 1.

In 1851, Riemann understood this and propositionosed the following "solution". We extend the domain of the function $g(z) = z^{\frac{1}{n}}$ as follows. For $0 \neq w$, we treat each $w_k = r(\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi))$ as being different numbers for $0 \leq k < n$. One way of doing this is considering $\mathbb{C} \times \{0, 1, \dots, n-1\}$, each of which sheet is "cut" along $[0, \infty)$. We identify the lower part of the cut on sheet k with the upper part of the cut on sheet $k+1$ mod n. The result is a n-sheeted surface, $[R]$, called the Riemann surface associated with $g(z) = z^{\frac{1}{n}}$. The point 0 is called a branch point because there is only one copy of it.

As a result, we get the following commutative diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F} & [R] \\ & \searrow^{f} & \downarrow{\scriptstyle \pi} \\ & & \mathbb{C} \end{array}$$

where $\pi : (z, k) \to z$ is projection, f is as above, and $F(z) = F(r(\cos(\theta + 2k\pi/n) + i\sin(\theta + 2k\pi/n)) = (r^n(\cos(n\theta) + i\sin(n\theta)), k)$. Notice that where f was n to 1, $F : \mathbb{C} \to [R]$ is 1 to 1 and continuous.

1.7 Embed $\mathbb{C}$ in $\mathbb{M}_2(\mathbb{R})$

Suppose we fix $w = a + bi$. Consider the map $M_w : \mathbb{C} \to \mathbb{C} : z \mapsto wz$. We can think of M_w as a map from $\mathbb{R}^2$ to itself. $M_w : (u, v) \mapsto (au - bv, av + bu)$. Thus M_w corresponds to a matrix (with repsect to the standard basis $\{(1,0),(0,1)\}$ for $\mathbb{R}^2$:

$$M_w \longleftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

This correspondence establishes an algebra isomorphism between $\mathbb{C}$ and a subalgebra of $\mathbb{M}_2(\mathbb{R})$.

2 Elementary functions

2.1 The complex exponential function

Recall the Taylor expansions of $\cos(x), \sin(x), e^x$,

$$\begin{aligned} \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \cdots \\ \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - \cdots \\ e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \end{aligned}$$

Now $e^x > 0$ and $(e^x)' = e^x$, and so $x \mapsto e^x$ is strictly increasing on $\mathbb{R}$. As such, we can talk about the inverse of e^x, which we call the (natural) logarithm, $\log(x)$, which satisfies:

$$e^{\log(x)} = x \text{ for } x > 0 \text{ and } \log(e^x) = x \text{ for all } x \in \mathbb{R}$$

We would like to make sense of $e^z, \cos(z), \sin(z)$ and $\log(z)$ for complex numbers z. However, we do not know anything about convergence of complex series! Informally, if the series were to converge, and if we could reorder the terms at will, then we would expect for all $y \in \mathbb{R}$ that

$$e^{iy} = \cos(y) + i\sin(y)$$

We sometimes write $\text{cis}(y)$ for $\cos(y) + i\sin(y)$.

Therefore we choose to define our "extended" exponential to satisfy this expression. Namely, for $x, y \in \mathbb{R}$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \quad e^{iy} = \cos(y) + i\sin(y)$$

Notice that both definitions agree where they overlap at zero.

If the complex-exponential is to behave like the real-exponential, we would also expect that $\exp(x + yi) = \exp(x)\exp(iy)$, and so we define the complex- exponential to achieve that.

2.1 Definition. For $z = x + iy \in \mathbb{C}$, we define $\exp(z) = e^x \text{cis}(y)$

2.2 Definition. A function $f : \mathbb{C} \to \mathbb{C}$ is said to be periodic with period ω if $f(z + \omega) = f(z)\ \forall z \in \mathbb{C}$

2.3 Proposition. *1.* $\exp(z + w) = \exp(z)\exp(w)$

2. $\exp(z) \neq 0$

3. $|\exp(x + iy)| = e^x$

4. $\exp(\cdot)$ *is periodic, and each period has the form* $2\pi ni$ *for some* $n \in \mathbb{Z}$.

5. $\exp(z) = 1$ *if and only if* $z \in (2\pi i)\mathbb{Z}$

PROOF: Exercise. □

We point out in passing that as y goes from 0 to 2π, $\exp(iy)$ revolves around the unit circle $\mathbb{T}$ in a counter-clockwise direction.

2.2 Sine and cosine

When $y \in \mathbb{R}$, we can use our definition of exp to solve for cos and sin.

$$\cos(y) = \frac{\exp(iy) + \exp(-iy)}{2} \quad \sin(y) = \frac{\exp(iy) - \exp(-iy)}{2i}$$

This observation leads to:

2.4 Definition. For $z \in \mathbb{C}$, we define

$$\cos(z) = \frac{\exp(iz) + \exp(-iz)}{2} \quad \sin(z) = \frac{\exp(iz) - \exp(-iz)}{2i}$$

2.5 Proposition. *1.* $\sin^2(z) + \cos^2(z) = 1$

2. $\sin(z + w) = \sin(z)\cos(w) + \sin(w)\cos(z)$

3. $\cos(z + w) = \cos(z)\cos(w) - \sin(z)\sin(w)$

PROOF: Exercise. □

2.3 The complex logarithm function

The discussion of Riemann surfaces for the power functions applies to the log function as well. $z \mapsto \exp(z)$ is periodic with (minimal) period $2\pi i$. In particular, if we fix $y_0 \in \mathbb{R}$ and set

$$B_{y_0} = \{x + yi \mid x \in \mathbb{R}, y_0 \leq y < y_0 + 2\pi\}$$

It is clear that $\exp(B_{y_0}) = \mathbb{C} \setminus \{0\}$. In particular, horizontal lines are mapped onto rays emmanating from the origin, and vertical line segments get mapped to circles centred at the origin. Not only this, but the restricted map is one to one and onto from this strip to $\mathbb{C} \setminus \{0\}$.

2.6 Definition. We define $\log : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ with range B_{y_0} (that is, $y_0 \leq \Im(\log z) < y_0 + 2\pi$) via $\log z = \log |z| + i \arg(z)$, where $\arg(z)$ takes values in B_{y_0}.

The Riemann surface for the log function looks like the the Riemann surface for $z \mapsto z^n$, except there are infinitely many sheets and zero is removed.

3 Properties of $\mathbb{C}$

3.1 $\mathbb{C}$ is complete

3.1 Theorem. *A sequence $(z_n)_{n=1}^{\infty}$ in $\mathbb{C}$ converges if and only if it is Cauchy.*

PROOF: Suppose that $(z_n)_{n=1}^{\infty}$ converges to $z \in \mathbb{C}$. Then given $\varepsilon > 0$, there is N such that for all $n \geq N$, $|z_n - z| < \frac{\varepsilon}{2}$. But then for $n, m \geq N$,

$$|z_n - z_m| \leq |z_n - z| + |z - z_m| < \varepsilon$$

and so $(z_n)_{n=1}^{\infty}$ is Cauchy.

Now suppose that $(z_n)_{n=1}^{\infty}$ is Cauchy. Let $\varepsilon_0 = 1$. Since $(z_n)_{n=1}^{\infty}$ is Cauchy, we can find N_0 such that for all $n, m \geq N_0$, $|z_n - z_m| < \varepsilon_0$. In particular, for $n \geq N_0$, $|z_n - z_{N_0}| < 1$, which implies that $|z_n| < |z_{N_0}| + 1$ and therefore $(z_n)_{n=1}^{\infty}$ is bounded by $\max\{|z_1|, \dots, |z_{N_0-1}|, |z_{N_0}| + 1\}$.

By the Bolzano-Weierstraß theorem we can find a subsequence $(z_{n_k})_{k=1}^{\infty}$ of $(z_n)_{n=1}^{\infty}$ which converges to some $z \in \mathbb{C}$. We claim that $(z_n)_{n=1}^{\infty}$ converges to z.

Let $\varepsilon > 0$. We can find K such that for all $k \geq K$, $|z_{n_k} - z| < \frac{\varepsilon}{2}$. We can also find N such that for all $n, m \geq N$, $|z_n - z_m| < \frac{\varepsilon}{2}$. Thus if $n, k \geq \max\{N, K\}$ then $n_k \geq k$ and

$$|z_n - z| \leq |z_n - z_{n_k}| + |z_{n_k} - z| < \varepsilon$$

and so $(z_n)_{n=1}^{\infty}$ converges. □

3.2 Riemann sphere

We can represent the elements of $\mathbb{C}$ as the points on a sphere as follows. Let Σ be a sphere of radius 1 tangent to the complex plane at the point 0. The diameter of Σ containing 0 intersects Σ at a point which we will call N, the north pole. Given $z \in \mathbb{C}$, let $\sigma(z)$ denote the intersection of Σ with the line segment from N to z. This establishes a bijection between $\mathbb{C}$ and $\Sigma \setminus \{N\}$. (σ is actually a homeomorphism.) We think of N as the "point at infinity" for $\mathbb{C}$, and we denote it simply ∞.

The map $\mathbb{C} \cup \{\infty\} \to \Sigma : z \mapsto \sigma(z) : \infty \mapsto N$ is called the stereographic projection of the extended complex plane onto Σ.

3.2 Theorem. *(Bolzano-Weierstraß) Every sequence in $\mathbb{C}$ has a limit point in the extended complex plane, $\mathbb{C} \cup \{\infty\}$.*

3.3 Continuous curves

Let $x : [a,b] \to \mathbb{R}$ and $y : [a,b] \to \mathbb{R}$ be two continuous functions. We define a continuous curve $\mathscr{C} = \{(x(t), y(t)) \mid t \in [a,b]\} \subseteq \mathbb{R}^2$, or with $z(t) = x(t) + iy(t)$, we think of $\mathscr{C}$ as lying in the complex plane. The point $z(a)$ (resp. $z(b)$) is called the initial (resp. final) point of $\mathscr{C}$. This assigns a direction to the curve (from the initial to the final point, duh). $\mathscr{C}$ is closed if the initial and final points are the same, otherwise it is called an arc.

3.3 Definition. A set $E \subseteq \mathbb{C}$ is called arcwise connected if for any two (distinct) points in E there exists an arc between them that lies entirely in E.

We say that a set $G \subseteq \mathbb{C}$ is a domain if it is open and arcwise connnected. A closed domain is a set of the form $\overline{G}$ (the closure of G), where G is a domain. A region is a set of the form $G \cup H$, where G is a domain and $H \subseteq \partial G$. It is possible for a region to be a domain or a closed domain.

3.4 Definition. Let $\mathscr{C} = \{z(t) = x(t) + iy(t) \mid t \in [a,b)\}$, where x and y are continuous on $[a,b)$. Suppose that $t_1 \neq t_2 \implies z(t_1) \neq z(t_2)$. Then $\mathscr{C}$ is called a Jordan curve if $\lim_{t \to b^-} x(t)$ and $\lim_{t \to b^-} y(t)$ both exist. If

$$\lim_{t \to b^-} x(t) = x(a) \text{ and } \lim_{t \to b^-} y(t) = y(a)$$

then $\mathscr{C}$ is called a closed Jordan curve.

3.5 Theorem. *(Jordan Curve Theorem) Let $\mathscr{C}$ be a closed Jordan curve. Then $\mathbb{C} \setminus \mathscr{C}$ is the union of two domains with $\mathscr{C}$ as their common boundry. The first domain G_1 is bounded and called the interior of $\mathscr{C}$, while the second domain G_2 is not bounded and called the exterior of $\mathscr{C}$. We shall assume that the positive orientation of a closed Jordan curve keeps the interior on the "left".*

3.4 Connnectedness

3.6 Definition. A domain G is said to be simply connected if, given any closed Jordan curve $\mathscr{C} \subseteq G$, the interior of $\mathscr{C}$ is contained in G. Otherwise, G is said to be multiply connected.

In the extended complex plane, a domain G is simply connected if given any Jordan curve in G, either the interior or the exterior is contained in G. This means that $\mathbb{C} \cup \{\infty\} \setminus \mathbb{D}$ is simply connected in the extended complex plane, but this set (minus ∞) is not simply connected in the normal complex plane.

3.7 Definition. Let $C_0, \dots, C_n$ be $n+1$ closed Jordan curves such that no two of them intersect and that $C_1, \dots, C_n$ lie in the interior of C_0. Suppose further that that the interiors of the curves are pairwise disjoint. Then the interior of C_0 minus the closures of the interiors of $C_1, \dots, C_n$ is a domain G. We say that G is $(n+1)$-connected. We say that G is simply connected if $n = 0$.

For example, the annulus is 2-connected and the disc is simply connected.

3.8 Definition. Let $G \subseteq \mathbb{C}$ be a domain and suppose that $f : G \to \mathbb{C}$ is a function. If $z_0 \in G$ then we say that f has a limit L at z_0 and we write $\lim_{z \to z_0} f(z) = L$ if $\forall\, \varepsilon > 0$, there exists $\delta > 0$ so that $0 < |z - z_0| < \delta$ implies that $|f(z) - L| < \varepsilon$. If $f(z_0) = \lim_{z \to z_0} f(z)$ then we say that f is continuous at z_0. If f is continuous at all points in G then f is said to be continuous on G.

Notice that with $f : G \to \mathbb{C}$ as above, we can define two functions $u : \mathbb{R}^2 \to \mathbb{R}$ and $v : \mathbb{R}^2 \to \mathbb{R}$ such that $u(x,y) = \Re f(x+iy)$ and $v(x,y) = \Im f(x+iy)$, so that $f(x+iy) = u(x,y) + iv(x,y)$. It follows that f is continuous at $z_0 = x_0 + iy_0$ if and only if u and v are continuous at (x_0, y_0).

3.9 Example. $f(z) = z^n$ is continuous on $\mathbb{C}$. (Prove this.)

If $\tilde{G}$ is a region which contains some of the boundry points of the domain G, and if $z_0 \in \tilde{G}$ and z_0 is on the boundry of G, then $f : \tilde{G} \to \mathbb{C}$ is said to be continuous at z_0 if $f(z_0) = \lim_{z \to z_0, z \in \tilde{G}} f(z)$. In a similar way, we can define the notion of continuity of a function defined on a curve $\mathscr{C}$.

3.10 Definition. Let $K \subseteq \mathbb{C}$ be a set and $f : K \to \mathbb{C}$ be a function. We say that f is uniformly continuous on K if $\forall\, \varepsilon > 0$ there exists $\delta > 0$ so that $|z - w| < \delta$ implies that $|f(z) - f(w)| < \varepsilon$. (The key point is that one choice of δ works for all pairs z, w.)

We wish to establish conditions on a set K which will ensure that every continuous function on K is in fact uniformly continuous.

3.5 Compactness

3.11 Definition. Let $K \subseteq \mathbb{C}$. We say that a collection $\{G_\alpha\}_{\alpha \in \Lambda}$ of sets is an open cover of K if each G_α is open and $K \subseteq \cup_{\alpha \in \Lambda} G_\alpha$. A finite subcover of K is a finite subset $\{G_{\alpha_1}, \ldots, G_{\alpha_n}\}$ such that $K \subseteq \cup_{k=1}^{n} G_{\alpha_k}$. We say that K is compact if every open cover of K admits a finite subcover.

3.12 Theorem. *(Heine-Borel) Suppose that $K \subseteq \mathbb{C}$ is closed and bounded. Then K is compact.*

PROOF: Suppose otherwise. Let $\{G_\alpha\}_{\alpha \in \Lambda}$ be an open cover of K that does not admit a finite subcover. Since K is bounded, we can find $M > 0$ such that $K \subset R_0 = \{x + iy \mid x, y \in [-M, M]\}$. We can partition R_0 into four closed subsquares R_{0i} of equal area. At least one of the $K \cap R_{0i}$'s cannot be covered by finitely many of the G_α's. Let R_1 be one of those squares. Continue this process to get a sequence $R_0 \supseteq R_1 \supseteq R_2 \supseteq \cdots$ such that $K \cap R_i$ cannot be covered by finitely many of the G_α's. Choose $z_m \in K \cap R_m$. Then this is a bounded sequence, so by the Bolzano-Weierstraß theorem, there is a subsequence $(z_{m_n})_{n=1}^{\infty}$ that converges to some $z_0 \in \mathbb{C}$. K is closed, so $z_0 \in K$. Since $K \subseteq \cup_{\alpha \in \Lambda} G_\alpha$, there exists $\alpha_0 \in \Lambda$ so that $z \in G_{\alpha_0}$. Since G_{α_0} is open, there exists some $\delta > 0$ so that the ball around z_0 of radius δ is contained in G_{α_0}. Moreover, $z_0 \cap_{m=1}^{\infty} R_m$ and $\mathrm{diam} R_m \to 0$. Thus for m sufficiently large, $z, w \in R_m$ implies that $|z - w| < \delta$, which implies that $R_m \subseteq G_{\alpha_0}$. This contradiction shows that K is compact. □

3.13 Theorem. *If $K \subseteq \mathbb{C}$ is compact and $f : K \to \mathbb{C}$ is continuous, then f is uniformly continuous on K.*

PROOF: Exercise. □

4 Analyticity

The definition of the derivative of a complex function at a point w is the same in form as that of the derivatives for real-valued functions. It is therefore quite surprising how different the two theories turn out to be.

