Graph Theory
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Contents

1 Colouring

Notation. For these notes, G will be a (finite) graph with n vertices, $V(G)$ is the vertex set of G, and $E(G)$ is the edge set of G.

1.1 Definition. A *colouring* of G is a function $\varphi : S \to C$, where S is $V(G)$ or $E(G)$ or $V(G) \cup E(G)$ and C is a set of colours. φ is said to be *proper* if "adjacent" elements have different values. A minimum number of colours required for a proper colouring is called a *chromatic number*

1.1 Vertex Colourings

1.2 Definition. G is *k-(vertex)-colourable* if there is $\varphi : V(G) \to \{1, \dots, k\}$ such that φ is a proper colouring. $\chi(G)$ is the *chromatic number of G*; the minimum k such that G is k-colourable.

1.3 Example. Suppose we want to create a class schedule so that if a student is taking two classes then those classes are not scheduled at the same time. The colours are the possible time slots and the graph has all classes as its vertices and an edge between two vertices if there is a student that is taking both of those classes. A conflict-free schedule is a proper colouring of this graph with these colours.

We would like to find some bounds on the chromatic number. Recall that

$$\alpha(G) = \max\{|U| : U \subseteq V(G), U \text{ independent}\}$$

is the independence number of G. Then $\chi(G) \geq \frac{n}{\alpha(G)}$, since if φ is a proper colouring of G with $\chi(G)$ colours then each *colour class* is an independent set. Each of these sets has size at most $\alpha(G)$, so

$$n = |V(G)| = \sum_{i=1}^{\chi(G)} |\varphi^{-1}(i)| \leq \chi(G)\alpha(G)$$

1.4 Example. (i) $\chi(C_{2k+1}) \geq \frac{2k+1}{\alpha(C_{2k+1})} \geq \frac{2k+1}{k}$ so $\chi(C_{2k+1}) \geq 3$. It is clear now that $\chi(C_{2k+1}) = 3$.

(ii) $\chi(C_{2k}) = 2$ by similar reasoning.

Another trivial inequality is that if $K_r \subseteq G$ then $\chi(G) \geq r$. Unfortunately, this lower bound is not usually close to the true value. (We shall see more of this later.)

An upper bound for the chromatic number χ of G is $\chi \leq \frac{1}{2} + \sqrt{2|E(G)| + \frac{1}{4}}$. Indeed, if φ is a minimal proper colouring of G, there must be an edge between each pair of colour classes of φ. Hence there are at least as many edge in G as there are pairs of colour classes of φ, so

$$E(G)| \geq \binom{\chi}{2} = \frac{\chi(\chi-1)}{2} \qquad \text{and so} \qquad \chi^2 - \chi - 2|E(G)| \leq 0$$

Greedy Algorithm:

INPUT: graph G
OUTPUT: colouring φ of G

(i) Set an ordering on $V(G)$ (say $V(G) = \{v_1, \dots, v_n\}$)

(ii) Recursively define φ by $\varphi(v_i) = \min\{\mathbb{N} \setminus \{\varphi(v_j) : j < i, v_i \sim v_j\}\}$

The ordering of the vertices affects the number of colours used (consider star graphs and trees). Recall that $\Delta(G)$ is the maximum vertex degree in G and $\delta(G)$ is the minimum vertex degree. The number of ineligible colours is at most Δ, so $\chi \leq \Delta + 1$. There are graphs that have $\chi = \Delta + 1$, some examples are odd cycles, complete graphs, and of course disjoint unions of these graphs. It turns out that these are the only examples, as we shall see.

