A Quick Introduction to Representation Theory
with Applications

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Outline

1. Introduction
2. Characters
3. Applications
Let $G$ be a finite group. A representation of $G$ is a pair $(V, \rho)$, where

- $V$ is a finite dimensional vector space over $\mathbb{C}$
- $\rho : G \rightarrow \text{GL}(V)$ is a group homomorphism

Every group homomorphism $G \rightarrow \mathbb{C}^*$ is a one dimensional representation. (Think of scalars as $1 \times 1$ matrices.)
Examples

- If $X$ is a finite $G$-set then let $V$ be the vector space with basis $\{e_x \mid x \in X\}$. Then $\rho : G \to \text{GL}(V)$ defined by
  \[ \rho_s(e_x) = e_{s \cdot x} \]
  is called the permutation representation associated with $X$.

- The regular representation is a special case of this.

- Let $(V, \rho)$ and $(W, \varsigma)$ be representations. The direct sum representation is
  \[ \rho \oplus \varsigma : G \to \text{GL}(V \oplus W) : s \mapsto \begin{bmatrix} \rho_s & 0 \\ 0 & \varsigma_s \end{bmatrix} \]
If \((V, \rho)\) is a representation and \(W \subseteq V\) is a linear subspace such that
\[
\rho_s W \subseteq W \quad \text{for all } s \in G
\]
then \(W\) is a subrepresentation of \(V\).

If \(V\) has no subrepresentations other than \(\{0\}\) and \(V\) then it is said to be irreducible.

Every representation can be decomposed into a direct sum of irreducible representations. (Rickhart’s Theorem)
Schur’s Lemma

**Lemma**

Suppose that $(V,\rho)$ and $(W,\varsigma)$ are irreducible representations and $f : V \rightarrow W$ is linear such that

$$\rho_s f(v) = f(\varsigma_s v) \quad \text{for all } s \in G \text{ and } v \in V$$

Then $f$ is either the zero map or an isomorphism.
Proof of Schur’s Lemma

- Suppose that $f$ is non-zero.
- $\ker f \subseteq V$ is a subspace. If $v \in \ker V$ then for any $s \in G$,
  \[ f(\rho_s v) = \zeta_s f(v) = \zeta_s 0 = 0 \]
  Therefore $\ker f$ is a subrepresentation of $V$.
  But $V$ is irreducible and $f$ is non-zero, so $\ker f = \{0\}$.
- $f(V) \subseteq W$ is a subspace. If $f(v) \in f(V)$ then for any $s \in G$,
  \[ \zeta_s f(v) = f(\rho_s v) \in f(V) \]
  Therefore $f(V)$ is a subrepresentation of $W$.
  But $W$ is irreducible and $f$ is non-zero, so $f(V) = W$. \qed
Characters: Definition

- Let $\rho : G \to \text{GL}(V)$ be a representation of $G$. The **character** of $\rho$ is $\chi_V$, defined by

$$\chi_V(s) = \text{Tr}(\rho_s)$$

- Notice that $\chi_V$ is a class function since

$$\chi_V(hsh^{-1}) = \text{Tr}(\rho_{hsh^{-1}}) = \text{Tr}(\rho_h \rho_s \rho_h^{-1}) = \text{Tr}(\rho_s) = \chi_V(s)$$

- For any $s \in G$, $\rho_s^{\frac{|G|}{|s|}} = \rho_s^{\frac{|G|}{|s|}} = \rho_e = I$ so the eigenvalues of $\rho_s$ are roots of unity. Hence $\chi_V(s)$ is a sum of roots of unity.
Proposition

Let \((V, \rho)\) and \((W, \varsigma)\) be representations of \(G\).

- \(\chi_V(e) = \dim V\)
- \(\chi_V(s^{-1}) = \chi_V(s)\)
- \(\chi_{V \oplus W} = \chi_V + \chi_W\)
The collection of all class functions $f : G \rightarrow \mathbb{C}$ is a vector space. It has an inner product

$$\langle f, g \rangle = \frac{1}{|G|} \sum_{s \in G} f(s)\overline{g(s)}$$

**Theorem**

*The characters of the irreducible representations of $G$ form an orthonormal basis of this vector space.*

Proof of this fact uses Schur’s Lemma.
The dimension of the space of class functions is equal to the number of conjugacy classes, so there are that many irreducible representations.

Suppose that $(V_1, \psi_i), \ldots, (V_h, \psi_h)$ are all of the irreducible representations of $G$.

If $\rho : G \rightarrow \text{GL}(V)$ is any representation then there are integers $a_1, \ldots, a_h \geq 0$ such that

$$V \cong a_1 V_1 \oplus \cdots \oplus a_h V_h$$

Hence $\chi_V = a_1 \chi_1 + \cdots + a_h \chi_h$, and so

$$a_i = \langle \chi_V, \chi_i \rangle$$
Character Tables

Notice that $\langle \chi_V, \chi_V \rangle = a_1^2 + \cdots + a_h^2$, so $\chi_V$ is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.

**Theorem**

It is enough to know the character of a representation to know it up to isomorphism (once you already know all of the irreducible representations).

- It is convenient to record this information in a table.
- Here is the character table for $S_3$:

<table>
<thead>
<tr>
<th></th>
<th>id</th>
<th>(1 2)</th>
<th>(1 2 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$U'$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$W$</td>
<td>2</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>
Calculation of the Character Table for $\mathfrak{S}_3$

- $U$ is the trivial representation, every group has it.
- $U'$ is the alternating representation. It is obtained by considering the sign of the permutation.
- $\mathfrak{S}_3$ acts on the set $\{1, 2, 3\}$, so consider the associated permutation representation $(V, \rho)$. We have

$$
\rho_{id} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \rho_{(1 \ 2)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \rho_{(1 \ 2 \ 3)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
$$
Calculation, continued

- \((V, \rho)\) is not irreducible, since the vector \(e_1 + e_2 + e_3\) is fixed by all elements of the group. The span of this vector forms a copy of \(U\) sitting inside of \(V\).

- Suppose we write \(V = W \oplus U\). Then \(\chi_V = \chi_W + 1\), so
  \[
  \chi_W(id) = 2 \quad \chi_W(1 2) = 0 \quad \chi_W(1 2 3) = -1
  \]

- \(W\) is irreducible since
  \[
  \langle \chi_W, \chi_W \rangle = \frac{1}{|S_3|} \sum_{s \in S_3} |\chi_W(s)|^2 = \frac{1}{6} (2 + 4 \cdot 0 + 2 \cdot 1) = 1
  \]
Using representation theory we can show

**Lemma**

*If $G$ has a conjugacy class of order a power of a prime then $G$ is not simple.*

**Theorem**

*If $|G|$ is divisible by at most two primes then $G$ is solvable.*
Suppose that $G$ is a minimal counterexample. Then $G$ is simple since a normal subgroup $N$ would give solvable groups $N$ and $G/N$, which would show $G$ is solvable.

Say $|G| = p^aq^b$, where $b > 0$. Let $H$ be the Sylow subgroup of size $q^b$. Then $H$ has non-trivial center.

Let $h$ be a non-identity element in the center of $H$ and let $K$ be the centralizer of $h$ in $G$.

$|K| = p^c q^b$, and $c < a$ since otherwise $h$ would be in the center of $G$. The size of the conjugacy class of $h$ is $|G|/|K| = p^{a-c}$.

This is a contradiction by the Lemma.
Jean-Pierre Serre

*Linear Representations of Finite Groups.*