4.1 Definition. Let $G \subseteq \mathbb{C}$ be a domain and let $w \in G$. Suppose that $f : G \to \mathbb{C}$ is a function. We say that f is differentiable at w if

$$f'(w) = \lim_{h \to 0} \frac{f(w+h) - f(w)}{h} = \lim_{z \to w} \frac{f(z) - f(w)}{z - w}$$

exists as a complex number. We say that f is analytic in G if f is differentiable at each point $w \in G$. f is said to be analytic at w if f is analytic in $V_\delta(w)$ for some $\delta > 0$. If f is analytic on the entire plane, we say that f is entire.

4.2 Example. 1. Let $k \geq 1$ and $f(z) = z^k$. Then for all $w \in \mathbb{C}$

$$f'(w) = \lim_{z \to w} \frac{f(z) - f(w)}{z - w} = \lim_{z \to w} \frac{z^k - w^k}{z - w} = k w^{k-1}$$

as you would expect. Thus f is entire.

2. For $z = x + iy \in \mathbb{C}$, define $g(z) = \overline{z} = x - iy$. Then

$$\lim_{h\to 0} \frac{g(w+h) - g(w)}{h} = \lim_{h\to 0} \frac{\overline{h}}{h}$$

which does not exist. Thus complex conjugation is not differentiable at any point.

3. Exercise: neither $\Re(\cdot)$ nor $\Im(\cdot)$ are differentiable at any point in $\mathbb{C}$.

Since the proofs of the algebraic propositionerties of the (real) derivative only depend on the definition of the derivative, and this is the same for complex-valued functions, the proofs carry over to this setting.

1. $[\lambda f]'(z) = \lambda(f'(z))$ for all $\lambda \in \mathbb{C}$
2. $[f + g]'(z) = f'(z) + g'(z)$
3. $[fg]'(z) = f'(z)g(z) + f(z)g'(z)$
4. $[f/g]'(z) = \frac{f'(z)g(z)-f(z)g'(z)}{(g(z))^2}$ when $g(z) \neq 0$
5. $[f \circ g]'(z) = f'(g(z))g'(z)$ if $g'(z), f'(g(z))$ exist

4.1 Complex differentials

Let $w = f(z)$ be a complex function. For $\Delta z = \Delta x + i\Delta y$ set $\Delta w = f(z + \Delta z) - f(z)$. Observe that f is differentiable at z if and only if

$$f'(z) = \lim_{\Delta z\to 0} \frac{\Delta w}{\Delta z}$$

exists. The differential of the function $w = f(z)$ is $dw := f'(z)dz$, where $dz = \Delta z$. As in the real case, this gives us the suggestive equation $f'(z) = \frac{dw}{dz}$.

In CALCULUS 3, we saw that a function $u : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at a point (x, y) if and only if there exists $A \in L(\mathbb{R}^2, \mathbb{R})$ (say $[A] = [a, b]$ with respect to the standard basis) such that

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{|u(x + \Delta x, y + \Delta y) - u(x, y) - A(\Delta x, \Delta y)|}{\|(\Delta x, \Delta y)\|_2} = 0$$

If we write Δu for $u(x_\Delta x, y + \Delta y) - u(x, y)$ this becomes

$$\Delta u = A(\Delta x, \Delta y) = \eta\|(\Delta x, \Delta y)\|_2$$

where $\eta \to 0$ as $\|(\Delta x, \Delta y)\|_2 \to 0$. When u is differentiable at (x, y), recall from mulitvariable calculus that

$$a = \frac{\partial u}{\partial x} \text{ and } b = \frac{\partial u}{\partial y}$$

We saw before that if $f : \mathbb{C} \to \mathbb{C}$ is a complex function, then $f(x+iy) = u(x, y)+iv(x, y)$ and f is continuous if and only if u and v are continuous. Consider $f_R(z) = \Re(z)$. Then $u(x, y) = x$ and $v(x, y) = 0$. The partials exist and are $1, 0$ respectively, and the second partials are continuous, so u, v are differentiable. There is a problem here. It appears as though one cannot choose the real and imaginary parts of a complex differentiable function arbitrarily.

4.2 The Cauchy-Riemann equations

4.3 Theorem. *Let $G \subseteq \mathbb{C}$ be a domain and $f : G \to \mathbb{C}$ a function. Let $z = x + iy \in G$. Then f is differentiable at z if and only if*

1. *u, v are differentiable at (x, y)*
2. *u, v satisfy the Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

PROOF: See text. □

4.4 Example. Let $G \subseteq \mathbb{C}$ be a domain and $\mathscr{C} \subseteq G$ be a Jordan curve. Let $z(t)$, $t \in [a, b]$ be a parametrization of $\mathscr{C}$ and fix $t_0 \in (a, b)$. If $\mathscr{C}$ has a tangent line τ at $z_0 = z(t_0)$, then $\Delta z = z(t_0 + \Delta t) - z(t_0)$ has a "limiting direction" in the sense that $\varphi = \lim_{\Delta t \to 0} \arg \Delta z$ exists. Since $\Delta t \to 0$ implies that $\Delta z \to 0$, we may say that $\varphi \lim_{\Delta z \to 0} \arg \Delta z$. Now suppose that $f : G \to \mathbb{C}$ is continuous and $f'(z_0)$ exists and is not zero. Then $w(t) = f(z(t))$ defines a new curve Γ in $\mathbb{C}$. Let $\Delta w = f \circ z(t_0 + \Delta t) - f \circ z(t_0) = f(z_0 + \Delta z) - f(z_0)$. Since f is continuous $\lim_{\Delta z \to 0} \Delta w = 0$. Since $f'(z_0) \neq 0$, $f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$. In particular,

$$\arg f'(z_0) = \lim_{\Delta z \to 0} \arg \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \arg \Delta w - \arg \Delta z$$

Equivalently, $\lim_{\Delta z \to 0} \arg \Delta w = \arg f'(z_0) + \lim_{\Delta z \to 0} \arg \Delta z$, and so Γ has a tangent line T at $w_0 = f(z_0)$ and the angle that T makes with $\mathbb{R}$ is $\arg f'(z_0) + \varphi$.

4.5 Example. Suppose that C_1 and C_2 are two curves in G which pass through z_0. Suppose that the tangent line C_i at z_0 meets $\mathbb{R}$ at an angle of φ_i. We define the angle between the curves to be $\varphi_2 - \varphi_1$. From above, if $f'(z_0) \neq 0$, then the angle between $\Gamma_1 = f C_1)$ and $\Gamma_2 = f(C_2)$ at $w_0 = f(z_0)$ is

$$\theta_2 - \theta_1 = (\varphi_2 + \arg f'(z_0)) - (\varphi_1 + \arg f'(z_0)) = \varphi_2 = \varphi_1$$

Thus f preserves angle between curves passing through z_0 when $f'(z_0) \neq 0$.

4.6 Definition. Let $G \subseteq \mathbb{C}$ be a domain, $f : G \to \mathbb{C}$ a continuous function, and $z_0 \in G$. If f preserves angles between curves passing through z_0, we say that f is conformal at z_0. If f preserves the magnitude of the angle between curves, we say that f is isogonal at z_0.

Thus if $f : G \to \mathbb{C}$ is continuous on a domain G, then f is conformal at each $z_0 \in G$ where $0 \neq f'(z_0)$ exists. It can be shown that if f is analytic in G and $0 = f'(z_0)$ then f is not conformal at z_0.

Since $f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$, it follows that $|f'(z_0)| = \lim_{\Delta z \to 0} \frac{|\Delta w|}{|\Delta z|}$, and so we can think of $|f'(z_0)|$ as the local "stretching" factor of f at z_0.

4.3 The Möbius Tranformation

Suppose that $a, b, c, d \in \mathbb{C}$ and $|c| + |d| \neq 0$. The map

$$f(z) = \frac{az + b}{cz + d}$$

is called a Möbius transformation (or a fractional linear transformation).

1. If $c = 0$ then f is linear and $f'(z) = \frac{a}{d}$ for all $z \in \mathbb{C}$
2. If $ad - bc = 0$ then either
   (a) $d = 0$, whence $c \neq 0$ and so $b = 0$. Therefore f is just a constant function (undefined at $z = 0$)
   (b) $d \neq 0$. In this case, $(az + b)d = (ad)z + (bd) = (bc)z + bd = b(cz + d)$, and so f is constant again.

   These are the so-called trivial cases.
3. Suppose now that $c \neq 0$ and that $ad - bc \neq 0$. If $z \neq -\frac{d}{c}$ then $f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0$. Thus f is conformal at z. $\lim_{z \to -\frac{d}{c}} az + b = \frac{-ad+bc}{c} \neq 0$, while the limit on the bottom is 0, and so $\lim_{z \to -\frac{d}{c}} f(z) = \infty$. On the other hand, $\lim_{z \to \infty} f(z) = \frac{a}{c}$.

If we think of $f(-\frac{d}{c}) = \infty$ and $f(\infty) = \frac{a}{c}$ then a routine calculation shows that f is analytic on the extended complex plane and f is a bijection with inverse (another Möbius transformation)

$$f^{-1}(z) = \frac{1}{ad - bc} \frac{dz - b}{-cz + a}$$

5 Integration

5.1 Definition. A curve C with parametric equation $z(t) = x(t) + iy(t)$, $t \in [a, b]$ is called smooth if $z'(t)$ exists, $z(t) \neq 0$ for all $t \in [a, b]$, and $z'(t)$ is continuous in t.

Let G be a domain and $f : G \to \mathbb{C}$ be a function. Suppose that $C \subseteq G$ is a smooth curve. Define the integral of f in exactly the way you would expect. (I'm serious. This should be the fifth time you've seen this, so I'm not going to explain it again.)

5.2 Proposition. *The arclength $\lambda_{\mathscr{C}}$ of a smooth curve $\mathscr{C}$ is independent of parametrization.*

PROOF: Suppose that $\mathscr{C} = \{z_1(t) \mid y \in [a, b]\} = \{z_2(t) \mid y \in [c, d]\}$. We are assuming that there exists a differentiable function $f : [a, b] \to [c, d]$ so that $f(a) = c, f(b) = d$ and $f'(t) \neq 0$ for any $t \in [a, b]$. Then

$$\lambda_{\mathscr{C}} = \int_c^d |z_2'(s)| ds = \int_a^b |z_2'(f(s))| df(s) = \int_a^b |z_2'(f(s))| f'(s) ds = \int_a^b |z_2'(f(s)) f'(s)| ds = \int_a^b |z_1'(s)| ds$$

so $\lambda_{\mathscr{C}}$ is independent of the parametrization. □

Next we investigate some familiar rules of integration from multivariable calculus. If $\mathscr{C} = \{z(t) \mid t \in [a, b]\}$ is a curve, then we denote by $\{z((b + a) - t) \mid t \in [a, b]\}$ by $\mathscr{C}^-$. Note that the graphs of these curves coincide, they are merely traversed opposite directions.

5.3 Theorem. *Let f be a continuous function on a smooth curve $\mathscr{C}$. Then*

$$\int_{\mathscr{C}} f(z) dz = -\int_{\mathscr{C}^-} f(z) dz$$

5.4 Theorem. *Let f, g be continuous functions on a smooth curve $\mathscr{C}$. Let $a, b \in \mathbb{C}$. Then*

$$\int_{\mathscr{C}} (af + bg)(z) dz = a \int_{\mathscr{C}} f(z) dz + b \int_{\mathscr{C}} g(z) dz$$

5.5 Theorem. *Let $f : \mathscr{C} \to \mathbb{C}$ be a continuous function on a piecewise smooth curve $\mathscr{C}$. Suppose that $|f(z)| \le M$ for all $z \in \mathscr{C}$. Then*

$$\left| \int_{\mathscr{C}} f(z) dz \right| \le M \lambda_{\mathscr{C}}$$

Fix a smooth curve $\mathscr{C} \subseteq \mathbb{C}$. Let

$$\mathbb{A}_{\mathscr{C}} = \{f : \mathscr{C} \to \mathbb{C} \mid f \text{ is continuous}\}$$

It is clear that $\mathbb{A}_{\mathscr{C}}$ is a vectors space. Moreover, it is a ring under pointwise multiplication. Such objects are called algebras. Let

$$T : \mathbb{A}_{\mathscr{C}} \to \mathbb{C} : f \mapsto \int_{\mathscr{C}} f(z) dz$$

Then T is a linear functional. The set of all linear functionals is referred to as the algebraic dual, sometimes denoted $\mathbb{A}_{\mathscr{C}}^{\#}$. We can define a norm on $\mathbb{A}_{\mathscr{C}}$, $\|f\|_\infty = \sup\{|f(z)| : z \in \mathscr{C}\}$. Then T is bounded with respect to this norm. And so on...

6 Cauchy's Theorem

If $n \neq -1$ and $\mathscr{C}$ is a closed smooth curve in $\mathbb{C}$ (with $0 \notin \mathscr{C}$ if $n < 0$) then $\int_{\mathscr{C}} z^n dz = 0$. In particular, if $n \ge 0$ and $\mathscr{C} \in \mathbb{C}$ is smooth then $\int_{\mathscr{C}} p(z) dz = 0$ for all polynomials p. As we shall now see, this is just one example of a much more general phenomenon.

6.1 Cauchy's integral theorem

6.1 Definition. A triangular contour $\mathscr{C}$ is a union $\mathscr{C}_1 \cup \mathscr{C}_2 \cup \mathscr{C}_3$ of curves where

1. the final point of $\mathscr{C}_i$ is the initial point of $\mathscr{C}_{i+1}$
2. $\mathscr{C}_i$ is given by $\{z_i(t) \mid t \in [a_i, b_i]\}$ where $z_k(t)$ is linear in t

Note that $\mathscr{C}$ is automatically piecewise smooth.

6.2 Lemma. *Suppose that $\mathscr{G} \subseteq \mathbb{C}$ is a simply connected domain, $\Delta \subseteq \mathscr{G}$ is a triangular contour, and $f : \mathscr{G} \to \mathbb{C}$ is an analytic function. Then*

$$\int_{\Delta} f(z) dz = 0$$

The fact that $\mathscr{G}$ is simply connected implies that the interior of Δ is contained in $\mathscr{G}$, which is crucial to our argument.

PROOF: Suppose that $\Delta = ABC$. Bisect each side of Δ, say P bisects AB, Q bisects BC, and R bisects CA. By connecting P to Q, Q to R, and R to P, we obtain 4 congruent triangular contours, $\Delta_1, \Delta_2, \Delta_3, \Delta_4$, each of which has perimeter (i.e. arclength) equal to half that of Δ (i.e. $\lambda_{\Delta_i} = \frac{\lambda_\Delta}{2}$). Using the orientation inherited from Δ, we can complete the direction on $\Delta_1, \Delta_2, \Delta_3$ so that they again form triangular contours. Then

$$\begin{aligned}\int_\Delta f(z)dz &= \int_{ABC} f(z)dz \\ &= \int_{PBQ} f(z)dz - \int_{PQ} f(z)dz + \int_{APR} f(z)dz - \int_{PR} f(z)dz + \int_{RQC} f(z)dz - \int_{RQ} f(z)dz \\ &= \int_{\Delta_1} f(z)dz + \int_{\Delta_2} f(z)dz + \int_{\Delta_3} f(z)dz - \int_{PRQ} f(z)dz\end{aligned}$$

Orient Δ_4 in such a way that $-\int_{PRQ} f(z)dz = \int_{\Delta_4} f(z)dz$. Then

$$\int_\Delta f(z)dz = \sum_{i=1}^4 \int_{\Delta_i} f(z)dz$$

Let $M = |\int_\Delta f(z)dz|$. We want to show that $M = 0$. Now $M \le \sum_{i=1}^4 |\int_{\Delta_i} f(z)dz|$, thus a least one of the Δ_i's satisfies

$$\left|\int_{\Delta_i} f(z)dz\right| \ge \frac{M}{4}$$

Choose one of them and call it $\Delta^{(1)}$. We can iterate the procedure to obtain a nested sequence of triangular contours $\{\Delta^{(n)}\}_{n=1}^\infty$ such that

$$\left|\int_{\Delta^{(n)}} f(z)dz\right| \ge \frac{M}{4^n},\ n \ge 1$$

Using an argument similar to the principal of nested rectangles, we can show that there exists a unique point $z_0 \in \bigcup_{n=1}^\infty \Delta^{(n)} \cap \mathscr{G}$. f is analytic on $\mathscr{G}$, so $f'(z_0)$ exists and

$$\lim_{z\to z_0} \left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| = 0$$

Equivalently,

$$f(z) - f(z_0) - f'(z_0)(z - z_0) = \varepsilon(z)(z - z_0)$$

where $\lim_{z\to z_0} \varepsilon(z) = 0$. Moreover,

$$\begin{aligned}\int_{\Delta^{(n)}} f(z) - f(z_0) - f'(z_0)(z - z_0)dz &= \int_{\Delta^{(n)}} f(z)dz - (f(z_0) - z_0 f'(z_0)) \int_{\Delta^{(n)}} 1\,dz - f'(z_0)\int_{\Delta^{(n)}} z\,dz \\ &= \int_{\Delta^{(n)}} f(z)dz\end{aligned}$$

Thus $\frac{M}{4^n} \le |\int_{\Delta^{(n)}} f(z)dz| = |\int_{\Delta^{(n)}} \varepsilon(z)(z - z_0)dz|$. Let $\eta > 0$. Since $\lim_{z\to z_0} \varepsilon(z) = 0$, we can find $\delta > 0$ such that $0 < |z - z_0| < \delta$ implies that $|\varepsilon(z)| < \eta$. Moreover, since the arclength of $\Delta^{(n)}$ goes to zero as $n \to \infty$, we can find $N_0 > 0$ so that $n \ge N_0$ implies that $\lambda_{\Delta^{(n)}} < \delta$. So if $n \ge N_0$, then for $z \in \Delta^{(n)}$, $|z - z_0| < \lambda_{\Delta^{(n)}}$, and so

$$\frac{M}{4^n} \le \left|\int_{\Delta^{(n)}} \varepsilon(z)(z - z_0)dz\right| \le \eta\left(\frac{\lambda_\Delta}{2^n}\right)\lambda_{\Delta^{(n)}} = \frac{\eta\lambda_\Delta^2}{4^n}$$

Thus $M \le \eta\lambda_\Delta^2$, so $M = 0$ since η was arbitrary. Hence $\int_\Delta f(z)dz = 0$. □

6.3 Definition. A polygonal closed curve is just a closed curve $L = L_1 \cup \cdots \cup L_n$, where the final point of L_k is the initial point of L_{k+1} and each L_k is a linear curve (i.e. a line segment).