1.5 Theorem (Greedy Bound).

$$\chi(G) \leq 1 + \max\{\delta(H) \mid H \textit{ is an induced subgraph of } G\}$$

PROOF: In the Greedy Algorithm, at any step the number of ineligible colours for $\varphi(v_i)$ is at most $d_{G_i}(v_i)$, where $G_i = G[v_1, \dots, v_i]$. Hence $\varphi(v_i) \leq 1 + d_{G_i}(v_i)$, so we are done if we find an ordering of the vertices of G such that $d_{G_i}(v_i) = \delta(G_i)$ for each i. But this is easy by choosing the order of the vertices in reverse. Let v_n be a vertex of minimum degree in G, and let v_i be a vertex of minimum degree in $G - \{v_n, \dots, v_{i+1}\}$. □

Remark (Lovasz). Notice that $\chi(K_n - \{u, v\}) = n - 2$. Suppose that G satisfies $\chi(G - \{u, v\}) = \chi(G) - 2$ for all u and v adjacent in G. Does this imply that G is complete?

1.6 Lemma. *Let G be*

(i) connected

(ii) not complete

(iii) $\Delta(G)$-regular, with $\Delta \geq 3$

(iv) have no cutvertices

Then G has vertices v_1, v_2, v_n such that $v_1v_2 \notin E$, $v_1v_n, v_2v_n \in E$, and the graph $G - \{v_1, v_2\}$ is connected.

PROOF: If G is not 3-connected then G has a two vertex cutset $\{u, v\}$. Let $v_n = v$ and notice that $G - \{v_n\}$ has a block-cutvertex tree with at least three vertices, so $G - \{v_n\}$ has at least two endblocks. G is 2-connected, so v_n is adjacent to at least one non-cutvertex in each endblock. Let v_1 and v_2 be two such neighbours in different endblocks. Then $v_1v_2 \notin E$. Finally, note that deleting v_1 or v_2 from their blocks leaves the rest of their blocks connected. Clearly then $(G - \{v_n\}) - \{v_1, v_2\}$ is connected. But since $\Delta \geq 3$, $d_G(v_n) \geq 3$, so $G - \{v_1, v_2\}$ is connected. Now if G is 3-connected then since it is not complete there are $v_1, v_2 \in V$ such that $v_1v_2 \notin E$. Choose two such vertices with the minimum distance between them. Let P be the shortest uv-path. P must have length 2, so let v_n be the middle vertex. □

1.7 Theorem (Brooks). *If G is connected and not an odd cycle or complete then $\chi(G) \leq \Delta(G)$.*

PROOF: We have already seen that odd cycles and complete graphs fail this test. Suppose that G is connected and neither an odd cycle nor a complete graph. Let $\Delta = \Delta(G)$.

Suppose that G is not Δ-regular. Then G has no Δ-regular subgraph. Indeed, if H were a Δ-regular subraph and $v \in V(G) \setminus V(H)$ then, since G is connected, there is a path from v to H. If u' is the first vertex of H on this path then u' would have degree at least $\Delta + 1$ in G, which is impossible. The Greedy Bound gives us the result in this case, since in every induced subgraph H $\delta(H) < \Delta(H) \leq \Delta$.

Now assume that G is Δ-regular has a cutvertex. Let v be a cutvertex of G and let $G_1, \dots, G_m$ be the connected "components" of $G - \{v\}$ (so that v is a vertex of each of $G_1, \dots, G_m$, and $G - \{v\}$ is the disjoint union of $G_1 - \{v\}, \dots, G_m - \{v\}$). Since G is connected, $d_{G_j}(v) \leq \Delta - 1$ for each $j = 1, \dots, m$. Hence none of $G_1, \dots, G_m$ are Δ-regular, so if their maximum vertex degree is Δ then they are examples of connected graphs that are neither complete nor odd cycles, so $\chi(G_j) \leq \Delta$ by the first case. If their maximum vertex degree is less than Δ then $\chi(G_j) \leq \Delta$ by the Greedy Bound. It follows that we may colour each of $G_1, \dots, G_m$ with colours $\{1, \dots, \Delta\}$, and without loss of generality we may assume that the colour if v is the same for all graphs. The union of these colourings is a colouring of G with Δ colours.

