Differential Geometry
Michaelmas 2006
Dr. G.P. Paternain*

Chris Almost†

See Dr. Paternain's website for the room and time of the next examples class (at the beginning of Lent term).

Contents

*g.p.paternain@dpmms.cam.ac.uk
†cda23@cam.ac.uk

Notes: (available on the internet)

(i) Alexei Kovalev (followed most closely)

(ii) Mihalis Dafermos (these have an emphasis on calculus of variations)

Book:

(i) J. Jost, *Riemannian Geometry & Geometric Analysis*, Springer, Universitext (chapters 1, 2, 3)

1 Manifolds

1.1 Definition and First Examples

1.1.1 Definition. A *topological space* M is a set with a specified class of *open* sets, such that

(i) $\varnothing$ and M are open;

(ii) the intersection of two open sets is open;

(iii) an arbitrary union of open sets is open.

M is *Haudorff* if given $p_1, p_2 \in M$ there are open sets U_i $(i = 1, 2)$ such that $p_i \in U_i$ and $U_1 \cap U_2 = \varnothing$. M is *second countable* if one can find a coutable collection $\mathscr{B}$ of open sets of M such that any open set $U \subseteq M$ can be written as a union of elements in $\mathscr{B}$.

1.1.2 Definition. Recall that a *continuous map* is a map such that the preimage of every open set is open. A *homeomorphism* is a continuous bijection with a continuous inverse.

1.1.3 Definition. A homeomorphism $\varphi : U \to V$, where $U \subseteq M$ is open and $V \subseteq \mathbb{R}^d$ is open is called a *chart* and U the *coordinate neighbourhood*.

Insert standard chart picture here with change of local coordinates.

Suppose you have charts $\varphi : U \to V$ and $\psi : U' \to V'$. Then $\varphi \circ \psi^{-1}$ is a map between subsets of $\mathbb{R}^d$.

1.1.4 Definition. A C^∞ *differentiable structure* (or a *smooth structure*) on M is a collection of coordinate charts $\varphi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{R}^d$ (same d for all α) such that

(i) $\bigcup_\alpha U_\alpha = M$;

(ii) any two charts are *compatible*, namely for all α, β the change of local coordinates $\varphi_\beta \circ \varphi_\alpha^{-1}$ is C^∞ on its domain $\varphi_\alpha(U_\beta \cap U_\alpha)$ (this is equivalent to requiring that it has continuous partial derivatives of all orders);

(iii) the collection of charts $\{(U_\alpha, \varphi_\alpha)\}$ is maximal with respect to property (ii), namely if a chart (U, φ) is compatible with $(U_\alpha, \varphi_\alpha)$ for all α then (U, φ) is included in the collection.

In this case d is the *dimension* of M, denoted $\dim M$.

Remark.

(i) The change of local coordinates is a *diffeomorphism*, a C^∞ map with C^∞ inverse.

(ii) We only really need to worry about (i) and (ii), since there is a unique way of extending a collection of charts satisfying (i) and (ii) to a maximal one: just add all the compatible charts (proof is a Zorn's Lemma arugment).

(iii) If we start just with bijective charts, we can mount a topology on M by defining $D \subseteq M$ to be open if and only if $\varphi(D \cap U)$ is open in $\mathbb{R}^d$ for every chart $\varphi : U \to V \subseteq \mathbb{R}^d$. This is the *topology induced by the C^∞ structure*.

1.1.5 Examples.

(i) $\mathbb{R}^d$ is a manifold, as is any open subset of $\mathbb{R}^d$.

(ii) $S^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n x_i^2 = 1\}$, with smooth structure given by stereographic projection from the north and south poles. This is the *canonical* smooth structure on S^n.

(iii) Real projective space $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^*$, with smooth structure given by the charts $(0 \le i \le n)$

$$\begin{aligned} \varphi_i : U_i = \{[x_0 : \cdots : x_n] \mid x_i \neq 0\} \to \mathbb{R}^n \\ : [x_0 : \cdots : x_n] \mapsto \left(\frac{x_1}{x_i}, \dots, \hat{i}, \dots, \frac{x_n}{x_i} \right) \end{aligned}$$

omitting the i^{th} component. It is clear that these charts are compatible. Check that $\mathbb{RP}^n$ is a compact manifold. (Hint: there is a continuous projection $S^n \to \mathbb{RP}^n$.)

In the 50's J. Milnor showed that S^7 has many different smooth structures. He won a Field's Medal for his work. In the 80's an even more startling fact was discovered: $\mathbb{R}^4$ has uncountably many smooth structures! The Smooth Poincaré Conjecture asks whether S^4 has an exoctic (non-canonical) smooth structure.

1.1.6 Definition. Let M and N be smooth manifolds and $f : M \to N$ be continuous. We say that f is a *smooth map* if for all $p \in M$ and charts (U, φ) containing p and (V, ψ) containing $f(p)$ the map $\psi \circ f \circ \varphi^{-1}$ is C^∞ on its domain of definition.

1.1.7 Definition. A continuous map $f : M \to N$ is a *diffeomorphism* if f is a smooth bijection with smooth inverse. In this case we say that M and N are *diffeomorphic*

1.1.8 Exercise. Show that if M and N are manifolds then $M \times N$ is a manifold in a canonical way.

1.1.9 Example (Lie groups). A *Lie group* is a group G which is also a smooth manifold such that the map $G \times G \to G : (\sigma, \tau) \mapsto \sigma\tau^{-1}$ is a smooth map.

(i) $GL(n, \mathbb{R})$ is a Lie group since it embeds naturally in $\mathbb{R}^{n^2}$ as an open set and matrix multiplication is given by polynomials in the coefficients.

(ii) $O(n) = \{A \in GL(n, \mathbb{R}) \mid AA^T = I\}$ is a Lie group, as we shall see below.

1.1.10 Exponential and Logarithm of Matrices. *Let A be an $n \times n$ real matrix. The exponential of A is defined to be*

$$\exp(A) = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots + \frac{1}{n!}A^n + \cdots .$$

It can be shown, using the Weierstrass M-test, that this series is uniformly convergent on compact subsets of $M(n, \mathbb{R})$. In fact the map $A \mapsto \exp(A)$ is C^∞. The exponential map has the following properties.

(i) $\exp(A^T) = (\exp(A))^T$*;*

(ii) $\exp(C^{-1}AC) = C^{-1}\exp(A)C$*;*

(iii) Warning: it does not hold in general that $\exp(A+B) = \exp(A)\exp(B)$*, but it does hold if A and B commute. In particular,* $\exp(A)\exp(-A) = I$.

Similarily, we define the logarithm of a matrix A to be

$$\log(A) = A - \frac{1}{2}A^2 + \frac{1}{3}A^3 + \cdots + \frac{(-1)^{n+1}}{n}A^n + \cdots .$$

This series also converges uniformly on compact sets, provided that $|A - I| < 1$, and $A \mapsto \log(A)$ is C^∞ for all A with $|A - I| < 1$. We have

(i) $\exp(\log(A)) = A$ *if* $|A - I| < 1$*;*

(ii) $\log(\exp(A)) = A$ *if* $|A| < \log 2$.

Take A such that $|A - I| < 1$. If $A \in O(n)$ then $\log(A) \in S$, the set of skew-symmetric matrices. Indeed, $AA = I$ implies that

$$\exp(\log(A))\exp((\log(A))^T) = I$$

by (i), so

$$\exp((\log(A))^T) = \exp(-\log(A)).$$

But $|\log(A)| \le \log|A| < 2$ since $|A - I| < 1$, so by (ii)

$$(\log(A))^T = -\log(A).$$

The dimension of the space of skew-symmetric matrices is $\frac{1}{2}n(n-1)$. Consider

$$\varphi : U \to S \cong \mathbb{R}^{\frac{1}{2}n(n-1)} : A \mapsto \log(A),$$

where $U = \{A \in O(n) \mid |A - I| < 1\}$. Then φ is a chart around the identity. For any $C \in O(n)$, let $\varphi_C(A) = \log(C^{-1}A)$ for A in a small open set around C. It remains to check that change of coordinates is a smooth map, but this follows from the definition of Lie group (in particular that multiplication is smooth).

1.2 Tangent Space and Differentials

The tangent space T_pM

For each point $p \in M$ we would like to define the space of *tangent vectors* to M at p, which we will denote T_pM. It should be a vector space of dimension $\dim M$, and whenever $f : M \to N$ is a smooth map, there should be an associated linear map $df_p : T_pM \to T_{f(p)}N$ that satisfies the chain rule, which will be called the *differential* of f

Intuitively, any tangent vector to a curve $\alpha : I \to M$, where $0 \in I \subseteq \mathbb{R}$ is an open interval and $\alpha(0) = p$, is a tangent vector to M at p, and any tangent vector should arise in this way. Let us first consider the simplest case, where the manifold under consideration is an open set $U \subseteq \mathbb{R}^n$. Let $p \in U$ and consider all smooth curves through p (i.e. all smooth curves $\alpha : I \to U$, where $0 \in I \subseteq \mathbb{R}$ is an open interval and $\alpha(0) = p$). We say that two such curves α_1 and α_2 are *equivalent* if $\alpha_1'(0) = \alpha_2'(0)$, since all we really care about is the tangent vector at p. Let T_pU be the set of equivalence classes. There is a natural bijection between $T_pU \to \mathbb{R}^n$ given by $[\alpha] \mapsto \alpha'(0)$, since each equivalence class is uniquely defined by its tangent vector at p. If $f : U \to \mathbb{R}^m$ is a smooth map then, from multivariable calculus,

$$\frac{d}{dt}\bigg|_{t=0} f(\alpha(t)) = df_p(\alpha'(0)),$$

where $df_p = (\frac{\partial f_i}{\partial x_j})_{ij}$ is the usual differential. It follows that $df_p([\alpha]) = [f \circ \alpha]$.

Now let M be a manifold and $p \in M$. Fix a chart (U, φ) around p. If α_1 and α_2 are two curves through p then we say that α_1 is *equivalent* to α_2 if

$$(\varphi \circ \alpha_1)'(0) = (\varphi \circ \alpha_2)'(0).$$

We need to check that this definition does not depend on the chart chosen. Let (V, ψ) be another chart around p and let $h = \psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ be the change of coordinates. If α_1 and α_2 are equivalent with respect to φ, then by applying $dh_{\varphi(p)}$ we get

$$\begin{aligned} dh_{\varphi(p)}((\varphi \circ \alpha_1)'(0)) &= dh_{\varphi(p)}((\varphi \circ \alpha_2)'(0)) \\ (h \circ \varphi \circ \alpha_1)'(0) &= (h \circ \varphi \circ \alpha_2)'(0) \\ (\psi \circ \alpha_1)'(0) &= (\psi \circ \alpha_2)'(0) \end{aligned}$$

so they are equivalent with respect to ψ as well.

1.2.1 Definition. T_pM, the *tangent space* to M at p is the set of all equivalence classes of smooth curves $\alpha : I \to M$, where $0 \in I \subseteq \mathbb{R}$ is an open interval and $\alpha(0) = p$. Two curves α_1 and α_2 through p are equivalent if $(\varphi \circ \alpha_1)'(0) = (\varphi \circ \alpha_2)'(0)$ for some chart (U, φ) containing p.

There is a linear structure on T_pM induced by the map $\Phi_\varphi : T_pM \to \mathbb{R}^n : [\alpha] \to (\varphi \circ \alpha)'(0)$ (i.e. there is a unique linear structure on T_pM such that Φ_φ is a linear isomorphism). Again, this linear structure is defined in terms of a particular chart, so again we need to check that it is well-defined. As above, we apply $dh_{\varphi(p)}$.

$$\begin{array}{ccc} & T_pM & \\ {\scriptstyle \Phi_\varphi} \swarrow & & \searrow {\scriptstyle \Phi_\psi} \\ \mathbb{R}^n & \xrightarrow[dh_{\varphi(p)}]{\cong} & \mathbb{R}^n \end{array}$$

Since h is a change of coordinates, $dh_{\varphi(p)}$ is a linear isomorphism, so the linear structures are isomorphic.

The differential df_p

Now we move to the definition of the differential of a smooth map. We take as the definition the result noted at the end of the discussion from the beginning of the section.

1.2.2 Definition. Let $f : M \to N$ be a smooth map between manifolds. The *differential* of f at p is

$$df_p : T_pM \to T_{f(p)}N : [\alpha] \mapsto [f \circ \alpha].$$

First we check that this defines a linear map. Let (U, φ) be a chart containing p and (V, ψ) be a chart containing $f(p)$. Then we have the following commutative diagram.

$$\begin{array}{ccc} T_pM & \xrightarrow{df_p} & T_{f(p)}N \\ {\scriptstyle \Phi_\varphi} \downarrow {\scriptstyle \cong} & & {\scriptstyle \cong} \downarrow {\scriptstyle \Phi_\psi} \\ \mathbb{R}^n & \xrightarrow[d(\psi \circ f \circ \varphi^{-1})_{\varphi(p)}]{} & \mathbb{R}^m \end{array}$$

Notice that $\Phi_\varphi = d\varphi_p$ with this definition of df_p, so we will no longer use the notation Φ_φ.

As in multvariable calculus, we have a Chain Rule for compositions of smooth maps.

1.2.3 Theorem (Chain Rule). *Let* $M \xrightarrow{f} N \xrightarrow{g} P$ *be smooth maps between manifolds, so that there are linear maps*

$$T_pM \xrightarrow{df_p} T_{f(p)}N \xrightarrow{dg_{f(p)}} T_{g(f(p))}P.$$

Then $d(g \circ f)_p = dg_{f(p)} \circ df_p$.

PROOF: Exercise. □

Expressions in local coordinates

Let (U, φ) be a chart around $p \in M$. Let $\{e_1, \dots, e_n\}$ be the canonical basis of $\mathbb{R}^n$. Pulling back this basis through the linear isomorphism $d\varphi_p : T_pM \to \mathbb{R}_n$ gives a "canonical" basis for T_pM (which depends on the chart chosen).

1.2.4 Definition. The *canonical basis* for T_pM with respect to a fixed chart (U, φ) containing p is $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$, where $\frac{\partial}{\partial x^i} := (d\varphi_p)^{-1}(e_i)$.

Notation. In class expressions of the form "$x = y_i$" will be written, by which it is meant "$x = \sum_{i=1}^n y_i$." Namely, whenever a lonely index occurs on the righthand side of an equality there is an implicit sum from 1 to the dimension of the space.

How does the canonical basis change when a different chart is used? Let $v \in T_pM$ and suppose $v = a_i \frac{\partial}{\partial x^i}$. Let (V, ψ) be another chart containing p and $h = \psi \circ \varphi^{-1}$ be the change of coordinates. Suppose that h changes x_i coordinates into x'_i coordinates via

$$x'_j = x'_j(x_1, \dots, x_n) = h(x_1, \dots, x_n),$$

and $v = a'_i \frac{\partial}{\partial x'^i}$.

Since $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ and $\{\frac{\partial}{\partial x'^1}, \dots, \frac{\partial}{\partial x'^n}\}$ are both bases of T_pM, there are (b_{ij}) such that

$$(d\varphi_p)^{-1}(e_i) = \frac{\partial}{\partial x^i} = b_{ij} \frac{\partial}{\partial x'^j} = b_{ij}(d\psi_p)^{-1}(e_j).$$

Apply $d\psi_p$ to both sides and the Chain Rule to see that $dh_{\varphi(p)}(e_i) = b_{ij}e_j$, so $(b_{ij}) = dh_{\varphi(p)} = (\frac{\partial x'_j}{\partial x_i})_{ij}$.

In particular, $\frac{\partial}{\partial x^i} = \frac{\partial x'_j}{\partial x_i} \frac{\partial}{\partial x'^j}$, and $a'_i = \frac{\partial x'_i}{\partial x_k} a_k$. Reiterating, the change of basis matrix changing from the canonical basis induced by φ to the canonical basis induced by ψ is the Jacobian at $\varphi(p)$ of the change of coordinate map $\psi \circ \varphi^{-1}$.