If $\mathscr{G} \subseteq \mathbb{C}$ is a simply connected domain, $f : \mathscr{G} \to \mathbb{C}$ is analytic, and $L \subseteq \mathscr{G}$ is a closed polygonal curve then $\int_L f(z)dz$ can be reduced to a sum of integrals over triangular contours (A.I. Markushevich). Because of this fact, we have

6.4 Corollary. *Suppose that $\mathscr{G} \subseteq \mathbb{C}$ is a simply connected domain. If $f : \mathscr{G} \to \mathbb{C}$ is analytic and $L \subseteq \mathscr{G}$ is a closed polygonal curve then*

$$\int_L f(z)dz = 0$$

Our next goal is to approximate smooth curves by polygonal curves. First we need the following lemma, proved on assignment 3.

6.5 Lemma. *Let $\mathscr{G} \subsetneq \mathbb{C}$ be a domain and $\mathscr{C} \subseteq \mathscr{G}$ be a continuous curve. Let $\rho = dist(\mathscr{C}, \partial \mathscr{G})$. Then*

1. *$\rho > 0$*
2. *if $D = \{w \in \mathbb{C} \mid |w - z| < \frac{\rho}{2}$ for some $z \in \mathscr{C}\}$ then D is a bounded domain and $\mathscr{C} \subseteq D \subseteq \overline{D} \subseteq \mathscr{G}$*

6.6 Theorem. *Let $\mathscr{G} \subseteq \mathbb{C}$ be a domain and $\mathscr{C} \subseteq \mathscr{G}$ be a piecewise smooth curve. Suppose that $f : \mathscr{G} \to \mathbb{C}$ is continuous. Given $\varepsilon > 0$, there exists a polygonal curve L so that*

1. *$L \subseteq \mathscr{G}$*
2. *L is inscribed in $\mathscr{C}$ (the vertices of L lie in $\mathscr{C}$ and the initial and final points of L are the same as the intial and final points of $\mathscr{C}$, respectively)*
3. *$\left| \int_{\mathscr{C}} f(z)dz - \int_L f(z)dz \right| < \varepsilon$*

PROOF: By the above lemma we can find a bounded domain D such that $\mathscr{C} \subseteq D \subseteq \overline{D} \subseteq \mathscr{G}$, and $\rho = \text{dist}(\mathscr{C}, \partial D) > 0$. Since f is continuous in $\mathscr{G}$ and $\overline{D}$ is compact, f is uniformly continuous on $\overline{D}$.

Let $\varepsilon > 0$ and let $\lambda_{\mathscr{C}}$ denote the arclength of $\mathscr{C}$. We can find $\delta > 0$ such that if $z, w \in \overline{D}$ and $|z - w| < \delta$ then $|f(z) - f(w)| < \frac{\varepsilon}{2\lambda_{\mathscr{C}}+1}$. Let $\gamma = \min\{\delta, \rho\}$. Choose a partition $\Pi = \{z_0, z_1, \ldots, z_m\}$ of $\mathscr{C}$ where the arclength $\lambda_{\mathscr{C}_i}$ of the subcurve $\mathscr{C}_i$ from z_{i-1} to z_i of $\mathscr{C}$ is less than γ for all $1 \leq i \leq m$. Let L be the polygonal curve obtained from Π by joining z_{i-1} to z_i for $1 \leq i \leq m$. Then it will satisfy the conclusions of the theorem.

1. Suppose $z \in L$. Then $z \in \overline{z_{i-1}z_i}$ for some i. Since $z_i \in \mathscr{C}$ and since

$$|z - z_i| \leq |z_{i-1} - z_i| < \gamma < \rho$$

   it follows that $z \in D$. Thus $L \subseteq D$.
2. This is clear.
3. Consider $S = \sum_{k=1}^{m} f(z_k)\Delta z_k$ which approximates the value of $\int_{\mathscr{C}} f(z)dz$. Now

$$\Delta z_k = z_k - z_{k-1} = \int_{\mathscr{C}_k} 1dz$$

and so $S = \sum_{k=1}^{m} \int_{\mathscr{C}_k} f(z_k)dz$. Hence

$$\begin{aligned}\left|\int_{\mathscr{C}} f(z)dz - S\right| &= \left|\sum_{k=1}^{n} \int_{\mathscr{C}_k} f(z) - f(z_k)dz\right| \\ &\leq \sum_{k=1}^{n} \left|\int_{\mathscr{C}_k} f(z) - f(z_k)dz\right| \\ &\leq \sum_{k=1}^{n} \lambda_{\mathscr{C}_k} \frac{\varepsilon}{2\lambda_{\mathscr{C}}+1} \\ &= \frac{\varepsilon}{2\lambda_{\mathscr{C}}+1} \sum_{k=1}^{n} \lambda_{\mathscr{C}_k} < \frac{\varepsilon}{2}\end{aligned}$$

since $\lambda_{\mathscr{C}_k} < \gamma$ implies that $|f(z) - f(z_k)| < \frac{\varepsilon}{2\lambda_{\mathscr{C}}+1}$ for all $z \in \mathscr{C}_k$. Similarily, we have

$$\left|\int_{L} f(z)dz - S\right| \leq \frac{\varepsilon}{2}$$

and so, by the triangle inequality,

$$\left|\int_{\mathscr{C}} f(z)dz - \int_{L} f(z)dz\right| < \varepsilon$$

□

As an imediate consequence of the above results, we get

6.7 Theorem. *(Cauchy's Integral Theorem) If $\mathscr{G} \subseteq \mathbb{C}$ is a simply connected domain, $f : \mathscr{G} \to \mathbb{C}$ is analytic and $\mathscr{C} \subseteq \mathscr{G}$ is a piecewise smooth closed curve, then*

$$\int_{\mathscr{C}} f(z)dz = 0$$

PROOF: Let $\varepsilon > 0$. By the above theorem, there exists a closed polygonal contour L inscribed in $\mathscr{C}$ such that

$$\left|\int_{L} f(z)dz - \int_{\mathscr{C}} f(z)dz\right| < \varepsilon$$

By a corollary above, $\int_L f(z)dz = 0$, so $|\int_{\mathscr{C}} f(z)dz| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, the result follows. □

6.8 Corollary. *Let $\mathscr{G} \subseteq \mathbb{C}$ be a simply connected domain and $f : \mathscr{G} \to \mathbb{C}$ be analytic. Suppose that $w_1, w_2 \in \mathscr{G}$ and let $\mathscr{C}_1, \mathscr{C}_2$ be two piecewise smooth curves in $\mathscr{G}$ with initial point w_1 and final point w_2. Then*

$$\int_{\mathscr{C}_1} f(z)dz = \int_{\mathscr{C}_2} f(z)dz$$

PROOF: Let $\mathscr{C}_2^-$ be the curve $\mathscr{C}_2$ traversed in the reverse direction. Then $\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2^-$ is a closed curve in $\mathscr{G}$. By Cauchy's Integral Theorem,

$$0 = \int_{\mathscr{C}} f(z)dz = \int_{\mathscr{C}_1} f(z)dz + \int_{\mathscr{C}_2^-} f(z)dz = \int_{\mathscr{C}_1} f(z)dz - \int_{\mathscr{C}_2} f(z)dz$$

□

Suppose that $\mathscr{C}_0, \dots, \mathscr{C}_n$ are $n+1$ piecewise smooth Jordan curves such that

1. $\mathscr{C}_i \cup \text{int}(\mathscr{C}_i) \subseteq \text{int}(\mathscr{C}_0)$ for $i = 1, \dots, n$
2. $\mathscr{C}_i \cup \text{int}(\mathscr{C}_i) \bigcap \mathscr{C}_j \cup \text{int}(\mathscr{C}_j) = \varnothing$ for $1 \le i < j \le n$

Then $D := \text{int}(\mathscr{C}_0) \cap (\bigcap_{i=1}^n \text{ext}(\mathscr{C}_i))$ is an $(n+1)$-connected domain. Suppose that f is a function which is analytic on $\overline{D}$. We claim that

$$\int_{\mathscr{C}_0} f(z)dz = \int_{\mathscr{C}_1} f(z)dz + \dots + \int_{\mathscr{C}_n} f(z)dz$$

Indeed, it can be seen that with these assumptions, we can find $n+1$ non-intersecting arcs $\gamma_1, \dots, \gamma_n$ which divide $\overline{D}$ into two closed regions $\overline{D_1}$ and $\overline{D_2}$ bounded by two closed piecewise smooth Jordan curves Γ_1 and Γ_2. Now f is analytic on $\overline{D}$ and therefore it is analytic on each of $\overline{D_1}$ and $\overline{D_2}$. But $\overline{D_i} = \Gamma_i \cup \text{int}(\Gamma_i)$, $i = 1, 2$, and since Γ_1 and Γ_2 are closed Jordan curves, their interiors are simply connected. So we can find simply connected domains $\mathscr{G}_i \supseteq \overline{D_i}$, $i = 1, 2$, so that f is analytic. Therefore

$$0 = \int_{\Gamma_1} f(z)dz = \int_{\Gamma_1} f(z)dz$$

by Cauchy's Integral Theorem. Hence

$$0 = \int_{\mathscr{C}_0} f(z)dz + \sum_{k=1}^{n} \int_{\mathscr{C}_k^-} f(z)dz$$

In particular, when there are only two curves $\mathscr{C}_0$ and $\mathscr{C}_1$ with $\text{int}(\mathscr{C}_1) \subseteq \text{int}(\mathscr{C}_0)$, then if $\mathscr{C}_0$, $\mathscr{C}_1$ are piecewise smooth Jordan curves and f is analytic on $\mathscr{C}_0 \cup \mathscr{C}_1 \cup (\text{int}(\mathscr{C}_0) \cap \text{ext}(\mathscr{C}_1))$ then $\int_{\mathscr{C}_0} f(z)dz = \int_{\mathscr{C}_1} f(z)dz$.

6.9 Example. 1. Suppose that $\mathscr{C}$ is a piecewise smooth Jordan curve and that $0 \in \text{ext}(\mathscr{C})$. Let $f(z) = \frac{1}{z}$. Then $f(z)$ is analytic everywhere except 0, so by assignment 3 we can find a simply connected domain $\mathscr{G}$ containing $\mathscr{C} \cup \text{int}(\mathscr{C})$ so that f is analytic on $\mathscr{G}$. By Cauchy's Integral Theorem, $\int_{\mathscr{C}} \frac{dz}{z} = 0$.

2. Now suppose that $0 \in \text{int}(\mathscr{C})$. Since $\text{int}(\mathscr{C})$ is open we can find some $r > 0$ such that $V_{2r}(0) \subseteq \text{int}(\mathscr{C})$. Let $\mathscr{C}_1 = \{re^{it} \mid 0 \le t \le 2\pi\}$. Since our function is analytic on $\mathscr{C} \cup \mathscr{C}_1 \cup (\text{int}(\mathscr{C}) \cap \text{ext}(\mathscr{C}_1))$, we see that $\int_{\mathscr{C}} \frac{dz}{z} = \int_{\mathscr{C}_1} \frac{dz}{z}$. Now for $z \in \mathscr{C}_1$, $dz = izdt$, and so $\int_{\mathscr{C}_1} \frac{dz}{z} = \int_0^{2\pi i} idz = 2\pi i$.
3. More generally, suppose that $\mathscr{C}$ is any piecewise smooth closed Jordan curve and $z_0 \in \text{int}(\mathscr{C})$. Given the substitution $w = z - z_0$, $dw = dz$, then $\int_{\mathscr{C}} \frac{dz}{z-z_0} = \int_{\mathscr{C}} \frac{dw}{w} = 2\pi i$.

6.2 Cauchy's integral formula

A key step in proved Cauchy's Integral Formula is provided by the next result, which acts like a complex version of the fundemental theorem of calculus

6.10 Theorem. *(Local Primitive Theorem) Suppose $\mathscr{G} \subseteq \mathbb{C}$ is a simply connected domain, $z_0 \in \mathscr{G}$, and $f : \mathscr{G} \to \mathbb{C}$ is analytic (see remark below). Then $F(z) = \int_{z_0}^{z} f(w)dw$, taken along any piecewise smooth curve from z_0 to z, defines a single-valued analytic function in $\mathscr{G}$ and $F'(z) = f(z)$ for all $z \in \mathscr{G}$.*

PROOF: Let $z \in \mathscr{G}$. The fact that $\int_{\mathscr{C}} f(w)dw = 0$ for all piecewise smooth closed curves in G implies that $\int_{\mathscr{C}_1} f(w)dw = \int_{\mathscr{C}_2} f(w)dw$ for all piecewise smooth curves $\mathscr{C}_1$ and $\mathscr{C}_2$ from z_0 to z. Let $R > 0$ so that $|h| < R$ implies that $z + h \in \mathscr{G}$. Let L_R be the line segment from z to $z + h$. Then

$$F(z+h) - F(z) = \int_{z_0}^{z+h} f(w)dw - \int_{z_0}^{z} f(w)dw = \int_{z}^{z+h} f(w)dw$$

Now $f(z) = \frac{f(z)}{h}\int_z^{z+h} dw = \frac{1}{h}\int_z^{z+h} f(z)dw$. Thus

$$\frac{F(z+h)-F(z)}{h} - f(z) = \frac{1}{h}\left(\int_z^{z+h} f(w) - f(z)dw\right)$$

Since f is analytic on $\mathscr{G}$, it is continuous at z. Thus, given $\varepsilon > 0$ we can find $0 < \delta < R$ so that $|w-z| < \delta$ implies $|f(w) - f(z)| < \varepsilon$. Since $\int_z^{z+h} f(w) - f(z)dw$ is independent of the curve from z to $z+h$, shall assume that we are integrating along the line segment from z to $z+h$. If $|h| < \delta$ and w lies on the line segment from z to $z+h$, then $|w-z| < \delta$, so we have

$$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| = \left|\frac{1}{h}\left(\int_z^{z+h} f(w) - f(z)dw\right)\right| \le \frac{1}{|h|}\varepsilon|h| = \varepsilon$$

Thus $F'(z) = \lim_{h\to 0} \frac{F(z+h)-F(z)}{h} = f(z)$. □

Remark. 1. Our proof of Cauchy's Integral Theorem used the fact that f was analytic on a simply connected domain $\mathscr{G}$ only where we needed $\int_\Delta f(z)dz = 0$ for all triangular contours Δ. If we only know that f is continuous on $\mathscr{G}$ but we assume that $\int_\Delta f(z)dz = 0$ for all such contours, then the proofs of the lemmas leading to Cauchy's Integral Theorem, and the theorem itself, did not require analycity. Thus we may also replace the assumption that f is analytic in the Local Primitive Theorem with the assumption that f is continuous and $\int_\Delta f(z)dz = 0$ for all triangular contours Δ. As we shall see below, this will imply that f is analytic on $\mathscr{G}$.

2. We refer to F above as the antiderivative or indefinite integral of f on G.

6.11 Proposition. *Let $\mathscr{G} \subseteq \mathbb{C}$ be a domain and suppose that $F_1, F_2 : \mathscr{G} \to \mathbb{C}$ are analytic with $F_1'(z) = F_2'(z)$ for all $z \in \mathscr{G}$. Then there exists $k \in \mathbb{C}$ such that $F_1(z) = F_2(z) + k$ for all $z \in \mathscr{G}$.*

PROOF: Let $F(z) = F_1(z) - F_2(z)$. Then F is analytic in $\mathscr{G}$ and we can write $F(x+iy) = u(x,y) + iv(x,y)$. Then

$$0 = F'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

So $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y}$. From mulitvariable calculus, it follows that there are $k_1, k_2 \in \mathbb{R}$ such that $u(x,y) = k_1$ and $v(x,y) = k_2$. Thus $F(z) = k_1 + ik_2$ for all $z \in \mathscr{G}$. □

We are now ready to prove one of the most important results in the course.

