A Quick Introduction to Representation Theory

with Applications

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Outline

Group Representations: Definition

- Let G be a finite group. A representation of G is a pair (V, ρ), where
  - V is a finite dimensional vector space over $\mathbb{C}$
  - $\rho : G \to \mathbf{GL}(V)$ is a group homomorphism
- Every group homomorphism $G \to \mathbb{C}^*$ is a one dimensional representation. (Think of scalars as 1×1 matrices.)

Examples

- If X is a finite G-set then let V be the vector space with basis $\{e_x \mid x \in X\}$. Then $\rho : G \longrightarrow \mathbf{GL}(V)$ defined by

$$\rho_s(e_x) = e_{s \cdot x}$$

  is called the permutation representation associated with X.
- The regular representation is a special case of this.
- Let (V, ρ) and (W, ς) be representations. The direct sum representation is

$$\rho \oplus \varsigma : G \longrightarrow \mathbf{GL}(V \oplus W) : s \mapsto \begin{bmatrix} \rho_s & 0 \\ 0 & \varsigma_s \end{bmatrix}$$

Subrepresentations and Irreducible Representations

- If (V, ρ) is a representation and $W \subseteq V$ is a linear subspace such that
$$\rho_s W \subseteq W \quad \text{for all } s \in G$$
then W is a subrepresentation of V.
- If V has no subrepresentations other than $\{0\}$ and V then it is said to be irreducible.
- Every representation can be decomposed into a direct sum of irreducible representations. (Rickhart's Theorem)

Schur's Lemma

Lemma

Suppose that (V, ρ) *and* (W, ς) *are irreducible representations and* $f : V \longrightarrow W$ *is linear such that*

$$\rho_s f(v) = f(\varsigma_s v) \quad \text{for all } s \in G \text{ and } v \in V$$

Then f *is either the zero map or an isomorphism.*

Proof of Schur's Lemma

- Suppose that f is non-zero.
- $\ker f \subseteq V$ is a subspace. If $v \in \ker V$ then for any $s \in G$,

$$f(\rho_s v) = \varsigma_s f(v) = \varsigma_s 0 = 0$$

  Therefore $\ker f$ is a subrepresentation of V.
  But V is irreducible and f is non-zero, so $\ker f = \{0\}$.
- $f(V) \subseteq W$ is a subspace. If $f(v) \in f(V)$ then for any $s \in G$,

$$\varsigma_s f(v) = f(\rho_s v) \in f(V)$$

  Therefore $f(V)$ is a subrepresentation of W.
  But W is irreducible and f is non-zero, so $f(V) = W$. □

Characters: Definition

- Let $\rho : G \to \mathbf{GL}(V)$ be a representation of G.
  The character of ρ is χ_V, defined by

$$\chi_V(s) = \mathrm{Tr}(\rho_s)$$

- Notice that χ_V is a class function since

$$\chi_V(hsh^{-1}) = \mathrm{Tr}(\rho_{hsh^{-1}}) = \mathrm{Tr}(\rho_h \rho_s \rho_h^{-1}) = \mathrm{Tr}(\rho_s) = \chi_V(s)$$

- For any $s \in G$, $\rho_s^{|G|} = \rho_{s^{|G|}} = \rho_e = I$ so the eigenvalues of ρ_s are roots of unity. Hence $\chi_V(s)$ is a sum of roots of unity.

Facts

Proposition

Let (V, ρ) and (W, ς) be representations of G.

- $\chi_V(e) = \dim V$
- $\chi_V(s^{-1}) = \overline{\chi_V(s)}$
- $\chi_{V \oplus W} = \chi_V + \chi_W$

Characters of Irreducible Representations

The collection of all class functions $f : G \longrightarrow \mathbb{C}$ is a vector space. It has an inner product

$$\langle f, g \rangle = \frac{1}{|G|} \sum_{s \in G} f(s)\overline{g(s)}$$

Theorem

The characters of the irreducible representations of G form an orthonormal basis of this vector space.

Proof of this fact uses Schur's Lemma.

Decomposing Representations

- The dimension of the space of class functions is equal to the number of conjugacy classes, so there are that many irreducible representations.
- Suppose that $(V_1, \psi_i), \dots, (V_h, \psi_h)$ are all of the irreducible representations of G.
- If $\rho : G \to \mathbf{GL}(V)$ is any representation then there are integers $a_1, \dots, a_h \geq 0$ such that

$$V \cong a_1 V_1 \oplus \cdots \oplus a_h V_h$$

- Hence $\chi_V = a_1\chi_1 + \cdots + a_h\chi_h$, and so

$$a_i = \langle \chi_V, \chi_i \rangle$$

Character Tables

Notice that $\langle \chi_V, \chi_V \rangle = a_1^2 + \cdots + a_h^2$, so χ_V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.

Theorem

It is enough to know the character of a representation to know it up to isomorphism (once you already know all of the irreducible representations).

- It is convenient to record this information in a table.
- Here is the character table for $\mathfrak{S}_3$:

	id	(1 2)	(1 2 3)
U	1	1	1
U'	1	-1	1
W	2	0	-1

Calculation of the Character Table for $\mathfrak{S}_3$

- U is the trivial representation, every group has it.
- U' is the alternating representation. It is obtained by considering the sign of the permutation.
- $\mathfrak{S}_3$ acts on the set $\{1,2,3\}$, so consider the associated permutation representation (V,ρ). We have

$$\rho_{id} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \rho_{(1\ 2)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \rho_{(1\ 2\ 3)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Calculation, continued

- (V,ρ) is not irreducible, since the vector $e_1 + e_2 + e_3$ is fixed by all elements of the group. The span of this vector forms a copy of U sitting inside of V.
- Suppose we write $V = W \oplus U$. Then $\chi_V = \chi_W + 1$, so

$$\chi_W(id) = 2 \quad \chi_W(1\ 2) = 0 \quad \chi_W(1\ 2\ 3) = -1$$

- W is irreducible since

$$\langle \chi_W, \chi_W \rangle = \frac{1}{|\mathfrak{S}_3|} \sum_{s \in \mathfrak{S}_3} |\chi_W(s)|^2 = \frac{1}{6}(2 + 4 \cdot 0 + 2 \cdot 1) = 1$$

Burnside's Theorem

Using representation theory we can show

Lemma

If G has a conjugacy class of order a power of a prime then G is not simple.

Theorem

If $|G|$ is divisible by at most two primes then G is solvable.

Proof of Burnside's Theorem

- Suppose that G is a minimal counterexample. Then G is simple since a normal subgroup N would give solvable groups N and G/N, which would show G is solvable.
- Say $|G| = p^a q^b$, where $b > 0$. Let H be the Sylow subgroup of size q^b. Then H has non-trivial center.
- Let h be a non-identity element in the center of H and let K be the centralizer of h in G.
- $|K| = p^c q^b$, and $c < a$ since otherwise h would be in the center of G. The size of the conjugacy class of h is $|G|/|K| = p^{a-c}$.
- This is a contradiction by the Lemma. □

For Further Reading

Jean-Pierre Serre
Linear Representations of Finite Groups.
GTM 42, Springer-Verlag, 1977.