Remark. $C^\infty(M, N)$ denotes the space of all C^∞ maps from M to N. Suppose $f \in C^\infty(M, \mathbb{R})$. A element $v \in T_pM$ induces a linear map $\hat{v} : C^\infty(M, \mathbb{R}) \to \mathbb{R} : f \mapsto df_p(v)$. But more than that, $\hat{v}$ satisfies the Leibniz rule

$$\hat{v}(fg) = f(p)\hat{v}(g) + \hat{v}(f)g(p),$$

so $\hat{v}$ is a so-called *derivation*. The tangent space can be defined to be the collection of derivations on $C^\infty(M, \mathbb{R})$. We may also write

$$\hat{v}(f) = a_i \frac{\partial}{\partial x^i} f \circ \varphi^{-1}.$$

1.3 Tangent Bundles

1.3.1 Definition. TM, the *tangent bundle*, is the disjoint union of all the tangent spaces, $\amalg_{p\in M} T_pM$. Let $\pi : TM \to M$ be the canonical projection. $\pi^{-1}(p)$ is the *fibre* at p. If (U,φ) is a chart around p, define the *bundle chart*

$$\varphi_T : \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^n : (p,v) \mapsto (\varphi(p), d\varphi_p(v)) = (\varphi(p), (a_1,\dots,a_n)).$$

Check that the collection of bundle charts induces a C^∞ structure on TM. The tangent bundle is a prime example of a *vector bundle*.

1.3.2 Definition. Similarily, T^*M, the *cotangent bundle*, is the disjoint union of the duals of all the tangent spaces.

Describe the charts on T^*M. With the proper charts, the cotangent bundle is also a vector bundle. In classical mechanics, the tangent bundle corresponds to the space of all positions and velocities, $(q,\vec{q})$, while the cotangent bundle corresponds to the space of all positions and momentums $(q,\vec{p})$. It is also know as the *phase space*.

1.3.3 Definition. A *vector field* on M is a smooth map $X : M \to TM$ such that $\pi \circ X = \text{id}$.

A vector field is a prime example of a *section* of a vector bundle. Suppose that (U,φ) is a chart around $p \in M$. Then $X(p) = (a_i(p)\frac{\partial}{\partial x^i})$ for some smooth functions $a_i : M \to \mathbb{R}$. Vector fields have *integral curves*, which are curves $\gamma : I \to M$ such that $\dot{\gamma}(t) = X(\gamma(t))$. (This follows from the existence and uniqueness of ODEs in $\mathbb{R}^n$.) If M is compact then we may take $I = \mathbb{R}$ (i.e. there is a solution to the ODE for all times $t \in \mathbb{R}$). A *flow* is a one parameter family of diffeomorphisms $\phi_t : M \to M$ $(t \in \mathbb{R})$ such that $\phi_{t+s} = \phi_t \circ \phi_s$ and $\phi_t(p) = \gamma_p(t)$, where γ_p is the unique solution to $\dot{\gamma}(t) = X(\gamma(t))$ around p. A flow is an action of $\mathbb{R}$ on M by diffeomorphisms.

1.3.4 Definition. A *smooth 1-form* ω on T^*M is a smooth map $\omega : M \to T^*M$ such that $\pi \circ \omega = \text{id}$. i.e. a smooth 1-form is a section of T^*M.

1.3.5 Proposition. *Let M be a smooth manifold of dimension n, and suppose that there are vector fields $X_1,\dots,X_n$ such that $\{X_1(p),\dots,X_n(p)\}$ is a basis of T_pM for all $p \in M$. Then TM is "isomorphic" to $M \times \mathbb{R}^n$.*

Here by "isomorphic" we mean diffeomorphic via a diffeomorphism with takes fibres to fibres in a linear fashion. Manifolds M satisfying 1.3.5 are called *parallelizable* in the literature of differential geometry.

PROOF: For any $(p,v) \in TM$, $\{X_1(p),\dots,X_n(p)\}$ is a basis of T_pM, so suppose v has coordinates $v = (a_iX_i(p))$. Let

$$\Phi : TM \to M \times \mathbb{R}^n : (p,v) \mapsto (p,(a_1,\dots,a_n)).$$

Clearly Φ is a bijection and maps fibres to fibres in a linear fashion, so we need only check that Φ and Φ^{-1} are smooth. Let (U,φ) be a chart around a point p. This induces a chart $(\pi^{-1}(U),\varphi_T)$ on TM around (p,v) for any $v \in T_pM$. On $M \times \mathbb{R}^n$ we take $\varphi \times \mathrm{id}$ around $(p,(a_1,\dots,a_n))$. Then for any $((x_i),(b_j)) \in \varphi(U)\times\mathbb{R}^n$, let $q = \phi^{-1}(x_i)$, so

$$(\varphi \times \mathrm{id}) \circ \Phi \circ \varphi_T^{-1}((x_i),(b_j)) = (\varphi \times \mathrm{id}) \circ \Phi \left(q, \left(b_j \frac{\partial}{\partial x^j} \right) \right).$$

Write $X_i = \sum X_{ij} \frac{\partial}{\partial x^j}$, where the X_{ij} depend on q and are smooth since X is smooth. Write $b_j \frac{\partial}{\partial x^j}$ as $a_i X_i(q)$ with respect to the basis $\{X_1(q),\dots,X_n(q)\}$, we have $b_j = \sum a_i X_{ij}$. Let (c_{ij}) be the inverse matrix of (X_{ij}), so that $a_i = \sum c_{ij} b_j$. Then c_{ij} depends smoothly on q and

$$(\varphi \times \mathrm{id}) \circ \Phi \left(q, \left(b_j \frac{\partial}{\partial x^j} \right) \right) = (\varphi \times \mathrm{id})(q,(c_{ij}b_j)) = ((x_i),(c_{ij}b_j)). \qquad \square$$

Parallelizability is special! It can be shown that every even dimensional sphere S^{2n} is not parallelizable. Lie groups are always parallelizable thanks to the left-regular representation.

Lie bracket of vector fields

Let $V(M)$ denote the space of all smooth vector fields on M. For $X, Y \in V(M)$, we will define the *Lie bracket* $[X,Y] \in V(M)$. But first, for $p \in M$ consider that $X(p) \in T_pM$, so we may regard X as an operator on $C^\infty(M,\mathbb{R})$, acting as

$$X(f)(p) = \widehat{X(p)}(f) = df_p(X(p)).$$

We say that X is a *first order linear operator*, and in any fixed local coordinates (U,φ) X acts as

$$X(f)(p) = a_i(p) \frac{\partial}{\partial x^i}(f \circ \varphi^{-1}),$$

or $X(p) = a_i(p) \frac{\partial}{\partial x^i}$.

1.3.6 Definition. Let $X(p) = a_i(p)\frac{\partial}{\partial x^i}$ and $Y(p) = b_i(p)\frac{\partial}{\partial x^i}$ be vector fields and define the *Lie bracket* of X and Y by its action on $C^\infty(M,\mathbb{R})$ as

$$[X,Y](f) = X(Y(f)) - Y(X(f)).$$

Notice that since f is smooth the mixed partials are equal, and

$$\begin{aligned}[X,Y](f) &= X(Y(f)) - Y(X(f)) \\ &= a_j \frac{\partial}{\partial x^j}\left(b_i \frac{\partial}{\partial x^i}(f\circ\varphi^{-1})\right) - \left(b_j \frac{\partial}{\partial x^j}\left(a_i \frac{\partial}{\partial x^i}(f\circ\varphi^{-1})\right)\right) \\ &= a_j \frac{\partial b_i}{\partial x_j}\frac{\partial}{\partial x^i}(f\circ\varphi^{-1}) + a_j b_i \frac{\partial^2}{\partial x_j \partial x_i}(f\circ\varphi^{-1}) \\ &\quad - \left(b_j \frac{\partial a_i}{\partial x_j}\frac{\partial}{\partial x^i}(f\circ\varphi^{-1}) + b_j a_i \frac{\partial^2}{\partial x_j \partial x_i}(f\circ\varphi^{-1})\right) \\ &= \left(a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j}\right)\frac{\partial}{\partial x^i}(f\circ\varphi^{-1}).\end{aligned}$$

Therefore $[X,Y]$ is also a first order linear operator, and we are justified in writing

$$[X,Y] = (a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j})\frac{\partial}{\partial x^i}.$$

1.3.7 Properties of Lie bracket. *For all vector fields $X,Y,Z \in V(M)$,*

(i) $[\cdot,\cdot]$ *is bilinear;*

(ii) $[X,Y] = -[Y,X]$ *(anti-commutative);*

(iii) $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$ *(Jacobi identity).*

The vector space $V(M)$ with $[\cdot,\cdot]$ is an example of a *Lie algebra*, (i.e. a vector space with a product satisfying (i), (ii), and (iii)).

Left invariant vector fields on Lie groups

1.3.8 Definition. Let G be a Lie group and $\mathfrak{g} = T_e G$. For $g \in G$, the *half-translation* by g is the map

$$L_g : G \to G : h \mapsto hg.$$

Half-translations are diffeomorphisms.

1.3.9 Proposition. *Any Lie group is parallelizable.*

Proof: Notice that

$$(dL_g)_e : T_e G = \mathfrak{g} \to T_g G,$$

so given $\xi \in \mathfrak{g}$ we may define a vector field $X_\xi(g) = (dL_g)_e(\xi)$. Then $X_\xi : G \to TG$ is a smooth vector field. If $\{\xi_1, \dots \xi_n\}$ is a basis of $\mathfrak{g}$ (where $n = \dim G$) then $\{X_{\xi_1}, \dots, X_{\xi_n}\}$ is a basis at every $g \in G$. By 1.3.5, G is parallelizable. □

Now consider

$$(dL_g)_h(X_\xi(h)) = (dL_g)_h((dL_g)_e(\xi)) = d(L_g \circ L_h)_e = d(L_{gh})e(\xi) = X_\xi(gh)$$

for $h \in G$, since $L_g \circ L_h(x) = L_g(hx) = ghx = L_{gh}(x)$. Therefore

$$(dL_g)_h(X_\xi(h)) = X_\xi(L_g(h)).$$

1.3.10 Definition. A *left-invariant vector field* is a vector field satisfying... Let $\ell(G)$ denote the set of all left-invariant vector fields. (they all arise as X_ξ for some ξ).

1.3.11 Theorem. *$(\ell(G),[\cdot,\cdot])$ is a Lie algebra for any Lie group G.*

PROOF: We have $\mathfrak{g} \to \ell(G) \hookrightarrow V(M) : \xi \mapsto X_\xi$, so we will show that $\ell(G)$ is a Lie subalgebra of $V(M)$.

For $\xi, \eta \in \mathfrak{g}$, $[X_\xi, X_\eta] \in \ell(G)$. Indeed, let $f \in C^\infty(G,\mathbb{R})$.

$$\begin{aligned}
(dL_g)([X_\xi,X_\eta])(f) &= [X_\xi,X_\eta](f \circ L_g) \\
&= X_\xi(X_\eta(f \circ L_g)) - X_\eta(X_\xi(f \circ L_g)) \\
&= X_\xi(dL_g(X_\eta)(f)) - X_\eta(dL_g(X_\xi)(f)) \\
&= X_\xi(X_\eta(f)) - X_\eta(X_\xi(f)) \\
&= [X_\xi,X_\eta](f)
\end{aligned}$$

Recalling for $F : M \to N$, $p \in M$, $v \in T_pM$, and $f \in C^\infty(M,\mathbb{R})$ we have

$$dF_p(v)(f) = df_{F(p)} \circ dF_p(v) = d(f \circ F)(v) = v(f \circ F)$$

□

1.4 Submanifolds

1.4.1 Definition. Let M and N be two manifolds. Suppose that $N \subset M$, and let $i : N \to M$ be the inclusion map. We say that N is an *embedded submanifold* if

(i) i is smooth;

(ii) $di_p : T_pN \to T_{i(p)}M$ is one-to-one for all $p \in N$ (i.e. i is an *immersion*); and

(iii) i is a homoeomorphism onto its image (i.e. $D \subseteq N$ is open in N if and only if $D = N \cap E$ for some open $E \subseteq M$).

1.4.2 Examples.

(i) Take $M = T^2$, the 2-dimensional torus, and $N = \mathbb{R}$, embedded as a line of irrational slope. Then N is dense in M, so (iii) fails though (i) and (ii) hold.

(ii) Take $M = \mathbb{R}^2$ and $N = \mathbb{R}$ embedded in the shape of a "ρ", but with an open end instead of a double point (insert a diagram). Again (iii) fails but (i) and (ii) hold.

We would like to decide when a set of equations define a submanifold.

1.4.3 Definition. Let $f : M \to N$ be a smooth map between two manifolds. Then f is a *submersion* if $df_p : T_PM \to T_{f(p)}N$ is onto for all $p \in M$. It is an *immersion* if df_p is one-to-one. A point $q \in N$ is a *regular value* if $df_p : T_pM \to T_qN$ is onto for every $p \in f^{-1}(|[q\})$ (i.e. f is a submersion for every p in the preimage of q).

1.4.4 Example. Let $\pi : \mathbb{R}^k \to \mathbb{R}^\ell : (a_1, \dots, a_k) \mapsto (a_1, \dots, a_\ell)$ be the canonical projection (where $k \geq \ell$). Then π is a submersion, the *canonical submersion*.

1.4.5 Theorem (Inverse Function Theorem). *If $f : M \to N$ be a smooth map between manifolds then f is a local diffeomorphism at p if and only if $df_p : T_pM \to T_{f(p)}N$ is a linear isomorphism.*

(f is a *local diffeomorphism* at p if there is an open neighbourhood U of p such that $f|_U : U \to f(U)$ is a diffeomorphism.)

PROOF: Use charts plus the inverse function theorem in $\mathbb{R}^n$. □

1.4.6 Theorem (Preimage Theorem). *Let $f : M \to N$ be a smooth map between manifolds. If $q \in N$ is a regular value, then $f^{-1}(\{q\})$ (when non-emtpy) is an embedded submanifold of M of dimension* $\dim M - \dim N$.

PROOF: The theorem will follow from the "local form of submersions:"

Claim. Let $f : M \to N$, where $\dim M = k$ and $\dim N = \ell$. If f is a submersion at p then there are charts (U, φ) around p and (V, ψ) around $f(p)$ such that $\psi \circ f \circ \varphi^{-1} = \pi$, the canonical submersion.

We have the picture

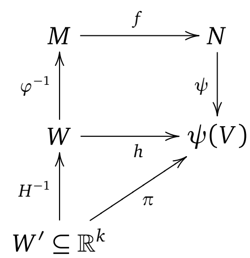
$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & N \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \psi} \\ \varphi(U) \subseteq \mathbb{R}^k & \xrightarrow[h]{} & \psi(V) \subseteq \mathbb{R}^\ell \end{array}$$

where $h = \psi \circ f \circ \varphi^{-1}$. Without loss of generality assume that $\varphi(p) = 0$, $\psi(f(p)) = 0$, and $h(0) = 0$. Then $dh_0 : \mathbb{R}^k \to \mathbb{R}^\ell$ is onto since f is a submersion at p. By a linear change of coordinates we may suppose that $dh_0 = [I_{\ell\times\ell}|0]_{\ell\times k}$. Define

$$H : \varphi(U) \to \mathbb{R}^k : (a_1, \dots, a_k) \mapsto (h(a_1, \dots, a_k), a_{\ell+1}, \dots, a_k).$$

Then by construction $dH_0 : \mathbb{R}^k \to \mathbb{R}^k$ is the identity map. Therefore H is a local diffeomorphism at 0, so there is a neighbourhood $W \subseteq \varphi(U)$ around 0 such that

$H|_W : W \to W' \subseteq \mathbb{R}^k$ is a diffeomorphism.



where $\pi := h \circ H^{-1}$ truly is the canonical submersion. Notice that $(\varphi^{-1}(W), H \circ \varphi)$ is a chart around p such that

$$\psi \circ f \circ (H \circ \varphi)^{-1} = \psi \circ f \circ \varphi^{-1} \circ H^{-1} = h \circ H^{-1} = \pi,$$

so the claim is proved.

To complete the proof, note that with the new charts (namely $(\varphi^{-1}(W), H \circ \varphi)$ and (V, ψ), making f into a canonical submersion) $x_{\ell+1}, \dots, x_k$ is coordinate system for $f \in (q)$ around p. $f^{-1}(q)$ is locally given by $x_1 = \cdots = x_\ell = 0$. (Check that this really finishes the proof.) □

1.4.7 Theorem (Whitney Embedding Theorem). *Any smooth manifold of dimension n can be embedded in $\mathbb{R}^{2n}$.*

2 Forms and Bundles

2.1 Differential Forms

Facts from multilinear algebra

Let V be a vector space over $\mathbb{R}$ of dimension n.

2.1.1 Definition. $A^p(V)$ is the set of alternating multilinear p-forms on V, i.e. $\omega \in A^p(V)$ if

$$\omega : \underbrace{V \times \cdots \times V}_{p \text{ copies}} \to \mathbb{R},$$

ω is linear in each entry, and

$$\omega(x_1, \dots, x_p) = \mathrm{sg}(\sigma)\omega(x_{\sigma(1)}, \dots, x_{\sigma(p)})$$

for all $\sigma \in \mathfrak{S}_n$.