6.12 Theorem. *(Cauchy's Integral Formula) Let $\mathscr{G} \subseteq \mathbb{C}$ be a domain. Suppose that $\mathscr{C}$ is a piecewise smooth Jordan curve with int$(\mathscr{C}) \subseteq \mathscr{G}$. If $z_0 \in$ int$(\mathscr{C})$ and $f : \mathscr{G} \to \mathbb{C}$ is analytic then*

$$f(z_0) = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{f(z)}{z - z_0}dz$$

PROOF: If z_0 lies inside $\mathscr{C}$ then $\frac{f(z)}{z-z_0}$ is analytic on $\mathscr{G}$, except for z_0 itself. Let γ_R denote the circle of radius R centered at z_0, where $R > 0$ is chosen sufficiently small that $\gamma_R \subseteq \text{int}\mathscr{C}$. By an above corollary,

$$\int_{\mathscr{C}} \frac{f(z)}{z-z_0}dz = \int_{\gamma_R} \frac{f(z)}{z-z_0}dz$$

Now $\int_{\gamma_R} \frac{1}{z-z_0} dz = 2\pi i$, so there exists $\delta > 0$ such that for any R with $0 < R < \delta$ then

$$\left| \int_{\gamma_R} \frac{f(z)}{z-z_0} dz - 2\pi i f(z_0) \right| = \left| \int_{\gamma_R} \frac{f(z)}{z-z_0} dz - f(z_0) \int_{\gamma_R} \frac{1}{z-z_0} dz \right| = \left| \int_{\gamma_R} \frac{f(z)-f(z_0)}{z-z_0} dz \right|$$

Since f is continuous at z_0, given $\varepsilon > 0$ we can choose $0 < \delta' < \delta$ such that $|w - z_0| < \delta'$ implies that $|f(w) - f(z_0)| < \varepsilon$. Then if $0 < R < \delta'$

$$\left| \int_{\gamma_R} \frac{f(z)}{z-z_0} dz - 2\pi i f(z_0) \right| = \left| \int_{\gamma_R} \frac{f(z)-f(z_0)}{z-z_0} dz \right| \le \frac{\varepsilon}{R} 2\pi R = 2\pi\varepsilon$$

Since $\int_{\gamma_R} \frac{f(z)}{z-z_0} dz$ is independent of $0 < R < \delta'$, it (and thus $\int_{\mathscr{C}} \frac{f(z)}{z-z_0} dz$) is equal to $2\pi i f(z_0)$. □

6.13 Theorem. *Let $\mathscr{G} \subseteq \mathbb{C}$ be a domain and suppose $f : \mathscr{G} \to \mathbb{C}$ is analytic. Let $\mathscr{C}$ be any closed piecewise smooth Jordan curve in $\mathscr{G}$ and $z_0 \in int(\mathscr{C}) \subseteq \mathscr{G}$. For any $n \in \mathbb{N}$,*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\mathscr{C}} \frac{f(z)}{(z-z_0)^{n+1}} dz$$

PROOF: By induction on n. When $n = 0$ we have Cauch'y Integral Formula, proved above. Suppose $n > 1$ and the theorem holds for $n - 1$ in all possible cases. We must show that

$$\frac{f^{(n-1)}(z_0+h) - f^{(n-1)}}{h} \to \frac{n!}{2\pi i} \int_{\mathscr{C}} \frac{f(z)}{(z-z_0)^{n+1}} dz$$

as $h \to 0$. Take $d > 0$ so small that $V_\delta(z_0) \subseteq \mathrm{int}(\mathscr{C})$ and take γ_r, a circle of radius $r > 0$, such that $\gamma_r \subseteq V_\delta(z_0)$. By a theorem,

$$\frac{n!}{2\pi i} \int_{\mathscr{C}} \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{n!}{2\pi i} \int_{\gamma_r} \frac{f(z)}{(z-z_0)^{n+1}} dz$$

So we must check that

$$\frac{f^{(n-1)}(z_0+h) - f^{(n-1)}}{h} \to \frac{n!}{2\pi i} \int_{\mathscr{C}} \frac{f(z)}{(z-z_0)^{n+1}} dz$$

as $h \to 0$. Since $h \to 0$, we may assume that $|h| < r$ so that $z_0 + h$ is inside γ_r. Now by the induction hypothesis

$$\begin{aligned}
\frac{f^{(n-1)}(z_0+h) - f^{(n-1)}}{h} &= \frac{1}{h} \frac{(n-1)!}{2\pi i} \int_{\gamma_r} \frac{f(z)}{(z-z_0-h)^n} - \frac{f(z)}{(z-z_0)^n} dz \\
&= \frac{(n-1)!}{2h\pi i} \int_{\gamma_r} f(z) h \frac{(z-z_0)^{n-1} + (z-z_0)^{n-2}(z-z_0-h) + \cdots + (z-z_0-h)^{n-1}}{(z-z_0)^n (z-z_0-h)^n} \\
&= \frac{(n-1)!}{2\pi i} \int_{\gamma_r} f(z) \frac{(z-z_0)^n + (z-z_0)^{n-1}(z-z_0-h) + \cdots + (z-z_0)(z-z_0-h)^{n-1}}{(z-z_0)^{n+1}(z-z_0-h)^n}
\end{aligned}$$

Now $\frac{n!}{2\pi i} \int_{\gamma_r} \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{(n-1)!}{2\pi i} \int_{\gamma_r} \frac{f(z)n(z-z_0-h)^n}{(z-z_0)^{n+1}(z-z_0-h)^n} dz$ Subtracting these and applying the triangle inequality gives that the difference is less than or equal to

$$\frac{(n-1)!}{2\pi i} \max_{\gamma_r} |f| \int_{\gamma_r} \left| \frac{(z-z_0)^n + (z-z_0)^{n-1}(z-z_0-h) + \cdots + (z-z_0)(z-z_0-h)^{n-1} - n(z-z_0-h)^n}{(z-z_0)^{n+1}(z-z_0-h)^n} \right| dz$$

which goes to 0 as $h \to 0$. Thus the theorem is proved by mathematical induction. □

6.14 Theorem. *Let $\mathscr{G}$ be a simply connected domain and $f : \mathscr{G} \to \mathbb{C}$ continuous and such that $\int_{\mathscr{C}} f(z)dz = 0$ for all piecewise smooth closed Jordan curves in $\mathscr{G}$. Then f is analytic.*

PROOF: Pick any $z_0 \in \mathscr{G}$. For $z \in \mathscr{G}$, let $g(z) = \int_{z_0}^{z} f(w)dw$, taken along any path γ from z_0 to z. This is independent of the path chosen because $\int_{\mathscr{C}} f = 0$ for all closed $\mathscr{C}$. We know g is analytic on $\mathscr{G}$ and $g'(z) = f(z)$. By the above theorem, g'' exists, so f' exists over $\mathscr{G}$. □

6.15 Example. *(Weierstraß M-test)* Let $f_n : A \to \mathbb{C}$ and let $s_n = \sum_{k=1}^{n} f_k$. We say that $\sum_{k=1}^{\infty} f_k$ converges uniformly when s_n converges uniformly. This happens if $\|f_n\|_A \le M_n$ for all n, for some constant M_n, and $\sum_{k=1}^{\infty} M_n$ converges. In particular, if f_n is continuous for each n, then so is $\sum_{k=1}^{\infty} f_k$.

6.16 Proposition. *If $f_n, f : \mathscr{G} \to \mathbb{C}$, where $\mathscr{G}$ is simply connected and f_n is analytic and $f_n \to f$ uniformly on every compact subset K of $\mathscr{G}$, then f is analytic.*

PROOF: $\mathscr{G}$ is covered by the closed discs inside it, and each disk is compact. To get f continuous on $\mathscr{G}$ it suffices to show that f is continuous on every closed disc in $\mathscr{G}$. But $f_n \to f$ uniformly on such discs. Since each f_n is continuous, f is continuous. To see that f is analytic use Morera. We must prove that $\int_{\mathscr{C}} f(z)dz = 0$ for all closed curves in $\mathscr{G}$. We know that $\int_{\mathscr{C}} f_n(z)dz = 0$ for all curves $\mathscr{C}$, by Cauchy's Theorem. Since $\int_{\mathscr{C}} f_n \to \int_{\mathscr{C}} f$ by uniform convergence on compact sets, we have $\int_{\mathscr{C}} f = 0$. □

6.17 Example. $n^z = e^{z \log n}$ is analytic on the whole plane. Let $\mathscr{G} = \{z \mid \Re z > 1\}$. show that the series $\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges to an analytic function on $\mathscr{G}$. It is enough to show that $s_k := \sum_{n=1}^{k} \frac{1}{n^z}$ converges uniformly on compact subsets of $\mathscr{G}$. If $K \subseteq \mathscr{G}$ is compact, then there is $\varepsilon > 0$ such that $\Re z \ge 1 + \varepsilon$ for all $z \in K$. Then for all $z \in K$, $|n^z| \ge n^{1+\varepsilon}$. From CALCULUS 2, $\sum_{k=1}^{\infty} \frac{1}{n^{1+\varepsilon}}$ converges, so by the M-test, $\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges.

In multivariable calculus, Green's Theorem tells us that if $\mathscr{G} \subseteq \mathbb{R}^2$ is a simply connected domain and if P, Q are continuously differentiable on some domain $\mathscr{G}_1 \supseteq \overline{\mathscr{G}}$ then

$$\iint_{\mathscr{G}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = \int_{\partial \mathscr{G}} P\,dx + Q\,dy$$

We can identify $\mathscr{G}$ with the subset $\mathscr{G}_{\mathbb{C}} = \{x + iy \mid (x, y) \in \mathscr{G}\}$ of $\mathbb{C}$. Let $\mathscr{C} = \partial \mathscr{G}$. If $f : \mathscr{G}_{\mathbb{C}} \to \mathbb{C}$ is analytic, we can write $f(x + iy) = u(x, y) + iv(x, y)$ and

$$\int_{\mathbb{C}} f(z)dz = \int_{\mathscr{C}} u\,dx - v\,dy + i \int_{\mathscr{C}} v\,dx + u\,dy$$

Suppose that u and v are continuously differentiable. Then by Green's Theorem

$$\int_{\mathscr{C}} u\,dx - v\,dy = \iint_{\mathscr{G}} -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} dA = 0 \text{ and } \int_{\mathscr{C}} v\,dx + u\,dy = \iint_{\mathscr{G}} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} dA = 0$$

by the Cauchy-Riemann equations. Hence $\int_{\mathbb{C}} f(z)dz = 0$. The issue is that we have to assume that u and v are continuously differentiable. In our original proof we needed only that u and v are differentiable and satisfy the Cauchy-Riemann equations. Since we concluded that f' is differentiable, we are able to conclude that u and v are continuously differentiable.

6.3 Harmonic Functions

6.18 Definition. Let $G \subseteq \mathbb{R}^2$ be a domain and $u : \mathcal{G} \to \mathbb{R}$ be a function. If u has continuous second partial derivatives on $\mathcal{G}$ and they satisfy the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

everywhere in $\mathcal{G}$, then u is said to be harmonic. Suppose u, v are hormonic in $\mathcal{G}$ and further suppose that they satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at every point in $\mathcal{G}$. Then u and v are said to be conjugate harmonic functions.

Our interest in harmonic functions stems from the following two theorems

6.19 Theorem. *Let $\mathcal{G} \subset \mathbb{C}$ be a domain and $f(x+iy) = u(x,y) + iv(x,y)$ be a complex-valued function on $\mathcal{G}$. Then f is analytic on $\mathcal{G}$ if and only if u and v are conjugate harmonic functions.*

PROOF: Suppose that f is analytic. Then u and v are differentiable and satisfy the Cauchy-Riemann equations. f' is analytic and

$$f' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

Thus all of these functions are differentiable and satisfy the Cauchy-Riemann equations, which is to say

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2}$$

so u is harmonic, and similarily for v. Now since f'' is analytic, all of the second partials are continuous as well, so u and v are harmonic conjugates.

Conversely, suppose that u and v are harmonic conjugates. Then u and v have continuous second partials, so in particular they have continuous partials. This in turn implies that u and v are differentiable. Since u and v satisfy the Cauchy-Riemann equations it follows that f is analytic (by Green's Theorem). □

6.20 Theorem. *Let $\mathcal{G} \subset \mathbb{R}^2$ be a simply connected domain and $u : \mathcal{G} \to \mathbb{R}$ be a harmonic function. Then there exists an analytic function $f : \mathcal{G}_{\mathbb{C}} \to \mathbb{C}$ such that $u = \Re f$.*

PROOF: Consider $g = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$. We claim that g is analytic. Since u is harmonic it has continuous second partials, so in particular $\frac{\partial u}{\partial x}$ and $-\frac{\partial u}{\partial y}$ are differentiable. Furthermore,

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \text{ by Laplace's equation, and } \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right)$$

so g satisfies the Cauchy-Riemann equations. Hence g is analytic. By the Local Primitive theorem there exists an analytic function $\tilde{f} = \tilde{u} + i\tilde{v}$ so that $\tilde{f}' = g$. But

$$\frac{\partial \tilde{f}}{\partial x} = \frac{\partial \tilde{u}}{\partial x} + i\frac{\partial \tilde{v}}{\partial y} = g = \frac{\partial u}{\partial x} - i\frac{\partial v}{\partial y}$$

Thus $\frac{\partial \tilde{u}}{\partial x} = \frac{\partial u}{\partial x}$. Also, by the Cauchy-Riemann equations, $\frac{\partial \tilde{u}}{\partial y} = -\frac{\partial \tilde{v}}{\partial x} = \frac{\partial u}{\partial y}$. Thus $\tilde{u} - u = k$ is constant. That is, $f := \tilde{f} - k = u + iv$ is analytic and $\Re f = u$. □

Remark. The function f above is completely determined (up to a constant).

6.4 Winding Number

6.21 Definition. Let $\mathscr{C}$ be a piecewise smooth closed curve in $\mathbb{C}$. For $z_0 \notin \mathscr{C}$, we define the index or winding number of $\mathscr{C}$ about z_0 to be

$$\text{Ind}_{\mathscr{C}}(z_0) = \frac{1}{2\pi i} \int_{\mathscr{C}} \frac{1}{z - z_0} dz$$

6.22 Theorem. *The function $z_0 \mapsto \text{Ind}_{\mathscr{C}}(z_0)$ is a continuous, integer valued function on $\mathbb{C} \setminus \mathscr{C}$, so that $\text{Ind}_{\mathscr{C}}(\cdot)$ is constant on the connected components of $\mathbb{C} \setminus \mathscr{C}$. On the unbounded component of $\mathbb{C} \setminus \mathscr{C}$, $\text{Ind}_{\mathscr{C}}(\cdot) = 0$.*

PROOF: See text, or algebraic topology course notes. □

7 Series

Let $\sum_{n=0}^{\infty} z_n$ be an infinite series where each $z_n \in \mathbb{C}$, $n \geq 1$. As in the real case, we $s_k = \sum_{n=n}^{k} z_n$, $k \geq 1$ and refer to s_k as the k^{th} partial sum of the series.

7.1 Definition. If $\lim_{k\to\infty} s_k \in \mathbb{C}$ then we say that the series converges to $s := \lim_{k\to\infty} s_k$ and write $s = \sum_{n=n}^{\infty} z_n$. Otherwise, we say that the series diverges. If $\lim_{k\to\infty} s_k = \infty$, we say the series propositionerly diverges (note that such a series will converge on the Riemann sphere). If $\lim_{k\to\infty} s_k$ does not exist, we say that the series is oscillatory.

Since $\mathbb{R} \subseteq \mathbb{C}$, every real series is a complex series.

7.2 Definition. As in the real case, we say that a series $\sum_{n=0}^{\infty} z_n$ converges absolutely if $\sum_{n=0}^{\infty} |z_n|$ converges. If $\sum_{n=0}^{\infty} z_n$ converges but is not absolutely convergent then we say that $\sum_{n=0}^{\infty} z_n$ is conditionally convergent.

The following are easy adaptations of the corollaryresponding results for real series.