Finally, assume that G is Δ-regular and has no cutvertex. If $\Delta = 0$ then $G = K_1$, which we have excluded. If $\Delta = 1$ then $G = K_2$, which we have also excluded. If $\Delta = 2$ then G is a cycle, so it must be an even cycle since we have excluded the odd cycles, and even cycles have chromatic number 2. Therefore we may assume $\Delta \geq 3$. Recall from the Greedy Algorithm that $\varphi(v_i) \leq 1 + d_{G[v_1,\dots,v_i]}(v_i)$. Order the vertices as in 1.6, where for $i = n, \dots, 4$, define v_{i-1} to be any vertex in $G - \{v_1, v_2, v_n, \dots, v_i]$ adjacent to one of $v_n, \dots, v_i$. Such a vertex exists since $G - \{v_1, v_2\}$ is connected. Now apply the greedy algorithm to G with this ordering. $\varphi(v_1) = 1$, $\varphi(v_2) = 1$, $\varphi(v_{i-1}) \leq \Delta$ $(4 \leq i \leq n)$ since v_{i-1} has an edge leading out of $G[v_1, \dots, v_{i-1}]$, and finally, v_n has two neighbours with colour 1, so $\varphi(v_n) \leq \Delta$. Therefore $\chi(G) \leq \Delta$. □

Given G with no subgraphs isomorphic to K_3, is there an upper bound on the chromatic number? Unfortunately, no. What follows is a fairly simple proof proof; there are proofs in which the graphs constructed by induction have considerably fewer vertices.

1.8 Theorem (Tutte). *For all k there is a graph of girth at least 4 and chromatic number at least k.*

PROOF: By induction on k. If $k = 1$ then K_1 has girth at least 4 and chromatic number $1 \geq k$. Now let $k \geq 2$ be arbitrary and take G such that $\chi(G) \geq k - 1$ and the girth of G is at least 4. Let $n = |V(G)|$ and $W = \{w_1, \dots, w_{(k-1)(n-1)+1}\}$ be a set of fresh vertices. For each $S \subseteq W$ with $|S| = n$, create a copy G_S of G (*i.e.* $\binom{|W|}{n}$ disjoint copies of G are created), and join each vertex of G_S to its correspoinding vertex in S. Call this new graph H. If there were a triangle in H then, since G has no triangles, this triangle would have to enter two distinct copies of G. But this is impossible since such a "triangle" would have to have at least 6 edges. To see that the chromatic number of H is at least k, suppose that it is properly $(k-1)$-colourable. Then W is coloured with at most $k - 1$ colours, so by the Pigeonhole principle there is $S \subseteq W$ with $|S| = n$ such that all vertices of S have the same colour. But then none of the vertices in G_S can be coloured that colour, and that would imply a $k - 2$ colouring of G, which is impossible. Therefore $\chi(H) \geq k$. □

1.9 Theorem. *There exist graphs with arbitrarily large girth and chromatic number.*

The first proofs of this were by probablistic methods, but eventually such graphs were constructed.

1.10 Theorem (Johansson, 1993). *There exists a constant c such that, for all triangle-free graphs G,*

$$\chi(G) \leq \frac{c\Delta(G)}{\log \Delta(G)}$$

1.2 Kneser graphs

1.11 Definition. The *Kneser graph* $KN(n,k)$ has all k-subsets of $\{1, \dots, n\}$ as vertices, where two vertices are adjacent if and only if they are disjoint.

1.12 Example. (i) $KN(5,1) \cong K_5$
(ii) $KN(4,2)$ is 3 copies of K_2.
(iii) $KN(5,2)$ is the Peterson graph.

It was conjectured by Kneser in 1955 that $\chi(KN(2k+\ell,k)) = \ell + 2$ for all $k \geq 1$ and $\ell \geq 0$. The proof of "$\leq$" is straightforward, but the other direction of the inequality would have to wait for Lovasz.