There is a *wedge product* (or *exterior product*) $A^p(V) \times A^q(V) \to A^{p+q}(V)$, defined by

$$(\omega \wedge \eta)(x_1, \dots, x_{p+q}) = \sum_{\sigma} \mathrm{sg}(\sigma)\omega(x_{\sigma_1}, \dots, x_{\sigma_p})\eta(x_{\sigma_{p+1}}, \dots, x_{\sigma_{p+q}})$$

where σ runs over all permutations of $\{1, 2, \dots, p+q\}$ such that $\sigma_1 < \cdots < \sigma_p$ and $\sigma_{p+1} < \cdots < \sigma_{p+q}$.

We have $A^0(V) = \mathbb{R}$ and $A^1(V) = V^*$ (the dual vector space).

2.1.2 Properties of Wedge Product.

(i) *It is bilinear and associative;*

(ii) *If* $\omega_1, \dots, \omega_k \in A^1(V)$ *then* $(\omega_1 \wedge \cdots \wedge \omega_k)(x_1, \dots, x_k) = \det(\omega_i(x_j))$;

(iii) *If* $\omega_1, \dots, \omega_n$ *is a basis of* V^* *then* $\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \mid i_1 < \cdots < i_p\}$ *is a basis of* $A^p(V)$;

(iv) $\dim A^p(V) = \binom{n}{p}$

(v) *If* $\omega \in A^p(V)$ *and* $\eta \in A^q(V)$, *then* $\omega \wedge \eta = (-1)^{pq} \eta \wedge \omega$;

(vi) *In particular, if* p *is odd then* $\omega \wedge \omega = 0$, *and specifically, the wedge product is anti-commutative on 1-forms.*

Remark. In the case $p = n$, so-called top dimensional forms, $\dim A^n(V) = 1$, so the choice of a basis is really a choice of determinant (or *volume form*).

2.1.3 Definition. Let $T : V \to W$ be linear. There is a natural map $T^* : A^p(W) \to A^p(V)$ defined by

$$(T^*\omega)(x_1, \dots, x_p) = \omega(Tx_1, \dots, Tx_p),$$

the *pullback* of T.

The pullback interacts nicely with the wedge product. Indeed,

$$T^*(\omega \wedge \eta) = T^*\omega \wedge T^*\eta.$$

Remark. For $T : V \to V$ with $n = \dim V$ and $\omega \in A^n(V)$, we have $T^*\omega = \lambda\omega$ where $\lambda = \det(T)$.

Back to manifolds

Let M be a manifold and (U, φ) be a chart giving a basis $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ of T_pM, for $p \in U$. The *dual basis* is $\{dx_1, \dots, dx_n\}$, a basis of T_p^*M. The notation is not accidental, $dx_i : T_p(M) \to \mathbb{R}$ really is the differential of the coordinate functions $x_i : U \to \mathbb{R}$, and we have $dx_i \frac{\partial}{\partial x^j} = \delta_{ij}$ (check).

2.1.4 Definition. A *differential p-form* on M is a function $x \mapsto \omega_x$, where $\omega_x \in A^p(T_xM)$, such that if (U, φ) is a chart and we write

$$\omega = \sum_{i_1 < \cdots < i_p} f_{i_1, \dots, i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

then the functions $f_{i_1, \dots, i_p}$ are smooth. The space of all p-forms will be denoted by $\Omega^p(M)$ (in particular $\Omega^0(M) = C^\infty(M, \mathbb{R})$).

Contrast the above definition with the definition of a vector field. We could have instead given $\amalg_{x\in M} A^p(T_xM) = A^p(M)$ the structure of a vector bundle over M and defined p-forms to be sections of this bundle.

Remark.

(i) We have a *wedge product* of differential forms: if $\omega \in \Omega^p(M)$ and $\eta \in \Omega^q(M)$ then $\omega \wedge \eta \in \Omega^{p+q}(M)$.

(ii) Smooth functions are 0-forms, so they may be wedged with p-forms. We will not use such pompous notation as "$f \wedge \omega$", and simply write $f\omega$.

2.1.5 Definition. Let $f : M \to N$ be a smooth map. Then there is a map $f^* : \Omega^p(N) \to \Omega^p(M)$, the *pullback* of f, defined by

$$(f^*\omega)_x(v_1, \dots, v_p) = \omega_{f(x)}(df_x(v_1), \dots, df_x(v_p)).$$

As in the case of the pullback of a linear map, $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$.

Exterior differentiation

2.1.6 Theorem. *There exists a unique linear operator $d : \Omega^p(M) \to \Omega^{p+1}(M)$ ($p \geq 0$) such that*

(i) if $f \in \Omega^0(M)$ then df coincides with the differential of f;

(ii) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg \omega} \wedge d\eta$ (the Leibniz rule);

(iii) $d(d\omega) = 0$ for all $\omega \in \Omega^p(M)$ (the chain condition).

PROOF (SKETCH): For smooth functions f we are forced to take $df = \frac{\partial f}{\partial x_i} dx_i$. Define

$$\begin{aligned} d(f\,dx_{i_1} \wedge \cdots \wedge dx_{i_p}) &= df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} + (-1)^{\deg f} f\,d(dx_{i_1} \wedge \cdots \wedge dx_{i_p}) \\ &= \frac{\partial f}{\partial x_i} dx_i dx_{i_1} \wedge \cdots \wedge dx_{i_p}, \end{aligned}$$

and extend d linearly to all p-forms.

Let's see why $d(d\omega) = 0$.

$$d\left(\frac{\partial f}{\partial x_i} dx_i dx_{i_1} \wedge \cdots \wedge dx_{i_p}\right) = \frac{\partial^2 f}{\partial x_i \partial x_j} dx_j \wedge dx_i dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

In the double sum over i, j there a lot of cancellation since f is smooth, $dx_j \wedge dx_i = -dx_i \wedge dx_j$ and $dx_i \wedge dx_i = 0$. In fact, nothing remains. Check that d is unique (which is clear since we were forced to define it the way we did) and well-defined. □

Consider the *de Rham complex*

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^p(M) \xrightarrow{d} \Omega^{p+1}(M) \xrightarrow{d} \dots$$

A p-form is *closed* if $d\omega = 0$, and *exact* if there is η such that $\omega = d\eta$. Notice that exact implies closed. The *de Rham cohomology* is

$$H^p_{dR}(M) = \frac{\{\omega \in \Omega^p(M) \mid d\omega = 0\}}{\{d\eta \mid \eta \in \Omega^{p-1}(M)\}}.$$

2.1.7 Example. Consider $M = \mathbb{R}^3$. Then a 1-form is something of the form $\omega = f\,dx + g\,dy + h\,dz$, so 1-forms are in one-to-one correspondence with vector fields $F = (f, g, h)$. A 2-form is something of the form

$$\eta = f\,dy \wedge dz + g\,dz \wedge dx + h\,dx \wedge dy,$$

so they are also in one-to-one correspondence with vector fields. Of course, a 3-form is just a multiple of the determinant. Now

$$\begin{aligned} d\omega &= \left(\tfrac{\partial f}{\partial x}dx + \tfrac{\partial f}{\partial y}dy + \tfrac{\partial f}{\partial z}dz\right) + \left(\tfrac{\partial g}{\partial x}dx + \tfrac{\partial g}{\partial y}dy + \tfrac{\partial g}{\partial z}dz\right) + \left(\tfrac{\partial h}{\partial x}dx + \tfrac{\partial h}{\partial y}dy + \tfrac{\partial h}{\partial z}dz\right) \\ &= \left(\tfrac{\partial g}{\partial x} - \tfrac{\partial f}{\partial y}\right) dx \wedge dy + \left(\tfrac{\partial f}{\partial z} - \tfrac{\partial h}{\partial x}\right) dz \wedge dx + \left(\tfrac{\partial h}{\partial y} - \tfrac{\partial g}{\partial z}\right) dy \wedge dz \end{aligned}$$

Therefore $d\omega$ corresponds exactly to the curl of F. Similarly,

$$d\eta = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\right) dx \wedge dy \wedge dz$$

so $d\eta$ corresponds to the divergence of F. (I really wish that I had taken vector calculus.)

2.2 Orientability and Integration

Orientable manifolds

Suppose that $h : U \to V$ is a diffeomorphism between open sets in $\mathbb{R}^n$. Then we can write coordinates for the tangent space to V as $y_j = h_j(x_1, \dots, x_n)$, where $x_1, \dots, x_n$ are coordinates for the tangent space of U. Then

$$h^*(dy_1 \wedge \dots \wedge dy_n) = \det(dh_x) dx_1 \wedge \dots \wedge dx_n.$$

(Recall that $\det(dh_x) = \det(\frac{\partial y_i}{\partial x_j})$ is the Jacobian.)

2.2.1 Theorem (Orientability). *Let M be an n-manifold. The following are equivalent.*

(i) There exists a nowhere vanishing smooth differential n-form.

(ii) There is a family of charts that cover M such that the determinant of the Jacobian of coordinate changes is positive on all overlaps.

(iii) $A^n(TM)$ is isomorphic to $M \times \mathbb{R}$.

2.2.2 Definition. M is *orientable* if it satisfies any (and hence all) of the three equivalent conditions.

PROOF: (i) if and only if (iii): Proved exactly as 1.3.5.

(i) implies (ii): Consider all charts (U, φ) such that $dx_1 \wedge \cdots \wedge dx_n = f\Omega$, where $f > 0$ and Ω is the non-vanishing n-form. We can cover M with such charts and if (U, φ) and (V, ψ) are two overlapping charts then

$$dx_1 \wedge \cdots \wedge dx_n = \det\left(\frac{\partial x_i}{\partial x'_j}\right) dx'_1 \wedge \cdots \wedge dx'_n$$

implies that $\det(\frac{\partial x_i}{\partial x'_j}) > 0$.

(ii) implies (i): We need to introduce partitions of unity. □

2.2.3 Theorem (Partitions of unity). *For any open cover $\bigcup_{\alpha \in A} U_\alpha = M$ there exists a countable collection of functions $\rho_i \in C^\infty(M, \mathbb{R})$ $(i = 1, 2, \dots)$ such that*

(i) for any i, the support $\overline{\{x \mid \rho_i(x) \neq 0\}}$ of ρ_i is compact and contained in some U_α;

(ii) the collection is locally finite: every point $x \in M$ has a neighbourhood W_x such that $\rho_i \neq 0$ on W_x only for finitely many i's;

(iii) $\rho_i \geq 0$ on M, and $\sum_i \rho_i(x) = 1$ for all $x \in M$.

The collection $\{\rho_i\}$ is a *partition of unity*, and is said to be *subordinate* to the cover $\{U_\alpha\}$.

PROOF: Deferred. □

PROOF (OF ORIENTABILITY, CONTINUED):
(ii) implies (i): Cover M with compatible charts $\{(U_\alpha, \varphi_\alpha)\}$ (i.e. the determinant of the Jacobian is positive on overlaps). Take a partition of unity $\{\rho_i\}$ subordinate to $\{U_\alpha\}$. Define $\omega_i = dx_1^i \wedge \cdots \wedge dx_n^i$, where the i indicates the (a?) chart that contains the support of ρ_i. Then $\rho_i \omega_i$ is a smooth form defined everywhere on M. Let $\Omega = \sum_i \rho_i \omega_i$, a smooth form on M since the sum is finite around any point. Ω is nowhere vanishing because of the third property of partitions of unity and the fact that the Jacobian is positive on overlaps. □

Integration of n-forms

2.2.4 Definition. Let M be an orientable n-manifold. Let $\omega \in \Omega^n(M)$ have compact support (i.e. $\overline{\{x \mid \omega_x \neq 0\}}$ is compact).

(i) If $M = U \subseteq \mathbb{R}^n$ is an open set then $\omega = f\,dx_1 \wedge \cdots \wedge dx_n$, and define

$$\int_U \omega := \int_U f(x_1, \dots, x_n) dx_1 \dots dx_n,$$

where the latter is the Riemann integral.

(ii) Suppose that the support of ω is contained in U, for some chart (U, φ). Then define

$$\int_M \omega := \int_{\varphi(U)} (\varphi^{-1})^* \omega.$$

(iii) Consider a partition of unity $\{\rho_i\}$ subordinate to a cover of M by compatible charts. Define

$$\int_M \omega := \sum_i \int_M \rho_i \omega.$$

The sum is finite since ω has compact support.

Suppose that $h : V \to U$ is a diffeomorphism (where $V \subseteq \mathbb{R}^n$ is open). Then

$$h^* \omega = h^*(f\,dx_1 \wedge \cdots \wedge dx_n) = (f \circ h) \det(dh) dy_1 \wedge \cdots \wedge dy_n.$$

Hence

$$\int_V h^* \omega = \int_V (f \circ h) \det(dh) dy_1 \dots dy_n,$$

and if $\det(dh) > 0$ then the change of variables formula implies that $\int_V h^* \omega = \int_U \omega$. In this case we say that h is *orientation preserving*.

In the second part of the definition, we need to check that if (V, ψ) is another chart containing the support of ω then the two possible definitions of the integral agree. Assume without loss of generality that $U = V$. Then if $h = \psi \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is the change of coordinates, we have

$$\int_{\psi(V)} (\psi^{-1})^* \omega = \int_{\varphi(U)} h^*((\psi^{-1})^* \omega) = \int_{\varphi(U)} (\psi^{-1} \circ h)^* \omega = \int_{\varphi(U)} (\varphi^{-1})^* \omega$$

In the third part one needs to check that the definition is independent of the partition of unity.

Stoke's theorem

Let M be an oriented n-manifold and ω an n-form on M.

Notation. We will often write M^n to remind us that M is an n-manifold. The notation *does not* mean the Cartesian product of M with itself n times.

2.2.5 Definition. $N \subseteq M^n$ is a *domain with smooth boundry* (or a *codimension zero submanifold with boundry*) if for all $p \in N$ there is a chart (U, φ) around p in M such that
$$\varphi(U \cap N) = \varphi(U) \cap \mathbb{R}^n_-. \qquad (*)$$
In this case we define
$$\partial N = \{p \in N \mid \varphi \text{ satisfies } (*) \text{ and } \varphi(p) \in \partial \mathbb{R}^n_-\},$$
the *boundry* of N, where
$$\mathbb{R}^n_- = \{x \in \mathbb{R}^n \mid x_1 \leq 0\} \qquad \text{and} \qquad \partial \mathbb{R}^n_- = \{x \in \mathbb{R}^n \mid x_1 = 0\}.$$

Remark. The boundry of N is an embedded submanifold of M. Indeed, we have the map
$$\varphi|_{U \cap \partial N} : U \cap \partial N \to \varphi(U) \cap \partial \mathbb{R}^n_-,$$
and $\partial \mathbb{R}^n_- = \mathbb{R}^{n-1}$, so it is a chart.

The orientation of M induces an orientation on ∂N. By flipping the sign of x_2 if necessary, we may cover M with charts that are positively oriented and satisfying $(*)$. The restricted charts from the remark above give an orientation to ∂N.

2.2.6 Definition. For $p \in \partial N$ and $x \in T_pM$, we say that v is *outward directed* if $d\varphi_p(v)$ has positive first coordinate. A basis $\{v_2 \dots, v_n\}$ of $T_p(\partial N)$ is *positively oriented* if and only if $\{v, \dots, v_n\}$ is a postively oriented basis of T_pM, where v is any outward directed vector.

Check that this is a well-definition.

2.2.7 Theorem (Stoke's Theorem).
Let N be a domain with smooth boundry in an oriented smooth n-manifold M. Let ∂N have the induced orientation. For every $\omega \in \Omega^{n-1}(M)$ with $N \cap supp(\omega)$ compact, we have
$$\int_{\partial N} i^* \omega = \int_N d\omega,$$
where $i : \partial N \to M$ is the inclusion map.

PROOF: Suppose first that we can prove the theorem when

(i) $\text{supp}(\omega) \subseteq U$; where

(ii) (U, φ) is a positively oriented chart satisfying $(*)$.

Then cover M with positively oriented charts satisfying $(*)$ and take a partition of unity $\{\rho_i\}$ subordinate to this covering. We have

$$\int_N d\omega = \int_N d\sum_i \rho_i\omega = \sum_i \int_N d(\rho_i\omega) = \sum_i \int_{\partial N} \rho_i\omega = \int_{\partial N} \sum_i \rho_i\omega = \int_{\partial N} \omega.$$

Now let $\eta \in \Omega_c^{n-1}(\mathbb{R}^n)$ be the $(n-1)$-form that is $(\varphi^{-1})^*\omega$ in $\varphi(U)$ and zero outside of $\varphi(U)$. Then

$$\int_{\partial N} \omega = \int_{\varphi(U)\cap\partial\mathbb{R}^n_-} (\varphi^{-1})^*\omega = \int_{\partial\mathbb{R}^n_-} \eta,$$

while

$$\int_N d\omega = \int_{\varphi(U)\cap\mathbb{R}^n_-} d\omega = \int_{\mathbb{R}^n_-} d\eta.$$

Therefore it suffices to prove the theorem when $M = \mathbb{R}^n$, $N = \mathbb{R}^n_-$, and $\omega \in \Omega_c^{n-1}(\mathbb{R}^n)$.