7.3 Lemma. *Let $\sum_{n=0}^{\infty} z_n$ be a complex series*

1. *If $\sum_{n=0}^{\infty} z_n$ converges then $\lim_{n\to\infty} z_n = 0$.*
2. *If $\sum_{n=0}^{\infty} |z_n|$ converges then $\sum_{n=0}^{\infty} z_n$ converges.*
3. *If $\sum_{n=0}^{\infty} z_n = a$ and $\sum_{n=0}^{\infty} w_n = b$ then $\sum_{n=0}^{\infty}(z_n + w_n) = a + b$.*
4. *If $\sum_{n=0}^{\infty} z_n$ is absolutely convergent and if $\sigma : \mathbb{N} \to \mathbb{N}$ is a bijection then $\sum_{n=0}^{\infty} z_{\sigma(n)}$ is absolutely convergent, with the same limit.*
5. *If $\sum_{n=0}^{\infty} z_n = a$ and $\sum_{n=0}^{\infty} w_n = b$ are absolutely convergent series then $\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} z_j w_{k-j}$ converges absolutely to ab.*

7.4 Lemma. *(The Comparison Test) If $|z_n| \leq r_n$, $n \geq 1$ and if $\sum_{n=0}^{\infty} r_n$ converges then $\sum_{n=0}^{\infty} z_n$ is absolutely convergent.*

7.1 Uniform Convergence

7.5 Definition. Let $\sum_{n=0}^{\infty} f_n$ be a series, each of whose terms is a function defined on a subset $E \subseteq \mathbb{C}$. For each $n \geq 1$, let $s_k = \sum_{n=0}^{k} f_n$ be the k^{th} partial sum of the series. Suppose that $s(z) = \lim_{k\to\infty} s_k(z)$ exists for all $z \in E$. Then $\sum_{n=0}^{\infty} f_n$ is said to be uniformly convergent on E if $\forall\, \varepsilon > 0$, there exists $N > 0$ so that $k \geq N$ implies $|s(z) - s_k(z)| < \varepsilon$ for all $z \in E$.

7.6 Theorem. *If the series $\sum_{n=0}^{\infty} f_n$ is uniformly convergent on a set E and if each f_n is continuous at some fixed point $z_0 \in E$, then $\sum_{n=0}^{\infty} f_n$ is also continuous at z_0.*

PROOF: Assignment 5. □

7.7 Theorem. *(Weierstraß M-test) Let $\sum_{n=0}^{\infty} f_n$ be a series and suppose $|f_n(z)| \leq a_n$ for all $z \in E$ and $n \geq 1$, where $\sum_{n=0}^{\infty} a_n < \infty$. Then $\sum_{n=0}^{\infty} f_n$ converges uniformly (and pointwise absolutely) on E.*

PROOF: That $\sum_{n=0}^{\infty} |f_n(z)|$ converges follows from the comparison test. To see that the series converges uniformly on E, let $\varepsilon > 0$ and choose $N > 0$ so that $n \geq N$ implies that $\sum_{k=n+1}^{\infty} a_k < \varepsilon$. Then for any $z \in E$, if we denote $s(z) = \sum_{n=0}^{\infty} f_n(z)$, we have

$$|s(z) - \sum_{k=0}^{n} f_k(z)| = |\sum_{k=n+1}^{\infty} f_k(z)| \leq \sum_{k=n+1}^{\infty} |f_k(z)| \leq \sum_{k=n+1}^{\infty} a_k < \varepsilon$$

Thus the series converges uniformly on E. □

7.8 Example. The series $f_1(z) = z$ and $f_n(z) = z^n - z^{n-1}$ for $n > 1$ does not converge uniformly on $\mathbb{D}$, even though it converges pointwise for each point in $\mathbb{D}$. Fix $0 < r < 1$ and suppose $z \in \overline{r\mathbb{D}}$. Then for $n \geq 1$ we have $|f_n(z)| \leq r^{n-1}(r+1)$. Thus

$$\sum_{n=1}^{\infty} |f_n(z)| \leq \sum_{n=0}^{\infty} r^{n-1}(r+1) = \frac{r+1}{1-r} < \infty$$

By the last theorem, $\sum_{n=1}^{\infty} f_n$ converges uniformly (and pointwise absolutely) on $\overline{r\mathbb{D}}$. Thus the series converges uniformly on any compact subset of $\mathbb{D}$. This is refered to as UCC (Uniform Convergence on Compacta).

7.2 Integration of Series

Clearly, if $\mathscr{C} \subseteq \mathbb{C}$ is a piecewise smooth curve and if $f_1, \ldots, f_n : \mathscr{C} \to \mathbb{C}$ are continuous, then

$$\int_{\mathscr{C}} \sum_{k=0}^{n} f_k(z)\, dz = \sum_{k=0}^{n} \int_{\mathscr{C}} f_k(z)\, dz$$

We extend this to infinite series as follows.

7.9 Theorem. *Let $\mathscr{C} \subseteq \mathbb{C}$ be a piecewise smooth curve. Suppose that $(f_n)_{n=0}^{\infty}$ is a sequence of functions from $\mathscr{C}$ to $\mathbb{C}$, each of which is continuous on $\mathscr{C}$. If $\sum_{n=0}^{\infty} f_n$ converges uniformly on $\mathscr{C}$, then*

$$\int_{\mathscr{C}} \sum_{n=0}^{\infty} f_n(z)\, dz = \sum_{n=0}^{\infty} \int_{\mathscr{C}} f_n(z)\, dz$$

PROOF: From assignment 5, we see that $\sum_{n=0}^{\infty} f_n$ is continuous on $\mathscr{C}$, and therefore it is integrable there. Let $s_n(z) = \sum_{k=0}^{n} f_k(z)$ for $z \in \mathscr{C}$ (s_n is clearly continuous). Let $\varepsilon > 0$ and choose $N > 0$ so that $n \geq N$ implies

$$\left| \sum_{k=0}^{\infty} f_k(z) - \sum_{k=0}^{n} f_k(z) \right| < \frac{\varepsilon}{\lambda_{\mathscr{C}} + 1}$$

for all $z \in \mathscr{C}$. Then for $n \geq N$,

$$\left|\int_{\mathscr{C}} \sum_{k=0}^{\infty} f_k(z)\,dz - \int_{\mathscr{C}} \sum_{k=0}^{n} f_k(z)\,dz\right| = \left|\int_{\mathscr{C}} \left(\sum_{k=0}^{\infty} f_k(z) - \sum_{k=0}^{n} f_k(z)\right) dz\right|$$
$$\leq \frac{\varepsilon}{\lambda_{\mathscr{C}}+1}\lambda_{\mathscr{C}} < \varepsilon$$

This says that $\int_{\mathscr{C}} \sum_{k=0}^{\infty} f_k(z)\,dz = \sum_{k=0}^{\infty} \int_{\mathscr{C}} f_k(z)\,dz$. □

7.3 Differentiation of Series

7.10 Lemma. *Suppose that $\sum_{n=0}^{\infty} f_n$ is a uniformly convergent series with sum $s(z) = \sum_{n=0}^{\infty} f_n(z)$, $z \in E \subseteq \mathbb{C}$. Let $\varphi : E \to \mathbb{C}$ be a bounded function. Then $\sum_{n=0}^{\infty} \varphi(z) f_n(z)$ is uniformly convergent with sum $\varphi(z)s(z)$ for $z \in E$.*

PROOF: Trivial. □

7.11 Theorem. *(Weierstraß) Let $\mathscr{G} \subseteq \mathbb{C}$ be a domain. For each $n \geq 1$, suppose that $f_n : \mathscr{G} \to \mathbb{C}$ is analytic. Suppose $s(z) = \sum_{n=0}^{\infty} f_n(z)$ exists for all $z \in \mathscr{G}$ and that the series is uniformly convergent on compact subsets of $\mathscr{G}$ (i.e. the series is UCC on $\mathscr{G}$). Then $s(z)$ is analytic on $\mathscr{G}$. Moreover, the series is infinitely differentiable term by term, so that*

$$s^{(k)}(z) = \sum_{n=0}^{\infty} f_n^{(k)}(z)$$

for all $k \geq 1$, for all $z \in \mathscr{G}$, and the convergence of each differentiated series is also uniform on compact subsets of $\mathscr{G}$.

PROOF: Let $z_0 \in \mathscr{G}$ and choose $r > 0$ small enough so that $\overline{V_r(z_0)} \subseteq \mathscr{G}$. Since $\overline{V_r(z_0)}$ is closed and bounded (i.e. compact) the series $\sum_{n=0}^{\infty} f_n^{(k)}(z)$ converges uniformly on $\overline{V_r(z_0)}$. In particular, it converges uniformly on $\gamma_r := \partial \overline{V_r(z_0)}$. If we let $\varphi_k(z) = \frac{1}{2\pi i} \frac{k!}{(z-z_0)^{k+1}}$, then for $z \in \gamma_r$, $|\varphi_k(z)| \leq \frac{k!}{2\pi r^{k+1}}$, a fixed constant. By the last lemma,

$$\frac{k!}{2\pi i} \frac{s(z)}{(z-z_0)^{k+1}} = \sum_{n=0}^{\infty} \frac{k!}{2\pi i} \frac{f_n(z)}{(z-z_0)^{k+1}}$$

converges uniformly on γ_r. By the last section, we may integrate term by term, so for each $k \geq 0$

$$\frac{k!}{2\pi i} \int_{\gamma_r} \frac{s(z)}{(z-z_0)^{k+1}} dz = \sum_{n=0}^{\infty} \frac{k!}{2\pi i} \int_{\gamma_r} \frac{f_n(z)}{(z-z_0)^{k+1}} dz$$

When $k = 0$

$$\frac{1}{2\pi i} \int_{\gamma_r} \frac{s(z)}{(z-z_0)} dz = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{\gamma_r} \frac{f_n(z)}{(z-z_0)} dz = \sum_{n=0}^{\infty} f_n(z_0) = s(z_0)$$

Thus $s(\cdot)$ satisfies Cauchy's Integral formula. It now follows from the proof of Professor Zorzitto's "infinite differentiability theorem" that $s(\cdot)$ is analytic and thus $s(\cdot)$ is infinitely differentiable. (Note: analyticity of the function in the statement of that theorem was only required to establish that the original function satisfied Cauchy's Integral formula.) With this in mind, we have for $k \geq 1$

$$s^{(k)}(z_0) = \sum_{n=0}^{\infty} f_n^{(k)}(z_0)$$

It remains to show that this series converges uniformly on compact sets. Let $L \subseteq \mathscr{G}$ be compact and $\varepsilon > 0$. Fix $z_0 \in L$ and choose $r = r(z_0) > 0$ so that $V_{2r}(z_0) \subseteq \mathscr{G}$. Then $\gamma_r(z_0) \subseteq V_{2r}(z_0) \subseteq \mathscr{G}$. Moreover, $\overline{V_{\frac{r}{2}}(z_0)} \subseteq \mathscr{G}$. Since the original series converges uniformly on compact sets and since $\gamma_r(z_0)$ is compact we can find some $N = N(z_0, \varepsilon)$ so that $n \geq N$ implies $|s(w) - s_n(w)| < \varepsilon$ for all $w \in \gamma_r(z_0)$. Then $n \geq N$ implies for any $z \in \overline{V_{\frac{r}{2}}(z_0)}$

$$\left|\frac{k!}{2\pi i}\int_{\gamma_r(z_0)} \frac{s(w)}{(w-z)^{k+1}}dw - \sum_{j=0}^{n} \frac{k!}{2\pi i}\int_{\gamma_r(z_0)} \frac{f_j(w)}{(w-z)^{k+1}}dw\right| = \left|\frac{k!}{2\pi i}\int_{\gamma_r(z_0)} \frac{s(w)-s_n(w)}{(w-z)^{k+1}}dw\right| \leq \frac{k!}{2\pi}2\pi r\frac{\varepsilon}{\left(\frac{r}{2}\right)^{k+1}}$$

since $\varepsilon > 0$ is arbitrary, this shows that $\sum_{n=0}^{\infty} f_n^{(k)}(z)$ converges uniformly to $s^{(k)}(z)$ for $z \in \overline{V_{\frac{r}{2}}(z_0)}$. Applying compactness yields the result. □

8 Power Series

We now consider a special class of series of functions known as power series. Let $a \in \mathbb{C}$. Suppose that $f_n(z) = c_n(z-a)^n$ where $c_n \in \mathbb{C}$ and $n \geq 0$. Then $\sum_{n=0}^{\infty} f_n(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ is called a power series at a. The region of convergence of the series is the set of all points where the series converges.

8.1 Example.

1. Let $a = 0$ and $c_n = n^n$ for $n \geq 1$ and $c_0 = 1$. The series $1 + z + 4z^2 + 27z^3 + \cdots$ converges only at the point $z = 0$. Indeed, if $z \neq 0$ then there is n_0 such that $n \geq n_0$ implies that $|n^n z^n| \geq 1$. Therefore the terms of the series do not tend to zero, so the series cannot converge.
2. Let $a = 0$, $c_0 = 1$ and $c_n = \frac{1}{n^n}$ for $n \geq 1$. The series $1 + z + \frac{z^2}{4} + \frac{z^3}{27} + \cdots$ converges for every point of $\mathbb{C}$ by the comparison test.

8.2 Lemma. *Let $a \in \mathbb{C}$ and $\sum_{n=0}^{\infty} c_n(z-a)^n$ be a power series. Suppose the series converges at some point $z_0 \neq a$. If $0 < |z_1 - a| < |z_0 - a|$, then the series converges absolutely at z_1.*

PROOF: Exercise. This is a trivial consequence of the Comparison Test. □

8.3 Proposition. *If $\sum_{n=0}^{\infty} c_n(z-a)^n$ converges for some $z_0 \neq a$ but diverges for some $z_1 \neq a$ then there is some $R > 0$ such that $|z-a| < R$ implies that $\sum_{n=0}^{\infty} c_n(z-a)^n$ converges absolutely and $|z-a| > R$ implies that $\sum_{n=0}^{\infty} c_n(z-a)^n$ diverges.*

8.1 Radius of Convergence

PROOF: Let $R = \sup\{r > 0 \mid \sum_{n=0}^{\infty} c_n(z-a)^n$ converges absolutely for all $|z-a| < r\}$. By the lemma above, $|z_0 - a| \leq R \leq |z_1 - a|$. Clearly the series converges for $|z-a| < R$. If $|z_2 - a| > R$ and $\sum_{n=0}^{\infty} c_n(z_2 - a)^n$ were to converge then by the lemma above we would have $R > |z_2 - a|$, a contradiction. □

We refer to the R above as the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n(z-a)^n$ and the circle $\gamma_R = \{z \in \mathbb{C} : |z-a| = R\}$ as the circle of convergence. Whether of not the series converges for points on the circle depends upon the series itself. If the series $\sum_{n=0}^{\infty} c_n(z-a)^n$ converges for all $z \in \mathbb{C}$ we say that $R = \infty$. If the series diverges for all $z \neq a$ then we say that $R = 0$.

8.4 Example. Consider the power series $\sum_{n=0}^{\infty} z^n$. This is a geometric series, as such, if $|z| < 1$ then the series converges to $\frac{1}{1-z}$. If $|z| \geq 1$ then the series diverges.

Note however that the series does not converge uniformly on $\mathbb{D}$. Indeed, suppose otherwise. Let $\varepsilon > 0$ and choose $N > 0$ such that $n \geq N$ implies

$$\left| \frac{1}{1-z} - \sum_{k=0}^{n} z^k \right| < \varepsilon \text{ for all } z \in \mathbb{D}$$

But $z \in \mathbb{D}$ implies

$$\left| \sum_{k=0}^{n} z^k \right| \leq \sum_{k=0}^{n} |z^k| \leq n+1$$

whereas $\lim_{\substack{z \to 1 \\ z \in \mathbb{D}}} \frac{1}{1-z} = \infty$. On the other hand, if $0 < r < 1$ we can choose $N_0 > 0$ such that $\sum_{N_0+1} r^n < \varepsilon$. If $|z| \leq r$ and $n \geq N_0$,

$$\left| \frac{1}{1-z} - \sum_{k=0}^{n} z^k \right| = \left| \sum_{k=0}^{\infty} z^k \sum_{k=0}^{n} z^k \right| \leq \sum_{k=n+1}^{\infty} |z|^k \leq \sum_{k=N_0+1}^{\infty} r^k < \varepsilon$$

Thus $\sum_{k=0}^{\infty} z^k$ converges uniformly on all disks $\overline{r\mathbb{D}}$, and hence on all compact subsets of $\mathbb{D}$.

It turns out that this behavior is typical power series.

8.5 Theorem. *If $\sum_{n=0}^{\infty} c_n(z-a)^n$ has radius of convergence $R > 0$ then the series converges uniformly on every compact subset of $V_R(a)$.*

PROOF: Assignment. □

8.6 Corollary. *If $\sum_{n=0}^{\infty} c_n(z-a)^n$ has radius of convergence $R > 0$ then the sum $s(z)$ is continuous for all $|z-a| < R$.*

In fact, much more is true. Notice that for each $n \geq 0$ then $f_n(z) = c_n(z-a)^n$ is analytic on $\mathbb{C}$.

8.7 Theorem. *Let $s(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ be a power series with radius of convergence $R > 0$. Then $s(z)$ is analytic on $V_R(a)$. Moreover, the series can be differentiated term by term to get $s'(z) = \sum_{n=1}^{\infty} nc_n(z-a)^{n-1}$ and this series has the same radius of convergence R.*

PROOF: As in the previous result, if $0 < r < R$ then by Theorem 8.5 the series converges uniformly on $\overline{V_r(a)}$ and hence on every compact subset of $V_R(a)$. By Weierstraß's Theorem we find that $s(z)$ is analytic on $V_R(a)$ and $s'(z) = \sum_{n=1}^{\infty} nc_n(z-a)^{n-1}$ converges uniformly on compact subsets of $V_R(a)$.