Suppose

$$\omega = \sum_{i=1}^n f_i(x)dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge d_n.$$

Choose $b > 0$ such that $\operatorname{supp}(f_i) \subseteq [-b, b]^n$ for $i = 1, \ldots, n$. Then

$$i^*\omega = \omega|_{\partial\mathbb{R}^n_-} = f_1(0, x_2, \ldots, x_n)dx_2 \wedge \cdots \wedge dx_n,$$

so

$$\int_{\partial\mathbb{R}^n_-} \omega = \int f_1(0, x_2, \ldots, x_n)dx_2 \ldots dx_n.$$

On the other hand,

$$d\omega = \sum_{i=1}^n (-1)^{i-1}\frac{\partial f_i}{\partial x_i}dx_1 \wedge \cdots \wedge dx_n,$$

so

$$\int_{\mathbb{R}^n} d\omega = \sum_{i=1}^n (-1)^{i-1}\int_{\mathbb{R}^n_-} \frac{\partial f_i}{\partial x_i}dx_1 \wedge \cdots \wedge dx_n.$$

When $2 \leq i \leq n$, we have

$$\int_{-\infty}^{\infty} \frac{\partial f_i}{\partial x_i}(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)dt = 0,$$

so

$$\int_{\mathbb{R}^n_-} \frac{\partial f_i}{\partial x_i}dx_1 \wedge \cdots \wedge dx_n = 0.$$

When $i = 1$,

$$\int_{-\infty}^{0} \frac{\partial f_1}{\partial x_1}(t, x_2, \ldots, x_n)dt = f_1(0, x_2, \ldots, x_n),$$

so

$$\int_{\mathbb{R}^n_-} \frac{\partial f_1}{\partial x_1} dx_1 \wedge \cdots \wedge dx_n = \int f_1(0, x_2, \ldots, x_n) dx_2 \wedge \cdots \wedge dx_n,$$

and the theorem is proved. □

2.3 Metrics

Riemannian metrics

2.3.1 Definition. Let M be an n-manifold. A *Riemannian metric* on M is a function $g : x \in M \mapsto g_x$, where g_x is a positive definite inner product on T_xM and such that, for any chart (U, φ), the functions $\{g_{ij}(x) = g_x(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \mid i, j = 1, \ldots, n\}$ are smooth. In this case the pair (M, ω) is a *Riemannian manifold*.

Remark.

(i) There are variations on this definition e.g. a semi-Riemannian metric requires only that the metric be non-degenerate.

(ii) We could also say that g is a smooth section of the bundle of symmetric bilinear forms with the positivity property.

(iii) In the case when $n = 2$ and M is a surface, in classiscal notation we write $g = \left[\begin{smallmatrix} E & F \\ F & G \end{smallmatrix}\right]$ or as $g_{ij} dx^i dx^j$.

A metric induces natural isomorphisms (though they depend on the metric chosen),

$$\mathscr{L}_g : TM \to T^*M \qquad \text{and} \qquad \mathscr{L}_g^{-1} : T^*M \to TM,$$

the *Legendre transform*, defined by $\mathscr{L}_g : (x, v) \mapsto (x, w \mapsto g_x(v, w))$.

Remark. Sometimes we write $\mathscr{L}_g = \flat$ and $\mathscr{L}_g^{-1} = \sharp$ for these isomorphisms and call them the *musical isomorphisms*. The reason for this notation is from physics, as in abstract index notation $\flat$ lowers indices and $\sharp$ raises indices. (If $v = v^i \frac{\partial}{\partial x^i}$ and $p = p_i dx^i$ then $p_i = g_{ij} v^j$, and $v^j = g^{ij} p_i$, where g^{ij} is the inverse of g_{ij}.)

Notation. For $v \in T_xM$, by convention we write $g_x(v, v) = |v|_x^2 = |v|^2$.

Symplectic forms

2.3.2 Definition. Let M be an n-manifold. A *symplectic form* is a non-degenerate smooth closed 2-form, i.e. $\omega \in A^2(TM)$, $d\omega = 0$, such that ω_x is a non-degenerate bilinear form for all $x \in M$. The pair (M, ω) is a *symplectic manifold*.

For an $n \times n$ matrix A, $A^t = -A$ implies that

$$\det A = \det(-A) = (-1)^n \det A,$$

so if A is invertible then n is even. Hence if (M, ω) is a symplectic manifold then M is even dimensional.

2.3.3 Example. Let $M = \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ (coordinates (q_i, p_j)). The canonical symplectic form on M is $\omega = \sum_{i=1}^n dq_i \wedge dp_i$. (Check this, take the derivative, etc.)

If ω is a symplectic form on M^{2n}, then $\omega \wedge \cdots \wedge \omega = \omega^n$ is a top-dimensional form. Hence if ω is non-degenerate then ω^n is never zero (prove this). This implies in particular that M is orientable. Any orientable surface is automatically a symplectic manifold (use the form that shows orientability).

Suppose now that (M^{2n}, ω) is a compact symplectic manifold. Then ω is *not* exact. Indeed, if $\omega = d\eta$ for some smooth 1-form η, then ω^n is a volume form, so $\int_M \omega^n \neq 0$ (in fact it will be greater than zero). But

$$d(\eta \wedge w^{n-1}) = d\eta \wedge \omega^{n-1} \pm \eta \wedge d(\omega^{n-1}) = \omega^n$$

since ω is closed (indeed, exact), so ω^n is exact. Applying Stoke's theorem,

$$\int_M \omega^n = \int_M d(\eta \wedge w^{n-1}) = 0$$

since M has no boundry in itself. Recall that $H^2_{dR}(M)$ is the quotient of the closed 2-forms by the exact 2-forms, so in the case where (M, ω) is a compact symplectic manifold $0 \neq [\omega] \in H^2_{dR}(M)$.

2.3.4 Theorem (de Rham). $H^k_{dR}(M) \cong H^k(M, \mathbb{R})$ *(topological cohomology).*

In the case of S^4, $H^2(S^4, \mathbb{R}) = 0$, so there is no symplectic form on S^4 and it cannot be made into a symplectic manifold.

2.3.5 Example. The cotangent bundle is a very important example of a symplectic manifold. Let N be any manifold and $M = T^*N$. A *canonical 1-form* λ is defined by

$$\lambda_{(x,p)}(\xi) = p(d\pi_{(x,p)}(\xi))$$

(recall that $\pi : T^*N \to N$ is the foot-point projection, so

$$d\pi_{(x,p)} : T_{(x,p)}(T^*N) \to T_xN).$$

With local coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$ on T^*N, we have $\lambda = p_i dq^i$. Locally take $\omega = -d\lambda = dq^i \wedge dp_i$. Then $(T^*N, -d\lambda)$ is a (non-compact) symplectic manifold.

2.3.6 Example (Classical mechanics). Let (M,ω) be a a symplectic manifold. For any smooth function $H : M \to \mathbb{R}$, dH is an exact 1-form. There exists a unique vector field X_H such that $dH_x(\cdot) = \omega_x(X_H(x),\cdot)$. This X_H is called the *Hamiltonian vector field* of H. If φ_t is the flow of X_H then this flow is the *Hamiltonian flow* of H.

Notice that

$$\frac{d}{dt}H(\varphi_t x) = dH_{\varphi_t x}(X_H(\varphi_t x)) = \omega_{\varphi_t x}(X_H(\varphi_t x), X_H(\varphi_t x)) = 0.$$

Therefore "energy is preserved along the gradients of the Hamiltonian flows."

Let (M,g) be a Riemannian manifold. For $\mathscr{L}_g(x,v) = (x,p)$, write $|p|_x =^{def} |v|_x$. Say $v = v^i \frac{\partial}{\partial x^i}$, $p = p_i dx^i$, so $|p|^2 = g^{ij}p_ip_j$ and $|v|^2 = g_{ij}v^iv^j$. Then define $H(x,p) = \frac{1}{2}|p|_x^2$ (the Hamiltonian) and let X_H be the vector field in T^*M, so that its integral curves are $\gamma : t \mapsto (x(t),p(t)) \in T^*M$, where $\dot{\gamma}(t) = X_H(\gamma(t))$. $t \mapsto x(t)$ is a *geodesic* curve in M.

Define a symplectic form $\tilde{\omega} = \mathscr{L}_g(-d\lambda)$ on TM. There is a canonical function $L : TM \to \mathbb{R} : (x,v) \mapsto \frac{1}{2}|v|_x^2$. There is an associated vector field on TM, with integral curves $\tilde{\gamma} : t \mapsto (x(t),v(t))$, with $\dot{\tilde{\gamma}}(t) = X_L(\tilde{\gamma}(t))$. Then $t\lambda x(t)$ is also a geodesic. (In fact, $v(t) = \dot{x}(t)$.)

(insert diagram)

Finally, potential forces are represented by smooth functions $V : M \to \mathbb{R}$ and incorportated by taking $H(x,p) = \frac{1}{2}|p|_x^2 + V(x)$ $(F = -\nabla V)$.

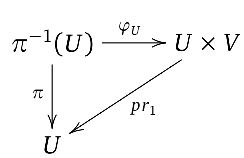
2.4 Bundles

Vector bundles

2.4.1 Definition. Let B be a smooth manifold. A manifold E together with a smooth submersion $\pi : E \to B$ is called a *vector bundle* of *rank* k over B if the following hold.

(i) There exists a k-dimensional vector space V (called the *typical fibre*) such that each fibre $E_p = \pi^{-1}(p)$ is a vector space isomorphic to V.

(ii) Any point $p \in B$ has a neighbourhood U such that there exists a diffeomorphism $\varphi_U : \pi^{-1}(U) \to U \times V$ which makes the diagram below commute.



(iii) $\varphi_U|_{E_p} : E_p \to V$ is a linear isomorphism.

B is called the *base space*, E is called the *total space*, φ_U is called a *trivialization*, and U is a *trivializing neighbourhood*.

2.4.2 Examples.

(i) The *trivial bundle* is $E = B \times V$ with $\pi(b, r) = b$.

(ii) Tangent bundles, cotangent bundles, exterior bundles $A^p(TM)$, ...

V may be an $\mathbb{R}$-vector space or a $\mathbb{C}$-vector space.

Suppose that $\{U_\alpha\}$ is a complete family of trivializing neighbourhoods, and $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$ are the corresponding trivializations. Notice that

$$\varphi_\beta \circ \varphi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times V \to (U_\alpha \cap U_\beta) \times V$$

has the form $\varphi_\beta \circ \varphi_\alpha^{-1}(b, v) = (b, \psi_{\beta\alpha}(b)v)$, where $\psi_{\beta\alpha} : U\alpha \cap U_\beta \to GL(V)$. Such $\psi_{\beta\alpha}$ is a *transition function*. Transition functions satisfy the *cocyle conditions*

(i) $\psi_{\alpha\alpha} = \mathrm{id}_{U_\alpha}$;

(ii) $\psi_{\alpha\beta}\psi_{\beta\alpha} = \mathrm{id}_{U_\alpha \cap U_\beta}$;

(iii) $\psi_{\alpha\beta}\psi_{\beta\gamma}\psi_{\gamma\alpha} = \mathrm{id}_{U_\alpha \cap U_\beta \cap U_\gamma}$.

In fact, given a collection of functions satisfying all the properties of transition functions, there is a unique bundle which has those functions as its transition functions.

2.4.3 Steenrod construction. *Let $E = \amalg_\alpha (U_\alpha \times \mathbb{R}^n)/\sim$, where $(x, v) \sim (y, w)$ if and only if $x = y$ and $w = \psi_{\beta\alpha}(x)v$, where $x \in U_\alpha$ and $y \in U_\beta$.*

Very often it will happen that $\psi_{\beta\alpha}$ takes values in subgroup $G < GL(V)$. When this happens we say that the bundle has *structure group G*.

2.4.4 Examples.

(i) For $E = B \times \mathbb{R}^n$, the transition function is $\psi_\alpha(x) = I_n$.

(ii) M is orientable if and only if the tangent bundle TM has structure group in $GL_+(k, \mathbb{R})$, linear maps with positive determinant.

(iii) The structure group of the Möbius band over S^1 is $\{\pm 1\} \cong \mathbb{Z}_2$.

(iv) $O(k)$, $SO(k)$, ...

Principal bundles

2.4.5 Definition. Let G be a Lie group and P be a smooth manifold. A *smooth action* (in this case a smooth *right* action) is just an action $P \times G \to P : (p, g) \mapsto pg$ which is also a smooth map. The action is a *free action* if $pg = p$ implies $g = 1$.

2.4.6 Definition. Let B be a smooth manifold. A *principal bundle* (or *principal G-bundle*) is a manifold P together with a smooth submersion $\pi : P \to B$ together with a smooth right free action satisfying the following conditions.

(i) $\pi(pg) = \pi(p)$ for all $p \in P$ and $g \in G$ (i.e. the fibres of π are the orbits of G);

(ii) for any $b \in B$ there is a neighbourhood U of b and a diffeomorphism $\varphi_U : \pi^{-1}(U) \to U \times G$ such that $pr_1 \circ \varphi_U = \pi|_{\pi^{-1}(U)}$;

(iii) the actions are "intertwined" by φ_U, i.e. for all $h \in G$, $\varphi_U(ph) = (b, gh)$, where $(b, g) = \varphi_U(p)$ and $\pi(p) = b \in U$.

As in the case of vector bundles, we have trivializing neighbourhoods and transition functions. Let $\{U_\alpha\}$ be a complete family of trivializing neighbourhoods, and consider the transition function defined by

$$\varphi_\beta \circ \varphi_\alpha^{-1}(b, g) = (b, \psi_{\beta\alpha}(b, g)).$$

As before, for each $b \in U_\alpha \cap U_\beta$, $\varphi_{\beta\alpha}(b, \cdot)$ is a map from G to G. Since the action is intertwined,

$$\psi_{\beta\alpha}(b, g)h = \psi_{\beta\alpha}(b, gh).$$

Taking $g = 1$, we have $\psi_{\beta\alpha}(b, 1)h = \varphi_{\beta\alpha}(b, h)$, so $\psi_{\beta,\alpha}(b, \cdot)$ (as a map from G to G) is left-translation on G by the element $\psi_{\beta\alpha}(b, 1)$. As is usual in geometry, we now simplify the notation by renaming

$$\varphi_{\beta\alpha} : U_\alpha \cap U_\beta \to G : b \mapsto \psi_{\beta\alpha}(b, 1),$$

the *transition function* associated with the principal bundle. Transition functions for principal bundles satisfy the cocycle conditions (namely, $\psi_{\alpha\beta} \circ \psi_{\beta\gamma} \circ \psi_{\beta\gamma} = \text{id}$). As in the case of vector bundles, if we know the transition functions then we can recover the bundle by the Steenrod construction. Take $P = \amalg_\alpha(U_\alpha \times G)/\sim$, where $(b, h) \sim (b', h')$ if and only if $b = b'$ and $h' = \psi_{\beta\alpha}(b)h$.

Suppose that $G \subseteq GL(k)$ (or, what amounts to the same thing, a representation of G in $GL(k)$) and we have a collection $\{\psi_{\beta\alpha} : U_\alpha \cap U_\beta \to G\}$ of transistion functions which satisfy the cocyle conditions. Then there is an associated vector bundle E and an associated principal bundle P. In this case E and P are said to be *associated bundles*

2.4.7 Definition. A *section* of a vector bundle E (resp. principal bundle P) is a smooth map $s : B \to E$ (resp. P) such that $\pi \circ s = \text{id}$.

2.4.8 Example (Hopf bundle). Let $B = \mathbb{CP}^1 = (\mathbb{C}^2 \setminus \{(0, 0)\})/\mathbb{C}^*$, the collection of (complex) lines through the origin in $\mathbb{C}^2$. We have charts $\mathbb{CP}^1 = U_1 \cup U_2$, where $U_i = \{[z_1 : z_2] \in \mathbb{CP}^1 \mid z_i \neq 0\}$, with coordinates $z : U_1 \to \mathbb{C} : [z_1 : z_2] \mapsto \frac{z_2}{z_1}$ and $\eta : U_2 \to \mathbb{C} : [z_1 : z_2] \mapsto \frac{z_1}{z_2}$. On $U_1 \cap U_2$ we have $\eta = \frac{1}{z}$.

Let E be the disjoint union of the (complex) lines through the origin in $\mathbb{C}^2$, a bundle over $\mathbb{CP}^1$ with trivial submersion $\pi : E \to \mathbb{CP}^1 : \ell \mapsto \ell$. E is the *tautological bundle* of $\mathbb{CP}^1$. A point in E will be written as $(\omega z_1, \omega z_2)$, with $|z_1|^2 + |z_2|^2 \neq 0$. Trivializations are

$$\varphi_1 : \pi^{-1}(U_1) \to U_1 \times \mathbb{C} : (\omega, \omega z) \mapsto ([1 : z], \omega\sqrt{1 + |z|^2})$$

and

$$\varphi_2 : \pi^{-1}(U_2) \to U_2 \times \mathbb{C} : (\omega\eta, \omega) \mapsto ([\eta : 1], \omega\sqrt{1 + |\eta|^2}).$$

It can be shown that, for $([1:z],\omega)\in(U_1\cap U_2)\times\mathbb{C}$,

$$\begin{aligned}\varphi_2\circ\varphi_1^{-1}([1:z],\omega) &= \varphi_2\left(\frac{\omega}{\sqrt{1+|z|^2}},\frac{\omega z}{\sqrt{1+|z|^2}}\right)\\ &= \varphi_2\left(\frac{|\eta|\omega}{\sqrt{1+|\eta|^2}},\frac{|\eta|\omega}{\eta\sqrt{1+|\eta|^2}}\right)\\ &= \left([\eta:1],\frac{|\eta|}{\eta}\omega\right)=\left([1:z],\frac{z}{|z|}\omega\right)\end{aligned}$$

Therefore $\psi_{21}([1:z])=\frac{z}{|z|}\in U(1)\subseteq\mathbb{C}^*$. This transition function gives a vector bundle or a principal bundle over $\mathbb{CP}^1$, both of which are called the *Hopf bundle*.

2.4.9 Example. S^3 embeds in $\mathbb{C}^2$ as $S^3=\{(z_1,z_2)\in\mathbb{C}^2\mid |z_1|^2+|z_2|^2=1\}$. For $e^{i\theta}\in U(1)$,

$$S^3\times U(1)\to S^3:((z_1,z_2),e^{i\theta})\mapsto(e^{i\theta}z_1,e^{i\theta}z_2)$$

is often called the *Hopf map*. The orbits are circles in S^3, and we have $S^3\xrightarrow{\pi}\mathbb{CP}^1=S^2$. This is an example of a non-trivial principal bundle. (I don't understand...)

Pullback bundles, morphisms, and automorphisms

For this section we will phrase everything in terms of vector bundles, but the definitions work, with appropriate modification, for principal bundles as well.

2.4.10 Definition. Let $E\xrightarrow{\pi}B$ be a vector bundle and $f:M\to B$ be a smooth map. Let $f^*E=\{(x,e)\in M\times E\mid\pi(e)=f(x)\}$, $F(x,e)=e$, and $\pi'(x,e)=x$. Then the following diagram commutes.

$$\begin{array}{ccc} f^*E & \xrightarrow{F} & E\\ \downarrow{\scriptstyle\pi'} & & \downarrow{\scriptstyle\pi}\\ M & \xrightarrow[f]{} & B\end{array}$$

f^*E is called the *pullback bundle* of E through f, a vector bundle over M.

From the point of view of transition functions, if

$$\psi_{\beta\alpha}:U_\alpha\cap U_\beta\to G\subseteq GL(V)\qquad\text{then}\qquad f^*\psi_{\beta\alpha}=\psi_{\beta\alpha}\circ f.$$

2.4.11 Definition. Suppose that $E\xrightarrow{\pi}B$ and $E'\xrightarrow{\pi'}B'$ are two vector bundles over the same vector space V and $f:B\to B'$ is smooth. A smooth map $F:E\to E'$ if a *bundle morphism* covering f if for any $b\in B$, F restricts to a linear map

$F|_{E_b} : E_b \to E'_{f(b)}$ and such that the following diagram commutes.

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ B & \xrightarrow[f]{} & B' \end{array}$$

If F is a diffeomorphism and $F|_{E_b} : E_b \to E'_{f(b)}$ is a linear isomorphism for all $b \in B$ then we say that F is a *bundle isomorphism*. An isomorphism $E \to E$ covering the identity map is a *bundle automorphism*. The groups of all autmorphisms is denoted $\mathrm{Aut}(E)$.

From a local point of view, take trivializing neighbourhoods U and U' such that $f(U) \subseteq U'$, $\varphi_U : \pi^{-1}(U) \to U \times V$, and $\varphi'_{U'} : (\pi')^{-1}(U') \to U' \times V$. Let $F_U = \varphi'_{U'} \circ F \circ \varphi_U^{-1}$, so that $F_U(b, v) = (f(b), h(b)v)$, where $h : U \to \mathscr{L}(V, V)$.

2.4.12 Example. Take $E = B \times V$, the trivial bundle. Then one chart suffices, so the local behavior determines the map, and we have $\mathrm{Aut}(E) = C^\infty(B, GL(V))$.

Remark.

(i) If the bundle has structure group $G \subseteq GL(V)$ then we often only care about $\mathrm{Aut}_G(E) \subseteq \mathrm{Aut}(E)$, where $\mathrm{Aut}_G(E)$ are those automorphisms for which when written in trivializations giving the G-struture, h above takes values in G.

(ii) Using bundle automorphisms one can change transition functions.

$$\psi'_{\beta\alpha} = h_\alpha \psi_{\beta\alpha} h_\alpha^{-1} \qquad \text{and} \qquad \varphi'_\alpha(e) = h_\alpha(\pi(e))\varphi_\alpha(e).$$

This change of trivializations is analogous to the change of basis in $GL(V)$.

2.5 Connections

First definition

We would like to generlize the fact that the derivative of a vector field $(v_i(x))$ in $\mathbb{R}^m$ is another vector field in $\mathbb{R}^m$, namely $(v'_i(x))$. If $E \xrightarrow{\pi} B$ is a vector bundle and $s : B \to E$ is a section then we would like a way of *differentiating* s in such a way that the derivative is also a section.

With the notation above, suppose $\dim B = n$ and $V = \mathbb{R}^m$. Let $U \subseteq B$ be a trivializing neighbourhood (so there is $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{R}^m$), and let x^k, $k = 1, \dots, n$ be local coordinates in U. Let a^j, $j = 1, \dots, m$ be canonical coordinates in $\mathbb{R}^m$. Using the trivialization, T_pE (for $p \in \pi^{-1}(U)$) has a basis $\{\frac{\partial}{\partial x^k}, \frac{\partial}{\partial a^j}\}$. For $\pi : E \to B$, $d\pi_p : T_pE \to T_{\pi(p)}B$ is a submersion and we call its kernel the *vertical subspace*. The vertical subspace is spanned by $\{\frac{\partial}{\partial a^j}\}$.

2.5.1 Definition. A subspace $S_p \subseteq T_pE$ is called a *horizontal subspace* if

$$S_p \cap \ker(d\pi_p) = \{0\} \qquad \text{and} \qquad S_p \oplus \ker(d\pi_p) = T_pE.$$

We will not simply take $S_p = \{\frac{\partial}{\partial x^j}\}$, since this depends heavily on the choice of local coordinates; notice that the vertical subspace does not depend on the local coordinates.

2.5.2 Lemma. *If $\theta^1, \dots, \theta^m \in (\mathbb{R}^{m+n})^*$ are linear functionals then*

$$\dim\left(\bigcap_{i=1}^{m} \ker \theta^i\right) = n$$

if and only if $\{\theta^1, \dots, \theta^m\}$ is linearly independent.

PROOF: Exercise (or possibly omitted). □

Let $\theta_p^1, \dots, \theta_p^m$ be m linearly independent covectors (i.e. elements of $(T_pE)^*$). Take $S_p = \bigcap_{i=1}^m \ker(\theta_p^i)$. $(T_pE)^*$ has basis $\{dx^k, da^j\}$, so we can write $\theta_p^i = f_k^i dx^k + g_j^i da^j$. If S_p is a horizontal subspace then $\theta_p^i(v) = 0$ for all i implies $v = 0$, for all vertical vectors v. Restating, if S_p is a horizontal subspace and $v = c^j \frac{\partial}{\partial a^j}$ then $0 = \theta_p^i(v) = g_j^i c^j$ for all i implies $c^j = 0$. It follows that S_p is horizontal if and only if $(g_j^i)_{ij}$ is an invertible $m \times m$ matrix. Say $(g_j^i)^{-1} = (d_j^i)$. Replace θ^i by $\tilde{\theta}^i$ by multiplying by (d_j^i), i.e. $\theta^i = d_j^i \theta^j = da^i + e_k^i dx^k$. This change does not alter S_p. Now let p vary, so the e_k^i become functions of p. If $e_k^i(p)$ are smooth functions of p then we say that S_p varies smoothly with p.

The following proposition summarizes the above discussion.

2.5.3 Proposition. *Let $S = S_p$ $(p \in E)$ be any smooth field of horizontal subspaces in TE. Let x^k, a^j be local coordinates on $\pi^{-1}(U)$ arising from a local trivialization (as above). Then $S_p = \bigcap_{i=1}^m \ker(\theta_p^i)$, where $\theta_p^i = da^j + e_k^j(x,a)dx^k$ for smooth functions $e_k^j(x,a)$ which are uniquely determined by the trivialization.*

2.5.4 Definition. A smooth field of horizontal subspaces S_p is a *connection* if in every local trivialization, the functions $e_k^i(x,a)$ are linear in the fibre variables, i.e. $e_k^i(x,a) = \Gamma_{jk}^i(x)a^j$, where $\Gamma_{jk}^i : U \subseteq B \to \mathbb{R}$ are smooth functions called the *coefficients of the connection S_p*.

Notation. We write $\theta^i = da^i + A_j^i a^j$, where $A_j^i = \Gamma_{jk}^i dx^k$. $(A_j^i)_{ij}$ is an $m \times m$ matrix of 1-forms on U.

Coming up, an "index festival"...

Transformation law for connections and the second definition

Let U' be another trivializing neighbourhood with trivialization $\varphi' : \pi^{-1}(U') \to \mathbb{R}^m$, and $x^{k\prime}$, $a^{i\prime}$ coordinates in $U' \times \mathbb{R}^m$.

Notation. A prime "′" refers to U' e.g. the transition matrix on $U \cap U'$ is $\psi_i^{i\prime}$ and its inverse is $\psi_{i\prime}^i$.

Let $x^{k\prime} = x^{k\prime}(x)$. Then $a^{i\prime} = \psi_i^{i\prime} a^i$, so

$$da^{i\prime} = d\psi_i^{i\prime} a^i + \psi_i^{i\prime} da^i = \frac{\partial \psi_i^{i\prime}}{dx^k} a^i + \psi_i^{i\prime} da^i,$$

and $dx^{k\prime} = \frac{\partial x^{k\prime}}{\partial x^k} dx^k$. Therefore

$$\theta^{i\prime} = da^{i\prime} + \Gamma^{i\prime}_{j\prime k\prime} a^{j\prime} x^{k\prime} = \psi_i^{i\prime} da^i + \left[\frac{\partial \psi_j^{i\prime}}{dx^k} + \Gamma^{i\prime}_{j\prime k\prime} \frac{\partial x^{k\prime}}{\partial x^k} \psi_j^{j\prime} \right] a^j dx^k,$$

so

$$\psi^i_{i\prime} \theta^{i\prime} = da^i + \left[\psi^i_{i\prime} \frac{\partial \psi_j^{i\prime}}{dx^k} + \psi^i_{i\prime} \Gamma^{i\prime}_{j\prime k\prime} \psi_j^{j\prime} \frac{\partial x^{k\prime}}{\partial x^k} \right] a^j dx^k.$$

From this is follows that

$$\Gamma^i_{jk} = \Gamma^{i\prime}_{j\prime k\prime} \psi^i_{i\prime} \psi_j^{j\prime} \frac{\partial x^{k\prime}}{\partial x^k} + \psi^i_{i\prime} \frac{\partial \psi_j^{i\prime}}{dx^k} \qquad (1)$$

so

$$A^i_j = \Gamma^i_{jk} dx^k \qquad \text{and} \qquad A^{i\prime}_{j\prime} = \psi_i^{i\prime} A^i_j \psi^j_{j\prime} + \psi_j^{i\prime} d\psi^i_{j\prime}. \qquad (2)$$

Rewritting, if A^φ is the matrix of 1-forms in U and $A^{\varphi\prime}$ is the matrix of 1-forms in U', then

$$A^{\varphi\prime} = \psi A^\varphi \psi^{-1} + \psi(d\psi^{-1}) = \psi A^\varphi \psi^{-1} - (d\psi)\psi^{-1}. \qquad (3)$$

2.5.5 Theorem. *Any system of functions Γ^i_{jk} ($1 \le k \le n$, $1 \le i, j \le m$) attached to local trivializations satisfying the transformation law* (1) *defines a connection A on E whose coefficients are Γ^i_{jk}.*

Third definition

Notation. For any vector bundle $E \xrightarrow{\pi} B$, $\Gamma(E)$ denotes the space of sections $s : B \to E$ (so $s(x) \in E_x$).

2.5.6 Definition. A *connection* is an $\mathbb{R}$-linear map $\nabla : \Gamma(E) \to \Gamma(T^*B \otimes E)$ such that

$$\nabla(fs) = df \otimes s + f \nabla s \qquad \text{Leibniz rule}$$

for $f \in C^\infty(B, \mathbb{R})$ and $s \in \Gamma(E)$.

Or better,

2.5.7 Definition. A *connection* is a map $\nabla : \Gamma(TB) \times \Gamma(E) \to \Gamma(E)$ which we write as $\nabla(X, s) = \nabla_X s$ and such that

(i) $\nabla_{fX+gY} s = f \nabla_X s + g \nabla_Y s$ for all $f, g \in C^\infty(B, \mathbb{R})$ and $X, Y \in \Gamma(TB)$;

(ii) $\nabla_X(s_1 + s_2) = \nabla_X s_1 + \nabla_X s_2$ and $\nabla_X(cs) = c \nabla_X s$ for all $s_i \in \Gamma(E)$ and $c \in \mathbb{R}$;

(iii) $\nabla_X(fs) = X(f)s + f\nabla_X s$ $(= df(X)s + f\nabla_X s)$.

Recall that if V and W are vector spaces then there is a canonical linear isomorphism

$$V^* \otimes W \to \mathrm{Hom}(V,W) : f \otimes w \mapsto \text{“}v \mapsto f(v)w\text{”}.$$

Whence $T^*B \otimes E \cong \mathrm{Hom}(TB, E)$, so a section $s : B \to T^*B \otimes E$ gives linear maps $s_x : T_xB \to E_x$.

We can think of $\Gamma(T^*B \otimes E)$ as the E-valued 1-forms, and we will denote it $\Omega^1_B(E)$ by analogy with the notation for forms. (Our old notation becomes $\Omega^1(B) = \Omega^1_B(B \times \mathbb{R})$.) Sections of E are thought of as E-valued 0-forms, denoted $\Omega^0_B(E)$

2.5.8 Definition. Let $\Omega^r_B(E)$ be the collection of E-valued r-forms, alternating multilinear maps

$$\underbrace{T_xB \times \cdots \times T_xB}_{r} \to E_x.$$

With these definitions, $\nabla : \Omega^0_B(E) \to \Omega^1_B(E)$. This will of course be generalized later.

The definitions 2.5.6 and 2.5.7 at the beginning of this section are see to be the same via the identification $\nabla_X(s) = (\nabla s)(X)$.

Bringing it all together

(Insert several puns on the word “connection” here.)

Let A be a connection (i.e. $A^i_j = \Gamma^i_{jk} dx^k$ with the correct transformation law). Work in a local trivialization, and define $\nabla := d_A$ by $d_A s = ds + As$. Locally we may write $s(x) = (s^1(x), \ldots, s^m(x))$ (a vector in $\mathbb{R}^m$), so

$$d_A(s^1, \ldots, s^m) = \left(\left(\frac{\partial s^1}{\partial x^k} + \Gamma^1_{jk} s^j \right) dx^k, \ldots, \left(\frac{\partial s^m}{\partial x^k} + \Gamma^m_{jk} s^j \right) dx^k \right).$$

Now d_A is well-defined since if we take another trivialization U' with transition matrix ψ from U' to U, then from above we have $A = \psi A' \psi^{-1} - (d\psi)\psi^{-1}$. We also have $s = \psi s'$. Checking,

$$\begin{aligned} d_A s = ds + As &= d(\psi s') + A(\psi s') = (d\psi)s' + \psi ds' + (\psi A' \psi^{-1} - (d\psi)\psi^{-1})(\psi s') \\ &= (d\psi)s' + \psi ds' + \psi A' s' - (d\psi)s' = \psi ds' + \psi A' s' = \psi(d_{A'} s') \end{aligned}$$

Remark. From the definition of ∇, it follows that ∇ is a *local operator*, i.e. if $s \in \Gamma(E)$ vanishes on an open set U then ∇s vanishes on U as well. Indeed, if $x \in U$ then let V be a neighbourhood of x such that $\overline{V} \subseteq U$. Let α be a C^∞ cut-off function such that $0 \le \alpha \le 1$, $\alpha|_{\overline{V}} \equiv 1$, and $\mathrm{supp}(\alpha) \subseteq U$. Then $\alpha s = 0$, so $0 = \nabla(\alpha s) = d\alpha \otimes s + \alpha \nabla s$ by linearity and the Leibniz rule. Evaluating at x shows $(\nabla s)(x) = 0$.