If R' is the radius of convergence of $s'(z)$, it follows that $R' \geq R$. Suppose that $R' > R$. Let $R < |z_0 - a| < R'$ and let $\mathscr{C}$ be the line segment connecting a to z_0. Then $\mathscr{C}$ is a compact subset of $V_{R'}(a)$, and so the series $s'(z)$ converges uniformly on $\mathscr{C}$. We may integrate $s'(z)$ term by term along $\mathscr{C}$ to yield

$$\int_{\mathscr{C}} \sum_{n=1}^{\infty} nc_n(z-a)^{n-1} dz = \sum_{n=1}^{\infty} \int_{\mathscr{C}} nc_n(z-a)^{n-1} dz = \sum_{n=1}^{\infty} c_n(z_0-a)^n$$

Thus this series converges and hence so does $\sum_{n=0}^{\infty} c_n(z_0-a)^n$, contradicting that $R < |z_0 - a|$. Hence $R' = R$. □

We can now attack the problem of finding the radius of convergence of a given power series.

8.8 Theorem. *(Cauchy-Hadamard) If $\sum_{n=0}^{\infty} c_n(z-a)^n$ is a power series and if $Q := \overline{\lim}(|c_n|^{\frac{1}{n}})$ then the radius of convergence of the series is $R = \frac{1}{Q}$ ($R = 0$ if $Q = \infty$ and $R = \infty$ if $Q = 0$).*

PROOF: We consider 3 cases.

Case 1: $Q = \infty$. In this case, the sequence $(|c_n|^{\frac{1}{n}})_{n=0}^{\infty}$ is unbounded. We wish to show that the series converges only at $z = a$. Suppose otherwise, that $\sum_{n=0}^{\infty} c_n(z_0 - a)^n$ converges for some $z_0 \neq a$. Then $\lim_{n\to\infty} c_n(z_0 - a)^n = 0$, and so in particular there exists an $M > 1$ such that

$$|c_n(z_0 - a)^n| < M \ \forall\, n \geq 0$$

But then

$$|c_n|^{\frac{1}{n}} \leq \frac{M^{\frac{1}{n}}}{|z_0 - a|} \leq \frac{M}{|z_0 - a|}$$

a contradiction. Hence the series converges only if $z = a$, so $R = 0$.

Case 2: $Q = 0$. In this case we wish to prove that $\sum_{n=0}^{\infty} c_n(z - a)^n$ for all $z \in \mathbb{C}$. Since $|c_n|^{\frac{1}{n}} \geq 0$ for all $n \geq 0$ and $\overline{\lim}_{n\to\infty} |c_n|^{\frac{1}{n}} = 0$, we must have $\lim_{n\to\infty} |c_n|^{\frac{1}{n}} = 0$. Fix $z_0 \in \mathbb{C}$ and choose $\varepsilon > 0$ such that $\varepsilon|z_0 - a| < \frac{1}{2}$. Choose $N = N(\varepsilon)$ so that $|c_n|^{\frac{1}{n}} < \varepsilon$ for all $n \geq N$. Then if $n \geq N$,

$$|c_n(z_0 - a)^n| \leq \left(\varepsilon|z_0 - a|\right)^n < \left(\frac{1}{2}\right)^n$$

By the comparison test, it follows that $\sum_{n=0}^{\infty} c_n(z - a)^n$ converges absolutely. Since $z_0 \in \mathbb{C}$ was arbitrary, $R = \infty$.

Case 3: $0 < Q < \infty$. Our goal now is to show that if $|z_0 - a| < \frac{1}{Q}$ then $\sum_{n=0}^{\infty} c_n(z - a)^n$ converges, whereas if $|z_0 - a| > \frac{1}{Q}$ then $\sum_{n=0}^{\infty} c_n(z - a)^n$ diverges. We know that Q is the largest limit point of $(|c_n|^{\frac{1}{n}})_{n=0}^{\infty}$. Thus, given $\varepsilon > 0$, we can find $N > 0$ such that $n \geq N$ implies that

$$|c_n|^{\frac{1}{n}} < Q + \varepsilon$$

If $|z_0 - a| < \frac{1}{Q}$ then $Q|z_0 - a| < 1$, so let

$$\varepsilon = \frac{1 - Q|z_0 - a|}{2|z_0 - a|}$$

Then for appropositionriate $N > 0$ we have that if $n \geq N$ then

$$\begin{aligned} |c_n|^{\frac{1}{n}} &< Q + \frac{1 - Q|z_0 - a|}{2|z_0 - a|} \\ &= \frac{1 + Q|z_0 - a|}{2|z_0 - a|} \\ |c_n|^{\frac{1}{n}}|z_0 - a| &< \frac{1 + Q|z_0 - a|}{2} =: q < 1 \end{aligned}$$

Again by the comparison test, $\sum_{n=0}^{\infty} c_n(z - a)^n$ converges.

Conversely, for $|z_0 - a| > \frac{1}{Q}$ and any $\varepsilon > 0$ we can find infinitely many n's such that

$$|c_n|^{\frac{1}{n}} > Q - \varepsilon$$

If $|z_0 - a| > \frac{1}{Q}$ then $Q|z_0 - a| > 1$, so let

$$\varepsilon = \frac{Q|z_0 - a| - 1}{|z_0 - a|} > 0$$

Then we have

$$|c_n|^{\frac{1}{n}} > Q - \frac{Q|z_0 - a| - 1}{|z_0 - a|} = \frac{1}{|z_0 - a|}$$

for infinitely many n's. Since the terms of $\sum_{n=0}^{\infty} c_n(z-a)^n$ do not go to zero, the series diverges. □

Remark. As previously noted, the behavior of a series on the circle of convergence depends upon the coefficients of the series. For example, let $f_1(z) = \sum_{n=0}^{\infty} z^n$, $f_2(z) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n}$, and $f_3(z) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n^2}$. Then they all have radius of convergence $R = 1$. On the other hand, f_1 diverges for every point on the circle of convergence, as the terms of the series don't go to zero. f_2 diverges at $z = 1$ but converges at $z = -1$, and in fact converges for all points on the circle of convergence except for $z = 1$. Finally, f_3 converges absolutely at every point on the circle of convergence.

9 Taylor Series

Let $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ be a power series with radius of convergence $R > 0$. As we saw in the last section, we may differentiate f and get a power series with the same radius of convergence. Hence for any $k \geq 0$

$$f^{(k)} = \sum_{n=0}^{\infty} (k+n)_k c_{k+n}(z-a)^n$$

By setting $z = a$ we see that

$$c_k = \frac{f^{(k)}(a)}{k!}$$

If a power series is related to an analytic function f by the equation above, we say that the series is a Taylor series for f at the point a. We refer to the c_n's as the Taylor coefficients. It is clear that if f is given by a power series with radius of convergence $R > 0$ then it is it's own Taylor series.

With f as above and $0 < r < R$, let γ_r be the circle of radius r centred at a (traversed once in the positive direction). By the Cauchy Integral formula,

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\gamma_r} \frac{f(z)}{(z-a)^{n+1}} dz$$

Combining this with the above equation gives us

$$c_n = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{(z-a)^{n+1}} dz$$

Suppose that $|f(z)| \leq M$ for all $|z - a| < R$. Then we obtain Cauchy's Inequalities

$$|c_n| \leq \frac{1}{2\pi} (2\pi r) \frac{M}{r^{n+1}} = \frac{M}{r^n}$$

This is true for each $r < R$, and therefore $|c_n| \leq \frac{M}{R^n}$.

We have seen that a power series with radius of convergence $R > 0$ converges to an analytic function $s(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ in the disk $|z - a| < R$. The following result may be considered a converse to this.

9.1 Theorem. *Suppose that f is analytic on $V_R(a)$. Then f admits a Taylor series expansion in $V_R(a)$, which is to say that the power series $\sum_{n=1}^{\infty} c_n(z-a)^n$ with coefficients*

$$c_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{(z-a)^{n+1}}dz$$

converges to $f(z)$ for all $z \in V_R(a)$.

PROOF: Let $x \in V_R(a)$ and with $0 < |w-a| < r < R$, let γ_r denote the circle of radius r centred at a. Then $w \in \mathrm{int}\gamma_r$. By Cauchy's Integral formula

$$f(w) = \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(z)}{z-w}dz$$

To obtain a power series from this, first observe that

$$\frac{1}{z-w} = \frac{1}{(z-a)-(w-a)} = \frac{1}{z-a}\frac{1}{1-\frac{w-a}{z-a}}$$

Notice that $\alpha := \left|\frac{w-a}{z-a}\right| < 1$ for all $z \in \gamma_r$. Hence

$$\frac{1}{z-w} = \frac{1}{z-a}\frac{1}{1-\frac{w-a}{z-a}} = \sum_{n=0}^{\infty}\frac{(w-a)^n}{(z-a)^{n+1}}$$

It follows that

$$\frac{1}{2\pi i}\frac{f(z)}{z-w} = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{f(z)}{(z-a)^{n+1}}(w-a)^n$$

Since $\alpha < 1$, $\sum_{n=0}^{\infty}\frac{1}{2\pi i}\frac{M\alpha^n}{r} < \infty$ and so the series above is uniformly convergent on γ_r by the Weiestraß M-test. Thus we can integrate along γ_r term by term. This gives

$$\begin{aligned}
f(w) &= \frac{1}{2\pi i}\int_{\gamma_r}\frac{f(z)}{z-w}dz \\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_{\gamma_r}\frac{f(z)}{(z-a)^{n+1}}dz\,(w-a)^n \\
&= \sum_{n=0}^{\infty}c_n(w-a)^n
\end{aligned}$$

Thus the series converges to f on $V_R(a)$. □

9.2 Definition. If a function f is analytic at a point $z_0 \in \mathbb{C}$ (which is to say that f is differentiable in some neighbourhood $V_\delta(z_0)$) then we say that z_0 is a regular point of f. Otherwise we say that z_0 is a singular point (or singularity) of f.

9.3 Theorem. *Suppose that f has a Taylor expansion $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ at the point a, with radius of convergence $R > 0$. If f is defined on $\gamma_R(a)$ then f has at least one singularity there.*

PROOF: Suppose that f has no singularities on $\gamma_R(a)$. For each $a \in \gamma_R(a)$ we can find an open disk $V_{\delta_z}(z)$ on which f is analytic. Since $\gamma_R(a)$ is compact, we can find finitely many $z_1, \dots, z_n \in \gamma_R(a)$ such that

$$\gamma_R(a) \subseteq \bigcup_{k=1}^{n} V_{\delta_{z_k}}(z_k) =: \mathcal{G}$$

Let $\rho := \operatorname{dist}(\gamma_R(a), \partial\mathcal{G}) > 0$. Then f is analytic inside $\gamma_{R+\frac{\rho}{2}}(a)$ and so has a convergent Taylor series inside $\gamma_{R+\frac{\rho}{2}}(a)$. But the coefficients of this series coincide with the coefficients of the original series since they (effectively) depend only on a. This contradicts the fact that the radius of convergence of the series is R. □

Consider the following situation with real series. Given the function $f(x) = \frac{1}{1+x^2}$, we can expand f as a series around the origin

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

which diverges for $|x| \geq 1$ and converges for $|x| < 1$. The behavior of $f(x)$ at $x = \pm 1$ is not exceptional in any way, and yet the series representation behaves badly. By considering $g(z) = \frac{1}{1+z^2}$ we see that $z = \pm i$ are singular points of g, and therefore by the above theorems we find that the power series will diverge $|z| > 1$, hence for $|x| > 1$.

9.1 Uniqueness Theorem for Power Series

We now turn our attention to the uniqueness of the coefficients in a Taylor expansion. Suppose that $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ and $g(z) = \sum_{n=0}^{\infty} d_n(z-a)^n$ have radii of convergence $R_f, R_g > 0$. Suppose furthermore that for some $0 < \delta < \min(R_f, R_g)$ we know that $f(z) = g(z)$ for all $|z-a| < \delta$. But then for all $n \geq 0$

$$c_n = \frac{f^{(n)}(a)}{n!} = \frac{g^{(n)}(a)}{n!} = d_n$$

9.4 Theorem. *(Uniqueness Theorem for Power Series) Let $E \subseteq \mathbb{C}$ be an infinite set and suppose $a \in \mathbb{C}$ is an accumulation point of E. Let $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ and $g(z) = \sum_{n=0}^{\infty} d_n(z-a)^n$ and suppose that these series converge on $V_R(a)$ for some $R > 0$. Suppose also that $f(e) = g(e)$ for all $e \in E$. Then $c_n = d_n$ for all $n \geq 0$.*

PROOF: Now f and g are analytic and therefore continuous on $V_R(a)$. Let $(z_n)_{n=1}^{\infty}$ be a sequence of points in E which converge to a. Since f and g are continuous at a

$$c_0 = f(a) = \lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} g(z_n) = g(a) = d_0$$

Let $f_1(z) = \sum_{n=0}^{\infty} c_{n+1}(z-a)^n$ and $g_1(z) = \sum_{n=0}^{\infty} d_{n+1}(z-a)^n$ and notice that the radius of convergence of f_1 and g_1 is the same as for f and g. Notice further that

$$f_1(z_n) = \frac{f(z_n) - c_0}{z_n - a} = \frac{g(z_n) - d_0}{z_n - a} = g_1(z_n)$$

for all $n \geq 0$. From the argument above we get that $c_1 = d_1$. Continuing by induction, we get $c_n = d_n$ for all $n \geq 0$. □

9.5 Theorem. *(Uniqueness Theorem for Analytic Functions) Suppose that $\mathcal{G} \subseteq \mathbb{C}$ is a domain and that $E \subseteq \mathcal{G}$ is an infinite set with accumulation point $a \in \mathcal{G}$. Suppose $f, g : \mathcal{G} \to \mathbb{C}$ are analytic and that $f(e) = g(e)$ for all $e \in E$. Then $f(z) = g(z)$ for all $z \in \mathcal{G}$.*

PROOF: Let $w \neq a$ be any point in $\mathscr{G}$. Choose a continuous path $\mathscr{C} = \{z(t) \mid t \in [0,1]\}$ such that $z(0) = a$ and $z(1) = w$. If $\mathscr{G} = \mathbb{C}$ let $r = 1$, otherwise let $r = \mathrm{dist}(\mathscr{C}, \partial\mathscr{G}) > 0$.

Since $z(t)$ is continuous and therefore uniformly continuous on $[0,1]$, we can find $\delta > 0$ such that $|t_1 - t_2| < \delta$ implies $|z(t_1) - z(t_2)| < r$. Let $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ be a partition of norm at most δ. Since $|t_i - t_{i-1}| < \delta$, the centres of the disks $V_r(z(t_i))$ and $V_r(z(t_{i-1}))$ are less than r apart, and so $V_r(z(t_i)) \cap V_r(z(t_{i-1})) \neq \varnothing$ for $i = 1, \dots, n$. Let $(e_n)_{n=1}^{\infty}$ be a sequence in E converging to a. Since f and g are analytic on $V_r(a)$, they admit Taylor series expansions $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ and $g(z) = \sum_{n=0}^{\infty} d_n(z-a)^n$ for $|z - a| < r$. Since $f(e_n) = g(e_n)$ for all $n \geq 1$, the uniqueness theorem for power series give us that $f(z) = g(z)$ for all $z \in V_r(a) = V_r(z(t_0))$.

Choose any point $z \in V_r(z(t_0)) \cap V_r(z(t_1))$. Then z is an accumulation point of the set $E_1 = V_r(z(t_0)) \cap V_r(z(t_1))$, on which $f(e) = g(e)$ for all $e \in E_1$. By the same reasoning as above, $f(u) = g(u)$ for all $u \in V_r(z(t_1))$.

Continue this process to find that $f(u) = g(u)$ for all $u \in V_r(z(t_k))$ for all $k = 0, \dots, n$. But then $f(w) = f(z(t_n)) = g(z(t_n)) = g(w)$. Since $w \in \mathscr{G} \setminus \{a\}$ was arbitrary and since both f and g are continuous at a, we conclude that $f = g$ on $\mathscr{G}$. □

9.6 Example.
1. Suppose that f is entire and $f\left(\frac{1}{n}\right) = 0$ for all $n \geq N$ then $f = 0$.
2. Suppose $f : V_2(0) \to \mathbb{C}$ is analytic and $f(z) = 0$ for all $z \in [0,1]$. Then $f = 0$.
3. In constrast, let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & x \leq \pi \\ \cos(x) + 1 & x > \pi \end{cases}$$

   Then f is differentiable on $\mathbb{R}$, $f = 0$ on $(-\infty, \pi]$, but $f \neq 0$. Thus the structure of analytic complex functions is stronger than that of real differentiable functions.