This implies that a connection ∇ on E also defines a connection on $E|_U$, where U is any open set of B.

2.5.9 Theorem. *Any connection ∇ arises as d_A for some A.*

PROOF: Take a trivialization over $U \subseteq B$. Local sections

$$\Gamma(E|_U) = \{s : U \to E \mid s(x) \in E_x\}$$

are just smooth functions $U \to \mathbb{R}^m$. We have a *frame* of sections $\{e_j\}$, the constant sections which map everything to $(0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^m$. Then we can write an arbitrary section as $s = s^j e_j$. Therefore

$$\nabla s = \nabla(s^j e_j) = ds^j \otimes e_j + s^j \nabla e_j.$$

But for some Γ, we can write $\nabla e_j = (\Gamma^i_{jk} dx^k) e_i$. Let $A^i_j = \Gamma^i_{jk} dx^k$. We now check that $\nabla s = d_A s$.

$$\nabla s = ds^j \otimes e_j + s^j \Gamma^i_{jk} dx^k e_i = (ds^i + s^j \Gamma^i_{jk} dx^k) e_i = d_A s$$

To finish the proof one has to check that the Γ^i_{jk} (in A) transform in the correct way. □

2.5.10 Example. Let $M^n \subseteq \mathbb{R}^{n+k}$, and let $s \in \Gamma(TM)$ (so s is a vector field). Then $s : M \to TM$ such that $s(x) \in T_x M \subseteq T_x \mathbb{R}^{n+k} = \mathbb{R}^{n+k}$. For $v \in T_x M$, it makes sense to write ds_x, and $ds_x(v) \in \mathbb{R}^{n+k}$. But this may have jumped out of $T_x M$, so orthogonal projection onto $T_x M$ brings it back. Define $(\nabla_v s)(x)$ to be the orthogonal projection of $d_x s(v)$ onto $T_x(v)$. As the notations suggests, $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ is a connection on TM. This is the *Levi-Civita* connection of the metric induced on M by $\mathbb{R}^{n+k}$.

Let $E \xrightarrow{\pi} B$ and $\alpha : I \to B$ be a smooth curve, and $s : I \to E$ be a section along α, i.e. $s(\alpha(t)) \in E_{\alpha(t)}$ for all $t \in I$. One can introduce an operator $\frac{D}{dt}$ which takes s into another section along α and such that

(i) $\frac{D}{dt}(s_1 + s_2) = \frac{Ds_1}{dt} + \frac{Ds_2}{dt}$;

(ii) $\frac{D}{dt}(fs) = \dot{f}s + f\frac{Ds}{dt}$, where f is a smooth function of t; and

(iii) If $s(t) = \omega(\alpha(t))$, where $\omega \in \Gamma(E)$, then $\frac{Ds}{dt} = \nabla_{\dot\alpha}\omega$.

Let $\alpha : I \to U \subseteq B$, where U is a trivializing neighbourhood, with trivialization $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{R}^m$, with $s(t) = (s^1(t), \dots, s^m(t))$ (so $s = s^j e_j$). We must have

$$\frac{D}{dt}(s^j e_j) = \dot{s}^j e_j + s^j \nabla_{\dot\alpha} e_j$$

and if $\alpha = (\alpha^1, \dots, \alpha^n)$ then $\frac{De_j}{dt} = \nabla_{\dot\alpha} e_j = \Gamma^i_{jk} dx^k(\dot\alpha) e_i = \Gamma^i_{jk} \dot\alpha^k e_i$. Bringing it all together,

$$\frac{D}{dt}(s^j e_j) = (\dot{s}^i + s_j \dot\alpha^k \Gamma^i_{jk}) e_i.$$

It can be shown that this is the unique local definition of $\frac{D}{dt}$.

2.5.11 Definition.

(i) $s \in \Gamma(E)$ is *parallel* (or *covariant-constant*) if $\nabla s = 0$.

(ii) If s is a section along a curve $\alpha : I \to B$ is *parallel* if $\frac{Ds}{dt} = 0$.

Notice that $\frac{Ds}{dt} = 0$ means that $\dot{s}^i + s_j \dot{\alpha}^k \Gamma^i_{jk} = 0$ for every i. This is a linear system of ODE's. Take $s_0 \in E_{\alpha(t_0)}$, so there is a unique parallel section s along α such that $s(t_0) = s_0$. This defines $P : E_{\alpha(t_0)} \to E_{\alpha(t_1)}$, $P(s_0) = s(t_1)$, a linear isomorphism. Such a thing is a *parallel transform*. The velocity vectors $\dot{s}$ of the parallel sections $t \mapsto s(t) \in E$ are live in the horizontal subspaces at $s(t)$ (exercise, use the forms $\theta^i = da^i + \Gamma^i_{jk} a^j dx^k$ to show that $\theta^i(\dot{s}) = 0$ if s is parallel).

Remark. If ∇^1 and ∇^2 are connections on $E \xrightarrow{\pi} B$, then $\nabla^1(fs) - \nabla^2(fs) = f(\nabla^1 s - \nabla^2 s)$, so $s \mapsto \nabla^1 s - \nabla^2 s$ is a C^∞-linear map from sections to E-valued 1-forms. We may think of having $\nabla^1 - \nabla^2 \in \Omega^1_B(\operatorname{End} E)$. Whence the space of connection is an affine space.

In a similar way we have $\Omega^r_B(\operatorname{End} E)$, the $\operatorname{End} E$-valued r-forms. In priniple, they act on E-valued sections (or ℓ-forms), getting $\Omega^r_B(\operatorname{End} E) \times \Omega^\ell_B(E) \to \Omega^{r+\ell}_B(E)$.

2.6 Curvature

"I'm not going to tell you what it is, I'm going to tell you where it lives..."

Recall the covariant derivative $d_A : \Omega^0_B(E) \to \Omega^1_B(E)$. We would like to extend it $d_A : \Omega^r_B(E) \to \Omega^{r+1}_B(E)$ analogously to how we extended d. For $\sigma \in \Omega^r_B(E)$ and $\omega \in \Omega^q(B)$ (and ordinary q-form), we require

$$d_A(\sigma \wedge \omega) = d_A \sigma \wedge \omega + (-1)^r \sigma \wedge d\omega.$$

Locally in a trivialization we get $d_A \sigma = d\sigma + A \wedge \sigma$. We have

$$\Omega^0_B(E) \xrightarrow{d_A} \Omega^1_B(E) \xrightarrow{d_A} \dots \xrightarrow{d_A} \Omega^i_B(E) \xrightarrow{d_A} \dots \xrightarrow{d_A} 0$$

But what is d_A^2?. For $\sigma \in \Omega^r_B(E)$, we have

$$\begin{aligned} d_A(d_A\sigma) &= d_A(d\sigma + A \wedge \sigma) \\ &= d(d\sigma + A \wedge \sigma) + A \wedge (d\sigma + A \wedge \sigma) \\ &= 0 + dA \wedge \sigma + (-1)A \wedge d\sigma + A \wedge d\sigma + A \wedge A \wedge \sigma \\ &= (dA + A \wedge A) \wedge \sigma \end{aligned}$$

Let $F = dA + A \wedge A$, so that $d_A(d_A\sigma) = F \wedge \sigma$. Notice that $d_A d_A(f\sigma) = f d_A d_A \sigma$, so $F \in \Omega^2_B(\operatorname{End} E)$. F is the *curvature*, the deviation of our chain from being a chain complex.

2.6.1 Definition. A *flat connection* is a connection with $F = 0$.

2.6.2 Example. Let $E = B \times \mathbb{R}$ (or $B \times \mathbb{R}^m$ if you wish). Then $\Omega^r_B(E) = \Omega^r(B)$ we may take d_A to simply be d, i.e. $d_A\omega = d\omega$, the *trivial connection* or *product connection*. Since $d^2 = 0$, the curvature is zero and the connection is flat.

Suppose that $A^i_j = \Gamma^i_{jk} dx^k$, and $A = A_k dx^k$, where A_k is an $m \times m$ matrix of functions. Then

$$\begin{aligned} F &= dA + A \wedge A \\ &= d(A_k dx^k) + (A_i dx^i) \wedge (A_k dx^k) \\ &= \frac{\partial A_k}{\partial x^i} dx^i \wedge dx^k + A_i A_k dx^i \wedge dx^k \\ &= \frac{1}{2}\left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k} + [A_i, A_k] \right) dx^i \wedge dx^k \end{aligned} \tag{C1}$$

Finally, d_A also extends naturally to $\Omega^r_B(\operatorname{End} E)$, but how? Let $\mu \in \Omega^r_B(\operatorname{End} E)$, $\sigma \in \Omega^\ell_B(E)$, so that $\mu \wedge \sigma \in \Omega^{r+\ell}_B(E)$. We would like to define d_A so that

$$d_A(\mu \wedge \sigma) = (d_A\mu) \wedge \sigma + (-1)^r \mu \wedge d_A\sigma.$$

But we may take that as the definition. Locally we have

$$\begin{aligned} (d_A\mu) \wedge \sigma &= d_A(\mu \wedge \sigma) - (-1)^r \mu \wedge d_A\sigma \\ &= d(\mu \wedge \sigma) + A \wedge \mu \wedge \sigma - (-1)^r \mu \wedge (d\sigma + A \wedge \sigma) \\ &= (d\mu) \wedge \sigma) + (-1)^r \mu \wedge d\sigma + A \wedge \mu \wedge \sigma \\ &\qquad - (-1)^r \mu \wedge d\sigma - (-1)^r \mu \wedge A \wedge \sigma) \\ &= (d\mu + A \wedge \mu - (-1)^r \mu \wedge A) \wedge \sigma \end{aligned}$$

Therefore $d_A\mu = d\mu + A \wedge \mu - (-1)^r \mu \wedge A$.

2.6.3 Theorem (Bianchi Identity). $d_A F = 0$

PROOF: Since $d_A d_A \sigma = F \wedge \sigma$,

$$(d_A F) \wedge \sigma = d_A(F \wedge \sigma) - F \wedge d_A\sigma = d_A(d_A d_A \sigma) - d_A d_A(d_A \sigma) = 0$$ □

3 Riemannian Metrics

3.1 Metric Connections

Let $E \xrightarrow{\pi} B$, A be a connection, and $\langle\cdot,\cdot\rangle$ a Riemannian metric on E over $\mathbb{R}$ (or a Hermitian metric on E over $\mathbb{C}$).

3.1.1 Definition. A *metric connection* (or *orthogonal connection* or *unitary connection*) is a connection such that for any $X \in V(B)$ and $s_1, s_2 \in \Gamma(E)$,

$$X\langle s_1, s_2\rangle = \langle \nabla_X s_1, s_2\rangle + \langle s_1, \nabla_X s_2\rangle.$$

Remark. If E is endowed with a Riemannian metric then we may choose trivializations $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^m$ such that the vectors $e_i(p)$ such that they are mapped by φ to the canonical basis in $\mathbb{R}^m$ are orthogonal (exercise: check this (use Gram-Schmidt method)). Such a trivialization is called an *orthogonal trivialization*.

3.1.2 Proposition. *A metric connection has skew-symmetric matrix of coefficients in any orthogonal trivialization.*

PROOF: Take e_i, an orthonormal set of sections in the trivialization. Then for any vector field X,

$$\begin{aligned} 0 &= X\langle e_i, e_j\rangle \\ &= \langle \nabla_X e_i, e_j\rangle + \langle e_i, \nabla_X e_j\rangle \\ &= \langle A_i^k(X)e_i, e_j\rangle + \langle e_i, A_j^k(X)e_j\rangle \\ &= A_i^j(X) + A_j^i(X). \end{aligned}$$ □

3.1.3 Corollary. *Let A be a metric curvature. Then in an orthogonal local trivialization, the matrix of F is also skew-symmetric.*

(In the form (C1) for curvature one can always swap i and k and get the negative of what one started with, but the corollary refers to taking the transpose, so we see a different kind of skew-symmetry.)

3.2 Levi-Civita connection

Let M be a manifold with a Riemannian metric g (also written $\langle\cdot,\cdot\rangle$). A *connection* on M is a connection on TM, $\nabla : \Gamma(TM) \to \Gamma(T^* \otimes TM)$. We have $\Gamma(TM) = V(M)$, the vector fields on M. Notice that we may consider $\langle X, Y\rangle$ a function on M, taking value, for $p \in M$, $\langle X(p), Y(p)\rangle_p$.

Recall the compatibility conditions for a connection to be a metric connection. They imply (?) that

$$Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle.$$

Notice that

$$\nabla_X(fY) - \nabla_{fY}(X) = X(f)Y + f(\nabla_X Y - \nabla_Y X),$$

while

$$[X, fY] = X(f)Y + f[X,Y],$$

so $T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$ is a tensor, called the *torsion* of ∇. Locally, if $T(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = T_{ij}^k \frac{\partial}{\partial x^k}$ then we may write $T(X,Y) = X^i Y^j T_{ij}^k \frac{\partial}{\partial x^k}$ by $C^\infty(M,\mathbb{R})$-linearity. If the torsion is zero then the connection is said to be *symmetric*.

3.2.1 Theorem. *For any Riemannian manifold, there is a unique connection* ∇ *such that* *(i)* ∇ *is compatible with* g*; and*

(ii) ∇ *is symmetric, i.e.* $\nabla_X Y - \nabla_Y X = [X,Y]$.

This connection is the Levi-Civita connection *of* M*.*

PROOF: Suppose that ∇ exists. Then

(i) $X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$;

(ii) $Y\langle X,Z\rangle = \langle \nabla_Y X, Z\rangle + \langle X, \nabla_Y Z\rangle$;

(iii) $Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$.

Then (i) + (ii) - (iii) gives

$$X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle = \langle [X,Z],Y\rangle + \langle [Y,Z],X\rangle + \langle [X,Y],Z\rangle + 2\langle X, \nabla_Y X\rangle \quad (4)$$

If ∇' is another connection then $\langle Z, \nabla_Y X\rangle = \langle Z, \nabla'_Y X\rangle$ for all Z, so $\nabla_Y X = \nabla'_Y X$ for all X,Y. To show existence, define ∇ by (4) and check that ∇ is compatible and symmetric. □

In local coordinates, if $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}$, then

$$\Gamma^k_{ij} g_{\ell k} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right).$$

3.3 Curvature revisited

From now on we deal with a Levi-Civita connection on a Riemannian manifold. Denote the curvature of ∇ by $R \in \Omega^2_M(\mathrm{End}\, TM)$. For $X,Y,Z \in V(M)$, $R(X,Y)Z \in V(M)$. This is $C^\infty(M,\mathbb{R})$-linear in all three variables. Further, $R(X,Y) = -R(Y,X)$.

Remark. If $M = \mathbb{R}^n$ and g is the usual inner product then $\Gamma^k_{ij} \equiv 0$, whence $R \equiv 0$. (So $\mathbb{R}^n$ is flat, surprise, surprise...)

Locally, write $R(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l})\frac{\partial}{\partial x^j} = R^i_{jkl}\frac{\partial}{\partial x^i}$. We have seen that $R^i_{jkl} = -R^i_{jlk}$. We have the expression

$$R = \frac{1}{2}\left(\frac{\partial A_l}{\partial x^k} - \frac{\partial A_k}{\partial x^l} + [A_k, A_l]\right) dx^k \wedge dx^l,$$

where $A = A_k dx^k$, so

$$R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right) = \frac{\partial A_l}{\partial x^k} - \frac{\partial A_k}{\partial x^l} + [A_k, A_l].$$

There is a direct formula for R^i_{jkl} in terms of the g_{ij} and their first and second derivatives.

3.3.1 Proposition. $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$.

The conclusion of this proposition may be taken as the definition of the curvature, given a Levi-Civita connection.