9.7 Definition. A zero of a function f is a point $z_0 \in \mathbb{C}$ such that $f(z_0) = 0$.

If $\mathscr{G}$ is a domain and $f : \mathscr{G} \to \mathbb{C}$ is analytic and $a \in \mathscr{G}$ is a zero of f, then f admits a Taylor expansion $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ which converges in the largest disk centred at a which is contained in $\mathscr{G}$. Since $f(a) = 0$, $c_0 = 0$. Unless $f \equiv 0$ on $\mathscr{G}$, there exists some $m_0 \in \mathbb{N}$ so that $c_{m_0} \neq 0$. In particular, if k is the smallest such index, then we say that a is a zero of order k for f. If $k = 1$ we also say that a is a simple zero.

As an easy corollary of the uniqueness theorem, every zero of a non-zero analytic function f on a domain $\mathscr{G}$ is isolated.

The following result is useful in proving the Maximum Modulus Principle.

9.8 Theorem. *Let $G \subseteq \mathbb{C}$ be a domain and suppose that there is $M \geq 0$ such that $|f(z)| = M$ for all $z \in \mathscr{G}$. Then f is constant on $\mathscr{G}$.*

PROOF: This was on the midterm. □

9.9 Theorem. *(Maximum Modulus Principle) Let $\mathscr{G} \subseteq \mathbb{C}$ be a domain and $f : \mathscr{G} \to \mathbb{C}$ be a non-constant analytic function. Then $|f|$ does not attain its maximum in $\mathscr{G}$.*

PROOF: Suppose otherwise, say $z_0 \in \mathscr{G}$ is such that $M := |f(z_0)| \geq |f(z)|$ for all $z \in \mathscr{G}$. Now $M > 0$, for otherwise f is (constantly) zero on $\mathscr{G}$. Choose $\delta > 0$ so that f is analytic on $V_\delta(z_0) \subseteq \mathscr{G}$. Let $0 < r < \delta$. By Cauchy's integral formula,

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{z - z_0} dz$$

In fact, with $z = z_0 + re^{i\theta}$, $dz = ire^{i\theta} d\theta$, so

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta$$

Thus

$$M = |f(z_0)| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta})| d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} M d\theta = M$$

Hence all of the inequalities are equalities. We claim that $|f(z_0 + re^{i\theta})| = M$, and this is obvious. The choice of r was arbitrary, so $|f(z)| = M$ for all $z \in V_\delta(z_0)$. Therefore f is constant on the disk, and by the uniqueness theorem f is constant on $\mathcal{G}$. This contradiction shows that f does not attain its maximum on $\mathcal{G}$. □

9.10 Corollary. *(Minimum Modulus Principle) Let $\mathcal{G} \subseteq \mathbb{C}$ be a domain and $f : \mathcal{G} \to \mathbb{C}$ be a non-constant analytic function that does not vanish on $\mathcal{G}$. Then $|f|$ does not attain its minimum in $\mathcal{G}$.*

PROOF: $\frac{1}{f}$ is analytic on $\mathcal{G}$. □

9.11 Corollary. *Let $\mathcal{G} \subseteq \mathbb{C}$ be a bounded domain and $f : \overline{\mathcal{G}} \to \mathbb{C}$ be an analytic function in $\mathcal{G}$ that is continuous on $\overline{\mathcal{G}}$ (resp. and non-vanishing on $\overline{\mathcal{G}}$). Then $\partial\mathcal{G}$ contains a point a so that $|f(a)| = \max\{|f(z)| \mid z \in \overline{\mathcal{G}}\}$ (resp. $|f(a)| = \min\{|f(z)| \mid z \in \overline{\mathcal{G}}\}$).*

PROOF: Since f is constant on $\overline{\mathcal{G}}$ and $\overline{\mathcal{G}}$ is compact, $|f|$ attains its max (min) at some point $a \in \overline{\mathcal{G}}$. If $a \in \mathcal{G}$ then f is constant so we may choose any point on the boundary. Otherwise we are done. □

9.12 Corollary. *Let $\mathcal{G} \subseteq \mathbb{C}$ be a bounded domain and $f : \overline{\mathcal{G}} \to \mathbb{C}$ be an analytic function in $\mathcal{G}$ that is continuous on $\overline{\mathcal{G}}$ and non-vanishing on $\overline{\mathcal{G}}$. Suppose that $|f||_{\partial\mathcal{G}}$ is constant. Then f is constant on $\mathcal{G}$.*

PROOF: f attains both its max and min on the boundary, hence they are equal and f is a constant function. □

10 Laurent Series

We now look at the case of series with negative exponents.

10.1 Theorem. *Given a series $\sum_{n=-\infty}^{0} c_n(z-a)^n$, let $\ell = \overline{\lim} |c_n|^{\frac{1}{n}}$. Then there are three possibilities:*

1. *$\ell = 0$ and the series convergec for all $a \neq z \in \mathbb{C} \cup \{\infty\}$*
2. *$0 < \ell < \infty$ and the series converges outside of the circle $|z - a| = \ell$ and diverges inside this circle.*
3. *$\ell = \infty$ and the series diverges for all $z \in \mathbb{C}$*

PROOF: If we set $w = \frac{1}{z-a}$, then the above series becomes $c_0 + c_{-1}w + c_{-2}w^2 + \cdots$ and this has radius of convergence $\frac{1}{\ell}$. The points $z = a$, $z = \infty$ are carried to $w = \infty$, $w = 0$, while the points outside the circle $|z - a| = \ell$ are carried to points inside the circle $|w| < \frac{1}{\ell}$ and visa versa. This observation proves the theorem. □

Clearly the map $w = \frac{1}{z-a}$ maps every closed bounded region $\overline{D}$ outside the circle $|z - a| = \ell$ into a closed bounded region $\overline{D}'$ inside the circle $|w| < \frac{1}{\ell}$. Since the series in w converges uniformly in $\overline{D}'$, it follows that $\sum_{n=-\infty}^{0} c_n(z-a)^n$ converges uniformly in $\overline{D}$. Since each term of the series is analytic outside of $|z - a| = \ell$, the sum $s(z)$ of the series is an analytic function at all points $|z - a| > \ell$. If $\ell = 0$, the circle $|z - a| = \ell$ is just the point a, in which case $s(\cdot)$ is analytic for all $z \neq a$.

10.2 Definition. A *Larent series* is a formal series $\sum_{n=-\infty}^{\infty} c_n(z-a)^n$ which is the sum of $\sum_{n=0}^{\infty} c_n(z-a)^n$, the regular part of the series, and $\sum_{n=-\infty}^{-1} c_n(z-a)^n$, the principal part. By definition, the series converges if and only if both the regular and principal parts converge.

The regular part of the series converges inside the disk of radius

$$R = \frac{1}{\overline{\lim}_{n\geq 0} |c_n|^{\frac{1}{n}}}$$

while the principal part of the series converges outside of the circle of radius

$$r = \overline{\lim} |c_n|^{\frac{1}{n}}$$

Thus the Larent series converges absolutely and uniformly on every compact set in the annulus $A = \{z \in \mathbb{C} \mid r < |z-a| < R\}$, provided $r < R$. Moreover, the sum $s(z)$ is analytic on A.

10.3 Theorem. *Let $s(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$ be a Laurent series with annulus of convergence $r < |z-a| < R$. If $r < \rho < R$ and $\gamma_\rho(a)$ is the circle centred at a of radius ρ then for each $n \in \mathbb{Z}$*

$$c_n = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{s(z)}{(z-a)^{n+1}} dz$$

PROOF: The series converges on the compact set $\gamma_\rho(a)$ and hence the same is true of each of the series

$$\frac{1}{2\pi i} \frac{s(z)}{(z-a)^{n+1}} = \frac{1}{2\pi i} \sum_{k=-\infty}^{\infty} c_k(z-a)^{k-n-1}$$

for each $n \in \mathbb{Z}$, since $\left|\frac{1}{2\pi i(z-a)^{n+1}}\right| = \frac{1}{2\pi\rho^{n+1}}$ for all $z \in \gamma_\rho(a)$. (We can multiply a uniformly convergent series by a bounded function and get another uniformly convergent series.) Since the series converges uniformly we can integrate term by term,

$$\frac{1}{2\pi i} \int_{\gamma_\rho} \frac{s(z)}{(z-a)^{n+1}} dz = \sum_{k=-\infty}^{\infty} c_k \frac{1}{2\pi i} \int_{\gamma_\rho} (z-a)^{k-n-1} dz$$

As for computing $\int_{\gamma_\rho} (z-a)^{k-n-1} dz$, let $z = a + \rho e^{i\theta}$ as θ runs over $[0, 2\pi]$. Then $dz = \rho i e^{i\theta} d\theta$ and

$$\int_0^{2\pi} (\rho e^{i\theta})^{k-n-1} \rho i e^{i\theta} d\theta = \int_0^{2\pi} \rho^{k-n} e^{i(k-n)\theta} i d\theta = \begin{cases} 0 & \text{if } k \neq n \\ 2\pi i & \text{if } k = n \end{cases}$$

Hence $\int_{\gamma_\rho} (z-a)^{k-n-1} dz = \delta_{k,n}$ and so $\frac{1}{2\pi i} \int_{\gamma_\rho} \frac{s(z)}{(z-a)^{n+1}} dz = c_n$. □

10.4 Theorem. *Let $A = \{z \in \mathbb{C} \mid r < |z-a| < R\}$ and suppose $f : A \to \mathbb{C}$ is analytic. Then f has a Laurent expansion at a. That is, the series $\sum_{n=-\infty}^{\infty} c_n(z-a)^n$ where*

$$c_n = \frac{1}{2\pi i} \int_{\gamma_\rho(a)} \frac{f(z)}{(z-a)^{n+1}} dz$$

converges to $f(z)$ for all $z \in A$.

PROOF: Let $w \in A$, and choose $r' > r$, $R' < R$ so that $0 < r < r' < |w-a| < R' < R$. Choose ρ so that $\gamma_\rho(w) \subseteq \{z \in \mathbb{C} \mid r' < |z-a| < R'\}$. Now the function $\frac{f(z)}{z-w}$ is analytic on $\{z \in \mathbb{C} \mid r' < |z-a| < R'\} \cap \{z \in \mathbb{C} \mid |z-w| > \rho\}$.

By Cauchy's integral formula

$$\frac{1}{2\pi i}\int_{\gamma_{R'}(a)}\frac{f(z)}{z-w}dz = \frac{1}{2\pi i}\int_{\gamma_{r'}(a)}\frac{f(z)}{z-w}dz + \frac{1}{2\pi i}\int_{\gamma_{\rho}(w)}\frac{f(z)}{z-w}dz$$
$$f(w) = \frac{1}{2\pi i}\int_{\gamma_{R'}(a)}\frac{f(z)}{z-w}dz + \frac{1}{2\pi i}\int_{\gamma_{r'}(a)}\frac{f(z)}{w-z}dz$$

We'll show that $\frac{1}{2\pi i}\int_{\gamma_{R'}(a)}\frac{f(z)}{z-w}dz$ (resp. $\frac{1}{2\pi i}\int_{\gamma_{r'}(a)}\frac{f(z)}{w-z}dz$) yields the non-negative (resp negative) powers of the series exapansion. If $z\in\gamma_{R'}(a)$ then

$$\frac{1}{z-w} = \sum_{n=0}^{\infty}\frac{(w-a)^n}{(z-a)^{n+1}}$$

If $z\in\gamma_{r'}(a)$ then

$$\frac{1}{w-z} = \sum_{n=0}^{\infty}\frac{(w-a)^n}{(z-a)^{n+1}} = \sum_{n=1}^{\infty}\frac{(w-a)^{-n}}{(z-a)^{-n+1}}$$

The uniform convergence of the (geometric) series on $\gamma_{R'}$ and $\gamma_{r'}$ respectively means that we can integrate term by term, giving the result. □

Suppose that $|f(z)|\le M$ for all $z\in A$. Then Cauchy's inequalities become

$$|c_n| \le \frac{1}{2\pi i}2\pi\rho\frac{M}{\rho^{n+1}} = \frac{M}{\rho^{n+1}}$$

for all $n\in\mathbb{Z}$ and $r<\rho<R$.

Let $z_0\in\mathbb{C}$ and suppose that for some $\delta>0$, f is analytic on $V_\delta(z_0)\setminus\{z_0\}$. Then z_0 is called an isolated singular point of f. Consider the Laurent expansion of f

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$$

valid in $V_\delta(z_0)\setminus\{z_0\}$. Then are 3 possibilities

1. If $c_n=0$ for all $n<0$ then z_0 is called a removable singularity.
2. There exists $M>0$ so that $c_n=0$ for $n<-M$. In this case z_0 is called a pole. The order of the pole is that largest power of $\frac{1}{z-z_0}$ which appears in this expansion.
3. If there are infinitely many $m<0$ such that $c_m\neq 0$ then z_0 is called an essential singularity of f.

If z_0 is a removable singularity, then $f(z)=\sum_{n=0}^{\infty}c_n(z-z_0)^n$, which clearly converges at z_0. If we define $g(z)=f(z)$ and $g(z_0)=c_0$ then g is analytic on the whole disk $V_\delta(z_0)$.

If z_0 is a pole of order m for f, then

$$(z-z_0)^m f(z) = c_{-m} + c_{-m+1}(z-z_0) + c_{-m+2}(z-z_0)^2 + \cdots$$

is an ordinary power series with non-zero constant term c_{-m}. Thus z_0 is a removable singularity of $(z-z_0)^m f(z)$. Verify that $\lim_{z\to z_0} f(z)=\infty$. We now study the relationship between zeroes and poles.

10.5 Theorem. *Let z_0 be a zero of order m of a function $0\neq f$ analytic at z_0. Then $\frac{1}{f}$ is analytic in a punctured neighbourhood of z_0, with a pole of order m at z_0.*

PROOF: We can write $f(z)=\sum_{k=m}^{\infty} c_k(z-z_0)^k$ with $c_m \neq 0$, and so $f(z)=(z-z_0)\varphi(z)$ where φ is analytic and non-zero at z_0. Thus $\frac{1}{f(z)}=\frac{(z-z_0)^{-m}}{\varphi(z)}$. Now $\frac{1}{\varphi}$ has a power series expansion at z_0 with non-zero constant coefficient, so z_0 is a pole of order m of $\frac{1}{f}$. □

The converse is also true, and the proof is similar.

10.6 Theorem. *Let f be analytic in a punctured neighbourhood of z_0, where z_0 is a pole of order m of f. If we set $\frac{1}{f(z_0)} := 0$ then $\frac{1}{f}$ is analytic at z_0 with a zero of order m.*

PROOF: Exercise. □

10.7 Corollary. *Suppose that f is analytic and non-vanishing in a punctured neighbourhood of z_0 and has an essential singularity at z_0. Then $\frac{1}{f}$ has an essential singularity at z_0.*

10.8 Theorem. *(Casorati-Weierstraß) Let f be analytic in a punctured neighbourhood of z_0 and suppose that z_0 is an essential singularity of f. If $A \in \mathbb{C}\cup\{\infty\}$ then there is a sequence $z_n \to z_0$ and $f(z_n)\to A$.*

This is a special case of *Picard's Theorem*: If f has an essential singularity at z_0 and $\delta > 0$ is small enough that f is analytic on $V_\delta(z_0)\setminus\{z_0\}$ then there is $w_0 \in \mathbb{C}$ such that for all $w_0 \neq w \in \mathbb{C}$, the equation $f(z)=w$ has infinitely many solutions inside of $V_\delta(z_0)\setminus\{z_0\}$. We will not prove Picard's theorem, but will prove the Casorati-Weierstraß theorem.

PROOF: There are two cases.

$A=\infty$: Suppose to the contrary that there is $\delta > 0$ such that $|f(z)| < M$ for all $0 < |z-z_0| < \delta$. By the Cauchy Inequalities,

$$|c_n| \le \frac{M}{\rho^n} \quad \forall\, n\in\mathbb{Z}\ \forall\, 0<\rho<\delta$$

where the c_n are the coefficients in the Laurent expansion of f at z_0. Now if $n<0$ then we can let $\rho\to 0$ to show that $c_n=0$. That is, the Laurent expansion of f at z_0 has no negative powers, so z_0 is a removable singular point of f, a contradiction. Thus $|f|$ is unbounded on every $V_\delta(z_0)\setminus\{z_0\}$, from which a sequence converging to z_0 such that the image under f diverges to ∞ may be found.

$A\in\mathbb{C}$: If every punctured neighbourhood of z_0 contains a point z so that $f(z)=A$, we are done. Otherwise, suppose that $\delta > 0$ is chosen so that $s \in V_\delta(z_0)\setminus\{z_0\}$ implies that $f(z)\neq A$. Then $\varphi(z)=\frac{1}{f(z)-A}$ has an essential singularity at z_0 (by the corollary above). By the first case, we may find a sequence $z_n \to z_0$ such that $\varphi(z_n)\to\infty$, which is to say that $f(z_n)\to A$. □

10.9 Example. Let $f(z)=e^{\frac{1}{z}}=1+\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{6z^3}+\cdots$, which has an essential singularity at $z=0$. For $A=\infty$ let $z_n=\frac{1}{n}$. Then $f(z_n)=e^n\to\infty=A$. For $A=0$ let $z_n=-\frac{1}{n}$. Then $f(z_n)=e^{-n}\to 0=A$. For $A\in\mathbb{C}\setminus\{0\}$, solving $e^{\frac{1}{z}}=A$ yields $z=\frac{1}{\log A}$. Let $\log_0$ denote the branch of the complex logarithm so that $\arg z$ lies in the interval $[0,2\pi)$. Then $z_0=\frac{1}{\log_0 A}$ and $f(z_0)=e^{\frac{1}{z_0}}=e^{\log_0 A}=A$. In fact $z_n := \frac{1}{\log_0 A+2\pi n i}$ is such that $z_n\to 0$ and $f(z_n)=A$ for all $n\ge 0$. Note that $f(z)=0$ has no solution in any $V_\delta(0)\setminus\{0\}$.