PROOF: It suffices to check this equation locally. Check first for $\{\frac{\partial}{\partial x^i}\}$, and then use $C^\infty(M,\mathbb{R})$-linearity. Use the fact that $\nabla_{\frac{\partial}{\partial x^i}} = \frac{\partial}{\partial x^i} + A_i$ (exercise). □

Warning: some authors/books will use the convention that

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

i.e. with the opposite sign.

Recall that $R(X,Y) = -R(Y,X)$, or $R^i_{jkl} = R^i_{jlk}$. Many times we will express the symmetry conditions in terms of the 4-tensor $\langle R(X,Y)Z, T\rangle$, locally given by $R_{ijkl} = g_{iq} R^q_{jkl}$.

3.3.2 Proposition (First Bianchi Identity).

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

PROOF: Since the left hand side is linear, it suffices to check the equation locally on a basis $\{\frac{\partial}{\partial x^i}\}$ with $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$. □

Locally this identity is $R^i_{jkl} + R^i_{ljk} + R^i_{klj} = 0$.

3.3.3 Proposition.

(i) $\langle R(X,Y)Z,T\rangle = -\langle R(Y,X)Z,T\rangle = -\langle R(X,Y)T,Z\rangle$

(ii) $\langle R(X,Y)Z,T\rangle = -\langle R(Z,T)X,Y\rangle$

Locally, $R_{ijkl} = -R_{ijlk} = -R_{jikl}$ and $R_{ijkl} = R_{klij}$.

PROOF:

(i) The first equality we have done before. The second equality is a consequence of 3.1.3 since the Levi-Civita connection is a metric connection, so the map $Z \mapsto R(X,Y)Z$ is anti-symmetric.

(ii) From the first Bianchi identity,

$$\langle R(X,Y)Z,T\rangle + \langle R(Y,Z)X,T\rangle + \langle R(Z,X)Y,T\rangle = 0,$$
$$\langle R(Y,Z)T,X\rangle + \langle R(Z,T)Y,X\rangle + \langle R(T,Y)Z,X\rangle = 0,$$
$$\langle R(Z,T)Y,Y\rangle + \langle R(T,X)Z,Y\rangle + \langle R(X,Z)T,Y\rangle = 0,$$
$$\langle R(T,X)Z,Z\rangle + \langle R(X,Y)T,Z\rangle + \langle R(Y,T)X,Z\rangle = 0.$$

Add and use part (i). □

3.4 Sectional, Ricci, and Scalar curvature

Sectional Curvature

For $u, v \in T_xM$ (linearly independent), let $|u \wedge v| = \sqrt{|u|^2|v|^2 - \langle u, v\rangle^2}$, and let $\sigma \subseteq T_xM$ be the 2-dimensional subspace spanned by u and v. Define

$$K_x(\sigma) := \frac{\langle R(u,v)v, u\rangle}{|u \wedge v|^2},$$

the *sectional curvature*. Check that $K_x(\sigma)$ depends only on σ and not the basis chosen. In fact $K_x(\sigma)$ determines R, i.e. if R' is another multilinear map with the same symmetry properties as R and $K = K'$ then $R = R'$.

Remark. If $\dim M = 2$ then there is only one plane to chose, spanned by $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\}$, so we get

$$K_x = \frac{\langle R(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2})\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^1}\rangle}{EG - F^2} = \frac{R_{1212}}{EG - F^2}$$

where $E = |\frac{\partial}{\partial x^1}|^2$, $G = |\frac{\partial}{\partial x^2}|^2$, and $F = \langle \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\rangle$. But there is a formula for R^1_{212} in terms of Γ^i_{jk}, so we get a formula for K in terms of the Christofel (sp?) symbols. It is the same expression for K that arises in the Teoreme Egregium of Gauss.

Ricci Curvature

For $u, v \in T_xM$, define

$$\mathrm{Ric}_g(u,v) := \mathrm{Tr}(w \mapsto R(w,u)v),$$

the *Ricci curvature*. It is a symmetric bilinear form and $\mathrm{Ric}_{ij} = R^q_{iqj} = g^{lq}R_{iljq}$ (symmetry follows from the symmetry of R).

Scalar Curvature

The *scalar curvature* at a point x is the trace of the symmetric linear form associated with Ric_g, i.e. write $\mathrm{Ric}_g(u,v) = \langle Q_x(u), v\rangle$, where $Q_x(u)$ is a symmetric map, and define $s_g(x) = \mathrm{Tr}(Q_x)$. Then $s_g(x) = g^{ik}\,\mathrm{Ric}_{ik}$.

An *Einstein manifold* is a Riemannian manifold (M, g) for which $\mathrm{Ric}_g = \lambda g$ for some $\lambda \in \mathbb{R}$. An Einstein manifold is *Ricci flat* if $\lambda = 0$. A *Ricci flow* is a flow of the form $\frac{\partial g_t}{\partial t} = -2\,\mathrm{Ric}_{g_t}$ $(- \lambda g_t$, here we think of the manifold being fixed, and the Riemannian metric varying with time, and λ is a "cosmological constant"). See the papers by M. Anderson in *Notices of the AMS*, 04.

3.5 Laplace(-Bertrami) operator

A bit of multilinear algebra

Let V be a real vector space with a positive definite inner product. Recall that $A^p(V)$ is the space of alternating p-forms, and if $\{\omega_1, \ldots, \omega_n\}$ is a basis of V^*, then $\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \mid i_1 < \cdots < i_p\}$ is a basis of $A^p(V)$. We may also think of $A^p(V) = \wedge^p(V^*)$, the p^{th} exterior power of V^*.

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of V with corresponding dual basis $\{\omega_1, \dots, \omega_n\}$ of V^*. This induces an inner product on V^* with respect to which $\{\omega_1, \dots, \omega_n\}$ is orthonormal. This induces also an inner product in $A^p(V)$ by declaring the basis $\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \mid i_1 < \cdots < i_p\}$ to be orthonormal. (Check that $\langle t_1 \wedge \cdots \wedge t_p, s_1 \wedge \cdots \wedge s_p\rangle = \det\langle t_i, s_j\rangle$.) If V is oriented then we have made a choice of a non-zero top form in $A^n(V)$. But this space is one dimensional, so there is a unique ω_g that has norm 1 and defined the orientation of V.

3.5.1 Definition. The *Hodge $*$ operator* is a linear operator

$$* : A^p(V) \to A^{n-p}(V) : \omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \mapsto \omega_{j_1} \wedge \cdots \wedge \omega_{j_{n-p}}$$

such that $\{\omega_{i_1}, \dots, \omega_{i_p}, \omega_{j_1}, \dots, \omega_{j_{n-p}}\}$ is a positively oriented basis of $A^n(V)$, e.g. $*(\omega_1 \wedge \cdots \wedge \omega_p) = \omega_{p+1} \wedge \cdots \wedge \omega_n$.

It's easy to see (check) that for $\alpha, \beta \in A^p(V)$, $\alpha \wedge *\beta = \langle \alpha, \beta\rangle \omega_g$. Also check that $** = (-1)^{p(n-p)}$.

Back to manifolds

Let (M^n, g) be an oriented Riemannian manifold. We can do the constructions above over each T_xM and T_x^*M and we get a volume form $\omega_g(x)$ for each $x \in M$, called the *Riemannian volume form*. We also get a Hodge $*$ operator $* : \Omega^p(M) \to \Omega^{n-p}(M)$ such that $\alpha \wedge *\beta = \langle \alpha, \beta\rangle_x \omega_g(x)$. Recall that $d : \Omega^p(M) \to \Omega^{p+1}(M)$.

3.5.2 Definition. Define $\delta : \Omega^p(M) \to \Omega^{p-1}(M)$ by $\delta := (-1)^{n(p+1)+1} * d*$ (think of divergence). The *Laplace operator* on $\Omega^p(M)$ is $\Delta : \Omega^p(M) \to \Omega^p(M)$, defined by $\Delta := d\delta + \delta d$.

(elliptic PDE)

3.5.3 Proposition. *Suppose that M is compact. Then*

$$\int_M \langle d\alpha, \beta\rangle \omega_g = \int_M \langle \alpha, \delta\beta\rangle \omega_g$$

for all $\alpha \in \Omega^{p-1}(M)$ and $\beta \in \Omega^p(M)$.

PROOF: We have

$$\begin{aligned} d(\alpha \wedge *\beta) &= d\alpha \wedge *\beta + (-1)^{p-1}\alpha \wedge d(*\beta) \\ &= d\alpha \wedge *\beta + (-1)^{p-1}\alpha \wedge (-1)^{(p-1)(n-p+1)} **d(*\beta) \\ &= d\alpha \wedge *\beta - \alpha \wedge *(\delta\beta) \\ \int_M d(\alpha \wedge *\beta) &= \int_M d\alpha \wedge *\beta - \int_M \alpha \wedge *(\delta\beta) \\ 0 &= \int_M \langle d\alpha, \beta\rangle \omega_g - \int_M \langle \alpha, \delta\beta\rangle \omega_g \end{aligned}$$

by Stoke's Theorem. □

3.5.3 means that for $\alpha, \beta \in \Omega^p(M)$, with respect to the inner product

$$(\alpha, \beta)_{L^2} := \int_M \langle \alpha, \beta \rangle \omega_g,$$

δ is the adjoint of d, i.e. $(d\alpha, \beta)_{L^2} = (\alpha, \delta\beta)_{L^2}$. Notice that Δ is self-adjoint with respect to this inner product.

3.5.4 Definition. A form $\omega \in \Omega^p(M)$ is *harmonic* if $\Delta\omega = 0$.

3.5.5 Corollary. *ω is harmonic if and only if $d\omega = 0$ and $\delta\omega = 0$ (the latter condition is referred to as* co-closed*).*

PROOF: One direction is obvious. Suppose that $\Delta\omega = 0$, so

$$0 = (\Delta\omega, \omega)_{L^2} = ((d\delta + \delta d)\omega, \omega)_{L^2} = (d\delta\omega, \omega) + (\delta d\omega, \omega) = (\delta\omega, \delta\omega) + (d\omega, d\omega).$$

Therefore both terms on the right hand side are zero since they are non-negative, and ω is both closed and co-closed since the L^2 inner product is positive definite. □

Recall that $H^p_{dR}(M, \mathbb{R})$ is the space of closed p-forms modulo the exact forms, so a class $a \in H^p_{dR}(M, \mathbb{R})$ is of the form $a = \{\alpha + d\beta \mid d\alpha = 0, \beta \in \Omega^{p-1}\}$. Let $\mathscr{H}^p = \ker \Delta$, the space of harmonic p-forms. There is a natural map $\mathscr{H}^p \to H^p_{dR}(M, \mathbb{R})$, which turns out to be an isomorphism.

Locally, let (U, φ) to be a positively oriented chart inducing local coordinates $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ and $\{dx^1, \dots, dx^n\}$. Then

$$|dx^1 \wedge \cdots \wedge dx^n|^2 = \det(\langle dx^i dx^j \rangle) = \det(g^{ij}),$$

so

$$dx^1 \wedge \cdots \wedge dx^n = \sqrt{\det(g^{ij})}\omega_g \qquad \text{and} \qquad \omega_g = \sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n.$$

(In classical terms, $\omega_g = \sqrt{EG - F^2} dx^1 \wedge dx^2$.) We write $\sqrt{g} := \sqrt{\det(g_{ij})}$.

In the special case $\Delta : \Omega^0 \to \Omega^0$, $\Delta f = d\delta f + \delta df = \delta df$. Suppose that ψ is a function with support contained in U. Then

$$\begin{aligned}
-\int \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x^i} \right) \psi \sqrt{g}\, dx^1 \cdots dx^n &= \int g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial \psi}{\partial x^j} \sqrt{g}\, dx^1 \cdots dx^n \\
&= \int \langle df, d\psi \rangle \omega_g \\
&= (\delta df, \psi) \\
&= (\Delta f, \psi) \\
&= \int \Delta f \psi \omega_g \\
&= \int \Delta f \psi \sqrt{g}\, dx^1 \wedge \cdots \wedge dx^n,
\end{aligned}$$

so $\Delta f = -\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(\sqrt{g}g^{ij}\frac{\partial f}{\partial x^i}\right)$. With $M = \mathbb{R}^n$ and $g_{ij} = \delta_{ij}$, we recover the usual Laplacian $\Delta f = -\sum \frac{\partial^2 f}{(\partial x^j)^2}$ (sometimes the minus sign is omitted in the literature).

3.5.6 Theorem (Hodge Decomposition Theorem, 1935).
Let M be a compact oriented manifold. Then $\dim \mathscr{H}^p < \infty$ for every $0 \le p \le n$, and

$$\begin{aligned}\Omega^p(M) &= \Delta\Omega^p(M) \oplus \mathscr{H}^p \\ &= d\delta\Omega^p(M) \oplus \delta d\Omega^p(M) \oplus \mathscr{H}^p \\ &= d\Omega^{p-1}(M) \oplus \delta\Omega^{p+1}(M) \oplus \mathscr{H}^p\end{aligned}$$

and the direct sums are L^2-orthogonal.

Remark. The first equality implies the other two. For orthogonality,

$$(d\delta\beta, \delta d\beta) = (d^2\delta\beta, d\beta) = 0$$

since $d^2 \equiv 0$.

3.5.7 Corollary. *Every de Rham cohomology class can be uniquely represented by a harmonic form.*

PROOF: For a class $a \in H^p_{dR}(M, \mathbb{R})$, $a = [\alpha]$ for some closed p-form α. By the Hodge Decomposition Theorem, we may write $\alpha = d\beta + \delta\nu + \omega$, where $\omega \in \mathscr{H}^p$, $\beta \in \Omega^{p-1}$, and $\nu \in \Omega^{p+1}$. Then $0 = d\alpha = d\delta\nu$, so $0 = (d\delta\nu, \nu) = (\delta\nu, \delta\nu)$, implying $\delta\nu = 0$. Therefore $\alpha = \omega + d\beta$, so $[\omega] = [\alpha] = a$.

If ω_1 and ω_2 were two harmonic p-forms giving the same class then $\omega_1 - \omega_2 = d\beta$ for some $\beta \in \Omega^{p-1}$, and $\delta d\beta = \delta(\omega_1 - \omega_2) = 0$, so $(d\beta, d\beta) = (\delta d\beta, \beta) = 0$, implying $d\beta = 0$ and $\omega_1 = \omega_2$. □

Therefore $\mathscr{H}^p \to H^p_{dR}(M, \mathbb{R}) : \omega \mapsto [\omega]$ is a linear isomorphism. Consequently $\dim H^p(M, \mathbb{R}) < \infty$ for all p when M is compact (and this is not an *a priori* trivial fact). These dimensions are called the *Betti numbers*.

Aside: Write

$$e(\alpha) := \frac{1}{2}|\alpha|^2_{L^2} = \frac{1}{2}\int \langle\alpha, \alpha\rangle\omega_g = \frac{1}{2}\int \alpha \wedge \alpha^*,$$

the *energy* of α. If α is closed then the cohomology class of α is represented by all forms of the form $\alpha + d\beta$. Write $e(t) = \frac{1}{2}|\alpha + t d\beta|^2_{L^2}$, so

$$e(t) = \frac{1}{2}\Big((\alpha, \alpha) + 2t(\alpha, d\beta) + t^2(d\beta, d\beta)\Big)$$

and

$$e'(0) = (\alpha, d\beta) = (\delta\alpha, \beta).$$

Then $e'(0) = 0$ for all β if and only if $\delta\alpha = 0$, so the harmonic p-forms are the critical points of e on the cohomology class.

Poincaré Duality

Define a pairing $\Omega^p(M)^{n-p}(M) \to \mathbb{R} : (\alpha, \beta) \mapsto \int_M \alpha \wedge \beta$. We claim that the pairing descends to cohomology $([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta$. But

$$\int_M (\alpha + dv) \wedge \beta = \int_M \alpha \wedge \beta + \int_M (dv) \wedge \beta = \int_M d(v \wedge \beta) = 0$$

by Stoke's Theorem, since $d(v \wedge \beta) = (dv) \wedge \beta \pm v \wedge d\beta$ and β is closed. We further claim that the pairing $H^p(M, \mathbb{R}) \times H^{n-p}(M, \mathbb{R}) \to R$ is non-degenerate. Indeed, if α is a harmonic p-form then $\beta = *\alpha$ is a harmonic $n - p$ form (check), so

$$([\alpha], [\beta]) \mapsto \int \alpha \wedge *\alpha = |\alpha|^2_{L^2} \neq 0$$

when $\alpha \neq 0$. Since each class has a harmonic representative, this shows the forms is non-degenerate. It follows that when M is compact and connected, $H^n(M, \mathbb{R}) = H^0(M, \mathbb{R}) = \mathbb{R}$.