11 Residues

11.1 Definition. Let z_0 be an isolated singular point of an analytic function f. The *residue of f at z_0* is the coefficient c_{-1} in the Laurent expansion of f at z_0, denoted $\mathrm{Res}(f;z_0)$.

If f has a pole or an essential singularity at z_0 then $\mathrm{Res}(f;z_0)$ may or may not be zero. If z_0 is a removable singularity of f then $\mathrm{Res}(f;z_0)=0$.

11.2 Theorem. *(Residue Theorem) Suppose that f is analytic inside and on a piecewise smooth closed Jordan curve $\mathscr{C}$, except for isolated singular points $z_1,\dots,z_N$ lying inside $\mathscr{C}$. Then*

$$\frac{1}{2\pi i}\int_{\mathscr{C}} f(z)dz = \sum_{k=1}^{N} \mathit{Res}(f;z_k)$$

PROOF: Let $\gamma_1,\dots,\gamma_N$ be circles centred at $z_1,\dots,z_N$, respectively, and small enough to be contained inside $\mathscr{C}$ and not to intersect each other. By (a corollary to) Cauchy's Theorem,

$$\int_{\mathscr{C}} f(z)dz = \sum_{k=1}^{N}\int_{\gamma_k} f(z)dz$$

Suppose that the Laurent series for f at z_k is

$$f(z) = \sum_{n=-\infty}^{\infty} c_n^{(k)}(z-a)^n \text{ for } k=1,\dots,N$$

Integrating term by term (as the series converges uniformly along γ_k), we obtain

$$\begin{aligned}\int_{\gamma_k} f(z)dz &= \int_{\gamma_k}\sum_{n=-\infty}^{\infty} c_n^{(k)}(z-a)^n dz\\ &= \sum_{n=-\infty}^{\infty} c_n^{(k)}\int_{\gamma_k}(z-a)^n dz\\ &= \sum_{n=-\infty}^{\infty} c_n^{(k)} 2\pi i\delta_{n,-1}\\ &= 2\pi i\mathrm{Res}(f;z_k)\end{aligned}$$

Therefore $\frac{1}{2\pi i}\int_{\mathscr{C}} f(z)dz = \sum_{k=1}^{N}\mathrm{Res}(f;z_k)$. □

11.3 Example. Evaluate the integral $\int_{|z|=2}\frac{e^z}{(z-1)^n}dz$. The only singularity of $f(z)=\frac{e^z}{(z-1)^n}$ is at $z=1$ and it is a pole of order n. Indeed

$$f(z) = \frac{e^z}{(z-1)^n} = \frac{ee^{z-1}}{(z-1)^n} = \sum_{i=0}^{\infty}\frac{e(z-1)^{i-n}}{i!}$$

and so $\int_{|z|=2}\frac{e^z}{(z-1)^n}dz = 2\pi i\mathrm{Res}(f;1) = \frac{2\pi i e}{(n-1)!}$.

Next we show how to calculate the residues at a pole without making explicit use of the Laurent expansions. First suppose that z_0 is a simple pole of f. Then $f(z)=\frac{c_{-1}}{(z-z_0)}+c_0+c_1(z-z_0)+c_2(z-z_0)^2+\cdots$ in some punctured neighbourhood of z_0. Thus $\mathrm{Res}(f;z_0)=c_{-1}=\lim_{z\to z_0}(z-z_0)f(z)$.

In particular, if $f(z)=\frac{g(z)}{h(z)}$ where $g(z_0)\neq 0$ and z_0 is a simple zero of h then z_0 is a simple pole of f, so

$$\mathrm{Res}(f;z_0)=\lim_{z\to z_0}(z-z_0)\frac{g(z)}{h(z)}=\lim_{z\to z_0}\frac{g(z)}{\frac{h(z)-h(z_0)}{z-z_0}}=\frac{g(z_0)}{h'(z_0)}$$

Suppose that z_0 is a pole of order $m>1$ for f. Then $f(z)=c_{-m}(z-z_0)^{-m}+\cdots+c_0+c_1(z-z_0)+\cdots$, and so $(z-z_0)^m f(z)=c_{-m}+c_{-m+1}(z-z_0)+\cdots+c_{-1}(z-z_0)^{m-1}$. But then $[(z-z_0)^m f(z)]^{(m-1)}=(m-1)!c_{-1}+\cdots$, so taking limits gives

$$\mathrm{Res}(f;z_0)=c_{-1}=\frac{1}{(m-1)!}\lim_{z\to z_0}[(z-z_0)^m f(z)]^{(m-1)}$$

In the example above, $\frac{e^n}{(z-1)^n}$ has a pole of order n at 1, so

$$\mathrm{Res}\left(\frac{e^n}{(z-1)^n};1\right)=\frac{1}{(n-1)!}\lim_{z\to 1}\left[(z-1)^n\frac{e^n}{(z-1)^n}\right]^{(n-1)}=\frac{e}{(n-1)!}$$

11.1 Applications of residues

11.4 Definition. The logarithmic residue of an analytic function f at a point a is $\mathrm{Res}(f'/f;a)$.

Suppose that a is a zero of order m of f. Then $f(z)=c_m(z-a)^m+\cdots$ in a neighbourhood of a, and so $f'(z)=mc_m(z-a)^{m-1}+\cdots$ in that neighbourhood. From this we get

$$\begin{aligned}\frac{f'(z)}{f(z)}&=\frac{mc_m(z-a)^{m-1}+\cdots}{c_m(z-a)^m+\cdots}\\&=\frac{m}{z-a}\left(\frac{c_m+\frac{m+1}{m}c_{m+1}(z-a)+\cdots}{c_m+c_{m+1}(z-a)+\cdots}\right)\\&=\frac{m}{z-a}\varphi(z)\end{aligned}$$

where $\varphi(z)$ is analytic on some neighbourhood of a and $\varphi(a)=1$. From this it follows that

$$\mathrm{Res}(f'/f;a)=\lim_{z\to a}(z-a)\frac{m}{z-a}\varphi(z)=m$$

the order of the zero at a. Similarily, if b is a pole of order n of f then $\mathrm{Res}(f'/f;b)=-n$.

11.5 Theorem. *Let $\mathscr{C}$ be a piecewise smooth closed Jordan curve and suppose that f is analytic on and inside $\mathscr{C}$, expect at finitly many poles $b_1,\dots,b_n$ inside $\mathscr{C}$. Suppose also that f has zeros $a_1,\dots,a_m$ inside $\mathscr{C}$. then*

$$\frac{1}{2\pi i}\int_{\mathscr{C}}\frac{f'(z)}{f(z)}dz=\sum_{k=1}^{m}\alpha_k-\sum_{j=0}^{n}\beta_j$$

where α_k (resp. β_j) is the order of the zero at a_k (resp. the order of the pole at b_j).

PROOF: The function f'/f only has poles where f has a pole or a zero, so

$$\frac{1}{2\pi i}\int_{\mathscr{C}}\frac{f'(z)}{f(z)}dz=\sum_{r=1}^{s}\text{Res}(f'/f\,;p_r)$$

where p_r is the r^{th} pole. The result follows from the calculations above. □

11.6 Theorem. *(Rouché) Suppose that f and g are analytic inside and on a piecewise smooth closed Jordan curve $\mathscr{C}$ and that $|f(z)|>|g(z)|$ at every point of $\mathscr{C}$. Then f and $f+g$ have the same number of zeroes (counted with multiplicity) inside $\mathscr{C}$.*

PROOF: Let $t\in[0,1]$. Since $|f(z)|>|g(z)|$ for all $z\in\mathscr{C}$, $(f+tg)(z)\neq 0$ for all $z\in\mathscr{C}$. Let

$$\varphi(t)=\frac{1}{2\pi i}\int_{\mathscr{C}}\frac{(f+tg)'(z)}{(f+tg)(z)}dz$$

By Theorem 11.5, $\varphi(t)$ is the number of zeroes of $f+tg$ inside $\mathscr{C}$ counted with multiplicity. Since φ is integer valued, if we can show that φ is continuous on $[0,1]$ then it must be constant. In particular, $\varphi(0)=\varphi(1)$ implies that the number of zeroes of f is equal to the number of zeroes of $f+g$ inside $\mathscr{C}$ (counted with multiplicity). To see that φ is continuous, fix $t\in[0,1]$. For $s\in[0,1]$,

$$\begin{aligned}|\varphi(t)-\varphi(s)|&=\frac{1}{2\pi}\left|\int_{\mathscr{C}}\frac{f'+tg'}{f+tg}-\frac{f'+sg'}{f+sg}\right|\\&=\frac{1}{2\pi}\left|\int_{\mathscr{C}}\frac{(g'f-f'g)(s-t)}{(f+tg)(f+sg)}\right|\\&\leq\frac{1}{2\pi}\lambda_{\mathscr{C}}|t-s|\max_{\mathscr{C}}\left|\frac{(g'f-f'g)(s-t)}{(f+tg)(f+sg)}\right|\end{aligned}$$

We can find some $M_1>0$ so that $|f+tg|\geq M_1$ on $\mathscr{C}$, since $f+tg$ is continuous and non-vanishing on the compact set $\mathscr{C}$. Also, $g'f-f'g$ is continuous on the compact set $\mathscr{C}$, so it is bounded by some $M_2>1$. If $|s-t|<\frac{1}{2}\frac{M_1}{M_2}$ then

$$|(f+sg)(z)|\geq|(f+tg)(z)|-|(s-t)g(z)|\geq\frac{1}{2}M_1>0$$

Thus $|s-t|<\frac{1}{2}\frac{M_1}{M_2}$ implies that

$$|\varphi(t)-\varphi(s)|\leq\frac{1}{2\pi}\lambda_{\mathscr{C}}\left(\frac{2M_2}{M_1^2}\right)|t-s|$$

so φ is Lipshitz and hence continuous. □

11.7 Example. How many zeros does the function $h(z)=z^8-4z^5+z^2-1$ have in the unit circle? Let $f(z)=-4z^5$ and $g(z)=z^8+z^2-1$. Then for $|z|=1$

$$|f(z)|=|-4z^5|=4>3=|z^8|+|z^2|+|-1|\geq|g(z)|$$

By Rouché's Theorem, f and $h=f+g$ have the same number of zeroes (including multiplicities) inside the the curve $|z|=1$, which is to say on the unit disc. f has 5 zeroes there, so h does as well.

11.8 Example. Show that $h(z) = 2 + z^2 - e^{iz}$ has precisely one simple zero in the upper open half plane. Let $f(z) = 2 + z^2$ and $g(z) = -e^{iz}$. Let $\mathscr{C}_R = \Gamma_R \cup [-R, R]$, where $\Gamma_R = \{Re^{i\theta} \mid \theta \in [0, \pi]\}$. Then $\mathscr{C}_R$ is a closed, piecewise smooth Jordan curve and f, g are entire. For $z \in [-R, R]$ we have

$$|f(z)| = |2 + z^2| \geq 2 > 1 = |e^{iz}| = |g(z)|$$

For $z \in \Gamma_R$ with $R > \sqrt{3}$ we have

$$|f(z)| = |2 + z^2| \geq |z|^2 - 2 = R^2 - 2 > 1 \geq |e^{-\Im z}| = |e^{iz}| = |g(z)|$$

By Rouché's Theorem, f and $h = f + g$ have the same number of zeroes inside of $\mathscr{C}_R$. Since R was arbitrary, f and h have the same number of zeroes on the whole upper open half plane. f has one zero there, so h does as well.

11.2 Evaluation of Improper Integrals

Techniques from complex analysis may be used to evaluate difficult looking improper real integrals with ease. We will end this section with two examples – see the textbook for many more.

11.9 Example. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^3}$, where $a > 0$.

Let $f(z) = \frac{1}{(z^2+a^2)^3}$ and $\mathscr{C}_R = \Gamma_R \cup [-R, R]$ (as above). Then f has only one singular point inside $\mathscr{C}_R$, namely $z_0 = ai$. Now

$$\operatorname{Res}(f; ai) = \lim_{z \to ai} \frac{1}{2!} \left[(z - ai)^3 \frac{1}{(z^2 + a^2)^3} \right]^{(2)} = \frac{1}{2} \frac{(-3)(-4)}{(z + ai)^5} \Big|_{z=ai} = \frac{3}{16a^5 i}$$

Therefore $\int_{\mathscr{C}_R} f(z)dz = 2\pi i \operatorname{Res}(f; ai) = \frac{3\pi}{8a^5}$. On the other hand,

$$\left| \int_{\Gamma_R} f(z)dz \right| = \left| \int_{\Gamma_R} \frac{dz}{(z^2 + a^2)^3} \right| \leq \lambda_{\Gamma_R} \sup_{\substack{|x+iy|=R \\ y>0}} \left| \frac{1}{(z^2 + a^2)^3} \right| \leq \frac{\pi R}{(R^2 - a^2)^3}$$

So $\lim_{R\to\infty} \int_{\Gamma_R} f(z)dz = 0$ and hence

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)^3} = \lim_{R\to\infty} \int_{-R}^{R} f(z)dz = \lim_{R\to\infty} \int_{\mathscr{C}_R} f(z)dz = \frac{3\pi}{8a^5}$$

11.10 Example. Evaluate $\int_0^{\infty} \frac{\sin x}{x} dx$.

Consider $f(z) = \frac{e^{iz}}{z}$. Then $\Im f(z) = \frac{\sin z}{z}$ when $z \in \mathbb{R} \setminus \{0\}$. Let $\mathscr{C}_{R,r} = \Gamma_R \cup [-R, -r] \cup \Gamma_r \cup [r, R]$, oriented in the anticlockwise direction. Then f is analytic on and inside $\mathscr{C}_{R,r}$ for all $R > r > 0$. By Cauchy's Integral Theorem $\int_{\mathscr{C}_{R,r}} f(z)dz = 0$. Taking limits as $R \to \infty$ and $r \to 0$ we get

$$0 = \int_{-\infty}^{0} \frac{e^{ix}}{x} dx + \lim_{r\to 0} \int_{\Gamma_r} \frac{e^{iz}}{z} dz + \int_{0}^{\infty} \frac{e^{ix}}{x} dx + \lim_{R\to\infty} \int_{\Gamma_R} \frac{e^{iz}}{z} dz$$

Consider $\int_{\Gamma_R} \frac{e^{iz}}{z} dz$. Notice that ou usual estimate yields that this integral is less than or equal to π, which is of little help to us. In stead we use integration by parts:

$$\begin{aligned}\int_{\Gamma_R} \frac{e^{iz}}{z}dz &= \int_{\Gamma_R} \frac{de^{iz}}{iz}dz \\ &= \frac{e^{iz}}{iz}\bigg|_{-R}^{R} - \int_{\Gamma_R} -\frac{e^{iz}}{iz^2}dz \\ &= \frac{e^{iR}+e^{-iR}}{iR} + \frac{1}{i}\int_{\Gamma_R} -\frac{e^{iz}}{z^2}dz \longrightarrow 0 \text{ as } R \to \infty\end{aligned}$$

since $|\int_{\Gamma_R} -\frac{e^{iz}}{z^2}dz| \leq \pi R \frac{1}{R^2}$.

As for $\lim_{r\to 0}\int_{\Gamma_r} \frac{e^{iz}}{z}dz$, notice that

$$\frac{e^{iz}}{z} = \frac{1}{z} + i + \frac{i^2 z}{2!} + \frac{i^3 z^2}{3!} + \cdots = \frac{1}{z} + \varphi(z)$$

where φ is analytic on a neighbourhood of zero. Hence φ is continuous on a neighbourhood of zero, and so bounded there, say by $M > 0$. Then

$$\lim_{r\to 0}\int_{\Gamma_r} \frac{e^{iz}}{z}dz = \lim_{r\to 0}\int_{\Gamma_r} \frac{dz}{z} + \lim_{r\to 0}\int_{\Gamma_r} \varphi(z)dz = -\pi i$$

since $\int_{\Gamma_r} \varphi(z)dz \leq \lambda_{\Gamma_r} M \to 0$ as $r \to 0$ and $\lim_{r\to 0}\int_{\Gamma_r} \frac{dz}{z} = \lim_{r\to 0} \frac{1}{re^{i\theta}} ire^{i\theta}d\theta = -\pi i$. Taking imaginary parts, we get

$$\int_{-\infty}^{0} \frac{\sin x}{x}dx + \int_{0}^{\infty} \frac{\sin x}{x}dx = \pi$$

And since these integrals are equal, $\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$.