Outline of the proof of the Hodge decomposition theorem

"A foray into the dungeons of the analysts. They are one who make the world tick. If you spend enough time down there you might like it. Let's hope they don't turn out the lights..."

We would like to show that $\Omega^p = \Delta\Omega^p \oplus \mathscr{H}^p$, and this direct sum is L^2-orthogonal. In some sense we need to solve the equation $\Delta\omega = \alpha$. Notice that for any φ, $\langle \Delta\omega, \varphi \rangle = \langle \alpha, \varphi \rangle$, so $(\Delta\omega, \varphi)_{L^2} = (\alpha, \varphi)_{L^2}$, or $(\omega, \Delta\varphi)_{L^2} = (\alpha, \varphi)_{L^2}$ since Δ is self-adjoint.

3.5.8 Definition. A linear functional $\ell : \Omega^p \to \mathbb{R}$ is said to be a *weak solution* of $\Delta\omega = \alpha$ if

(i) ℓ is bounded, i.e. $|\ell(\beta)| \leq C|\beta|$ for some C constant and all $\beta \in \Omega^p$.

(ii) $\ell(\Delta\varphi) = (\alpha, \varphi)_{L^2}$ for all $\varphi \in \Omega^p$.

If ω is an honest solution then $\ell_\omega(\varphi) = (\omega, \varphi)_{L^2}$ is a weak solution. Can we find even weak solutions? If we can one, can we get an honest solution out of it?

3.5.9 Theorem (Regularity Theorem). *Any weak solution of $\Delta\omega = \alpha$ is of the form ℓ_ω for some $\omega \in \Omega^p$.*

3.5.10 Theorem (Compactness Theorem). *Let $\alpha_n \in \Omega^p$, $n \geq 1$. If there is a constant C such that $|\alpha_n|, |\Delta\alpha_n| < C$ for all n then α_n has a Cauchy subsequence.*

First note that the Compactness Theorem implies that $\mathscr{H}^p$ is finite dimensional. Indeed, if not then there would exist an infinite orthonormal sequence of orthonormal harmonic forms, which would contradict Compactness. Therefore we may write $\Omega^p = \mathscr{H}^p \oplus (\mathscr{H}^p)^\perp$. Note that $\Delta\Omega^p \subseteq (\mathscr{H}^p)^\perp$, since $(\Delta\varphi, \omega) = (\varphi, \Delta\omega) = 0$ when $\omega \in \mathscr{H}^p$. It remains to prove that given $\alpha \in (\mathscr{H}^p)^\perp$ we can find $\omega \in \Omega^p$ such that $\Delta\omega = \alpha$. To find a weak solution, proceed as follows. Define $\ell|_{\Delta\Omega^p}$ as $\ell(\Delta\varphi) = (\varphi, \alpha)_{L^2}$. This is well-defined since for $\varphi_1, \varphi_2 \in \Omega^p$ with $\Delta\varphi_1 = \Delta\varphi_2$, $\varphi_1 - \varphi_2 \in \mathscr{H}^p$, so $(\varphi_1, \alpha) = (\varphi_2, \alpha)$ for all $\alpha \in \varphi_1 - \varphi_2 \in (\mathscr{H}^p)^\perp$. We must also check that $\ell|_{\Delta\Omega^p}$ is bounded (one can prove this again using the Compactness Theorem). The Hahn-Banach theorem allows us to extend ℓ to all of Ω^p, and the Regularity Theorem gives us an honest solution.

Δ is an *elliptic operator*. Elliptic operators have finite dimensional spaces of solutions and the Regularity Theorem holds for them.

Ellipticity

We will work in $\mathbb{R}^n$ for the moment. Consider a function $u : \mathbb{R}^n \to \mathbb{R}$ and a general linear partial differential operator (PDO) $P = \sum_{|\alpha|\le k} a_\alpha(x)\partial^\alpha$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \sum_i \alpha_i$, and $\partial^\alpha = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$. The "symbol of P" is given by replacing $\frac{\partial}{\partial x^j}$ with $i\xi_j$, so $P(x,\xi) = \sum_{|\alpha|\le k} a_\alpha(x)(i\xi)^\alpha$, where $y^\alpha = \prod_j y_j^{\alpha_j}$. The "principal symbol" is $p_k(x,\xi) = \sum_{|\alpha|=k} a_\alpha(x)(i\xi)^\alpha$. For example, if $P = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ then $P_2(x,\xi) = -|\xi|^2$. Associated with the "wave operator" $\frac{\partial^2 u}{\partial x_1^2} = \frac{\partial^2 u}{\partial x_2^2}$ the symbol is is $-\xi_1^2 + \xi_2^2$. P is *elliptic* at x if $P_k(x,\xi) \ne 0$ for all $\xi \ne 0$. P is *elliptic* it is elliptic at all x.

Recall that

$$\Delta f = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j}\left(g^{ij}\sqrt{g}\frac{\partial f}{\partial x^i}\right).$$

The higher order term is $g^{ij}\frac{\partial^2 f}{\partial x^i \partial x^j}$. (g^{ij}) is positive semi-definite, so Δ is elliptic (under an appropriate extension of the definition of ellipticity to vector bundles over manifolds).

3.5.11 Theorem (Bochner, 1946). *Let M be a compact, orientable, connected Riemannian manifold. Suppose that* $\mathrm{Ric} \ge 0$ *(i.e. this symmetric, bilinear form is positive semi-definite). Then every harmonic 1-form is parallel.*

For $\omega \in \mathscr{H}^1$, $\omega_x(v) = \langle X(x), v\rangle$, where X is the dual vector field. ω is parallel if and only if $\nabla X = 0$. If $\omega_x = 0$ then $\omega \equiv 0$ (since ω is parallel), so $\mathscr{H}^1 \to T_x^*M : \omega \mapsto \omega_x$ is an injective linear map. Therefore $\dim \mathscr{H}^1 \le \dim M$. In particular, $\dim H^1(M, \mathbb{R}) \le \dim M$.

Suppose Σ_g is a surface of genus g. Then $\Sigma_g \times S^n$ ($n \ge 2$) is an $(n+2)$-dimensional manifold, so if there is a positive semi-definite Ricci curvature then $2g = \dim H^1 \le n+2$. For large enough g we get a family of neat examples.

PROOF (OF BOCHNER'S THEOREM): If ω is a harmonic 1-form then the following identity holds for the dual vector field X.

$$-\Delta(\tfrac{1}{2}|X|^2) = |\nabla X|^2 + \mathrm{Ric}(X,X).$$

Integrating over M yields

$$0 = (-\Delta(\tfrac{1}{2}|X|^2), 1)_{L^2} = \int -\Delta(\tfrac{1}{2}|X|^2)\omega_g = \int |\nabla X|^2 \omega_g + \int \operatorname{Ric}(X,X)\omega_g.$$

Whence $\nabla X = 0$ since $\operatorname{Ric}(X,X) \geq 0$. □

This ends the examinable material of the course.

3.6 Yang-Mills Equations

Let M^n be a compact oriented manifold and E a vector bundle of rank k over M. Suppose that we have a Riemannian metric on M and a Riemannian metric on E, and a metric compatible connection D, i.e. for $s_1, s_2 \in \Gamma(E) = \Omega^0_M(E)$ and $X \in V(M)$,

$$X\langle s_1, s_2\rangle = D_X s_1, s_2\rangle + \langle s_1, D_X s_2\rangle,$$

also called an orthogonal connection. In an orthogonal trivialization $D = d + A$, and $A(X)$ is a skew-symmetric matrix. Consider the structure group $O(k)$, and notice that $A(X) \in \mathfrak{o}(k)$, the Lie algebra of $O(k)$. If F_D is the curvature of D then $F_D \in \Omega^2_M(\operatorname{End} E)$, and F_D is also skew-symmetric. Recall that $\Omega^p_M(\operatorname{End} E) = \Gamma(A^p(M) \otimes \operatorname{End} E)$ is the collection of $\operatorname{End} E$ valued p-forms. Let $\Omega^p_M(\operatorname{Ad} E)$ be the $\operatorname{Ad}(E)$-valued p-forms, where $\operatorname{Ad} E$ is the follection of bundles over M given by elements in $\operatorname{End} E$ which are skew-symmetric. E.g. $F_D \in \Omega^2_M(\operatorname{Ad} E)$, and if D_1 and D_2 are two metric compatible connections then $D_1 - D_2 \in \Omega^1_M(\operatorname{Ad} E)$.

We will now define an L^2-inner product on $\Omega^p_M(\operatorname{Ad} E) = \Gamma(A^p(M) \otimes \operatorname{Ad} E)$. If A and B are skew-symmetric matrices then $A \cdot B = -\operatorname{tr}(AB)$ is a natural inner product (it is the Killing form on $\mathfrak{l}(k)$). For $\omega \in \Omega^p(M)$ and $s \in \Gamma(\operatorname{Ad} E)$, define

$$\langle \omega_1 \otimes s_1, \omega_2 \otimes s_2\rangle_x = -\operatorname{tr}(s_1 s_2)\langle \omega_1, \omega_2\rangle_x$$

and as before define

$$(\omega_1 \otimes s_1, \omega_2 \otimes s_2)_{L^2} = \int_M \langle \omega_1 \otimes s_1, \omega_2 \otimes s_2\rangle_x \omega_g$$

We also have a $*$-operator acting $\Omega^p_M(\operatorname{Ad} E) \to \Omega^{n-p}_M(\operatorname{Ad} E)$ as $\omega \otimes s \mapsto (*\omega) \otimes s$.

3.6.1 Definition. The *Yang-Mills functional* is

$$YM : D \to (F_D, F_D)_{L^2} = \int_M \langle F_D, F_D\rangle_x \omega_g.$$

Harmonic forms pop up as critical points of the energy function. What are the critical points of the Yang-Mills functional? Let $A \in \Omega^1_M(\operatorname{Ad} E)$, so we have

$$\begin{aligned} F_{D+tA}(\sigma) &= (D+tA)((D+tA)(\sigma)) \\ &= (D+tA)(D\sigma + tA)(\sigma) \\ &= D^2\sigma + tD(A\sigma) + tA(D\sigma) + t^2 A \wedge A(\sigma) \\ &= D^2\sigma + t((DA)\sigma - A \wedge D\sigma) + tA(D\sigma) + t^2 A \wedge A(\sigma) \\ &= (F_D + tDA + t^2(A \wedge A))\sigma) \end{aligned}$$

since we can extend $D : \Omega^p_M(\operatorname{Ad} E) \to \Omega^{p+1}_M(\operatorname{Ad} E)$ to satisfy a Leibniz rule as we have earlier in the course. Therefore

$$\frac{d}{dt}\bigg|_{t=0} YM(D+tA) = 2(DA, F_D)_{L^2}.$$

For $D : \Omega^p_M(\operatorname{Ad} E) \to \Omega^{p+1}_M(\operatorname{Ad} E)$, define $D^* : \Omega^p_M(\operatorname{Ad} E) \to \Omega^{p-1}_M(\operatorname{Ad} E)$ to be the L^2-adjointof D. Then critical points of the Yang-Mills functional are exactly those D for which $D^* F_D = 0$. This is the *Yang-Mills equation*. Recall also the Bianchi Identity which gives $DF_D = 0$.

In the case $p = 2$, $** = (-1)^{2(n-2)} = 1$. One may check that when the dimension of M is even then $D^* = - * D *$. Whence in this case the Yang-Mills equation reduces to $D * F_D = 0$.

Remark.

(i) The same construction can be done (and often is in Physics) for a Hermitian inner product and unitary connections.

(ii) When $k = 2$ in the $O(2)$ orthogonal case or $k = 1$ in the $U(1)$ unitary case, we get Maxwell's equations. We get Hodge theory for 2-forms ($D * F_D = 0$ corresponds to F_D being co-closed and $DF_D = 0$ corresponds to F_D being closed).

Locally, if $g_{ij} = \delta_{ij}$ (say if $M = \mathbb{R}^4$) then if $D = d + A$, where $A = A_i dx^1$ and $F_D = F_{ij} dx^i \wedge dx^j$, then

$$D^* F_D = \left(-\frac{\partial F_{ij}}{\partial x^i} - [A_i, F_{ij}] \right) dx^j = 0$$

so the Yang-Mills equations are

$$\frac{\delta F_{ij}}{\partial x^i} + [A_i, F_{ij}] = 0 \text{for } j = 1, \dots, n$$

where $F_{ij} = \frac{\delta A_j}{\partial x^i} - \frac{\delta A_i}{\partial x^j} + [A_i, A_j]$.

Again, the set up is as follows. $E \to M$ is a vector bundle, we have Riemannian metrics on E and M, and a metic/compatible/orthogonal connection D. We extend D to $\Omega^p_M(\operatorname{Ad} E) \to \Omega^{p+1}_M(\operatorname{Ad} E)$.

3.6.2 Definition. The *Gauge group* is $\mathfrak{g} = \mathrm{Aut}_G(E)$, the collection of automorphisms h of E such that for each $x \in M$, $h_x : E_x \to E_x$ is orthogonal, where G is the structure group ($O(k)$ in our case).

The important point is that $\mathfrak{g}$ acts on the space of metric connections. If $h \in \mathrm{Aut}_G(E)$ and D is a metric connection then define $h^*D = h^{-1} \circ D \circ h$, i.e. $h^*D(s) = h^{-1}D(hs)$ for all $s \in \Gamma(E)$. Locally one may check that if $D = d + A$ then $h^*(A) = h^{-1}dh + h^{-1}Ah$ and $h^*F = h^{-1} \circ F \circ h$. Further,

$$YM(h^*D) = (F_{h^*D}, F_{h^*D})_{L^2} = (F_D, F_D)_{L^2} = YM(D)$$

since each element of $\mathrm{Aut}_G(E)$ is an isometry. Hence YM is invariant under the Gauge group. It follows that critcal points, i.e. Yang-Mills connections, are mapped to one another under the action of $\mathfrak{g}$.

3.6.3 Definition. The *moduli space* $\mathscr{M}$ of M is the space of Yang-Mills connections modulo the action of $\mathfrak{g}$.

3.6.4 Example. When $k = 2$ and $E = M \times \mathbb{R}^2$ is the trival rank 2 vector bundle over a compact oriented manifold Riemannian manifold M, the structure group is $SO(2)$. The associated Lie algebra is skew-symmetric 2×2 matrices and so is isomorphic to $\mathbb{R}$. Then we may write $D = d + A$ globally since E is trivial, where $A \in \Omega^1_M(\mathrm{Ad}\, E)$. Then $A(x) = \left[\begin{smallmatrix} 0 & a_x \\ -a_x & 0 \end{smallmatrix}\right]$ where a_x is a 1-form, so we may think of A as an ordinary 1-form, and $\Omega^1_M(\mathrm{Ad}\, E) = \Omega^1(M)$. For $s \in \Gamma(E)$, $s : M \to \mathbb{R}$, so $Ds = ds + \left[\begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} s_1 \\ s_2 \end{smallmatrix}\right]$, and

$$F_D = dA + A \wedge A = dA = \begin{bmatrix} 0 & da \\ -da & 0 \end{bmatrix}.$$

The Yang-Mills equation gives that $\delta da = 0$, so da is a harmonic 2-form. By triviality, $\delta da = 0$ and da harmonic imply that $da = 0$, so a is a closed 1-form. Hence any Yang-Mills connection is given by a closed 1-form on M, and *visa versa*.

What is the action of $\mathfrak{g} = SO(2)$? For $x \in M$, $h(x) \in SO(2)$ and $h^*(A) = h^{-1}dh + h^{-1}Ah$, and $h(x)$ is orthogonal, so we may write $h(x) = \exp \left[\begin{smallmatrix} 0 & u \\ -u & 0 \end{smallmatrix}\right]$. Thus $h^*(A) = du + A$ since $SO(2)$ is Abelian. As we let u run over $\mathbb{R}$, we see that $\mathscr{M} = H^1(M, \mathbb{R})$. In particular, if $H^1(M, \mathbb{R}) = \{0\}$ then $\mathscr{M}$ is just a single point.

3.6.5 Theorem (Donaldson, mid 80's). *Let M be a compact simply connected 4-manifold. We have a pairing*

$$\Gamma : H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) \to \mathbb{R} : ([\alpha], [\beta]) \mapsto \int \alpha \wedge \beta,$$

and if Γ is definite then there is a basis such that it is $\pm I$.

Consider in particular an $SU(2)$-bundle over M with $\int h(F \wedge F) = -8\pi^2$. He proved that in this case, $\mathscr{M}$ is compact, 5-dimensional, and is smooth a.e. except for some "core singularities" and has M as a boundry. And so on. . .

Index