1.13 Lemma. $\chi(KN(2k+\ell,k)) \leq \ell + 2$

PROOF: Starting at "1" and moving up to "$\ell+1$", colour each $S \subseteq \{1,\dots,n\}$ that contains "i" the colour i, unless it has been previously coloured. The uncoloured vertices are k-subsets of $\{\ell+2,\dots,2k+\ell\}$, a set of size $2k-1$. Hence the remaining vectices form an independent set and can be coloured with the remaining colour $\ell+2$. □

1.14 Theorem (Lovasz). $\chi(KN(2k+\ell,k)) \geq \ell+2$

PROOF (BÁRÁNY, 1978): Let S_n be the n-sphere, and for any $a \in S_n$, let $H(a)$ be the open hemisphere centered at a. Borsuk's Theorem states that if $n+1$ open sets cover S_n then at least one of them contains a pair of antipodal points. Gale's Theorem states that there are $2k+\ell$ points in S_ℓ such that every hemisphere contains at least k of them.

Take the $2k+\ell$ Gale points in S_ℓ, so that each $H(a)$ contains at least k of them. Suppose for contradiction that $\chi(KN(2k+\ell,k)) \leq \ell+1$, so we can colour it properly with $\{1,\dots,\ell+1\}$. Define

$$A_i = \{a \in S_\ell \mid H(a) \text{ contains a } k\text{-subset of colour } i\}$$

By the choice of Gale points, each $a \in S_\ell$ is in some A_i, so $A_1,\dots,A_{\ell+1}$ is a cover of S_ℓ. Each A_i is clearly open, so we may apply Borsuk's Theorem to see that some A_i contains both a and $-a$. Hence $H(a)$ and $H(-a)$ both contain a k-subset of colour i, say $X_1 \subseteq H(a)$ and $X_2 \subseteq H(-a)$. But $H(a)$ and $H(-a)$ are disjoint, so X_1 and X_2 are also disjoint, and by the definition of the Kneser graph they are adjacent in $KN(2k+\ell,k)$, which contradicts that the colouring was proper. □

1.3 Edge Colourings

1.15 Definition. G is *k-(edge)-colourable* if there is $\varphi : E(G) \to \{1,\dots,k\}$ such that φ is a proper colouring. $\chi'(G)$ is the *chromatic index of G*—the minimum k such that G is k-colourable.

Each edge colour class is a matching of G.

1.16 Theorem (König, 1916). $\chi'(G) = \Delta(G)$ *if G is bipartite.*

PROOF: Add edges to G to make it $\Delta(G)$-regular. By Hall's Theorem this new G has a perfect matching. Colour this matching some colour and then remove it. The resulting graph is $\Delta(G)-1$ regular, so by induction it is $\Delta(G)-1$ edge colourable. □

Notice that $\chi'(G) = \chi(L(G))$, where $L(G)$ is the *line graph* of G. (It is defined by taking $V(L(G)) = E(G)$ and defining two vertices to be adjacent if they have a common endpoint.) Now $\Delta(L(G)) \leq 2\Delta - 1$, and by Brooks' theorem, most of the time $\chi(L(G)) \leq 2\Delta - 2$, so usually $\chi'(G) \leq 2\Delta(G) - 2$. We can do better.

1.17 Theorem (Vizing). $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$.

PROOF: Clearly $\chi'(G) \geq \Delta(G)$, since the vertex of maximum degree has all $\Delta(G)$ of its edges adjacent.

Let G be a counterexample to the upper bound with the minimal number of edges. We will $\Delta+1$ (edge)-colour G for contradiction. $|E(G)|$ must be at least one, so let $xy_1 \in E(G)$. Then $\Delta(G - xy_1) \leq \Delta(G)$, so by minimality $G - xy_1$ can be $\Delta+1$ coloured. Let φ be a $\Delta+1$ colouring of the edges of $G - xy_1$. For each vertex $v \in V(G)$, define miss(v) to be the smallest colour not on any edge incident with v. Define y_i to be a vertex adjacent to x such that the colour on xy_i is miss(y_{i-1}), for $i = 2,3,\dots,m$, where m is maximal (*i.e.* miss(y_m) is not on any unused edge incident with x).

If miss(y_m) is not on any edge incident with x then shift the colours so that miss(y_i) is moved to the edge xy_i for $i = 1,\dots,m-1$. this is clearly legal at x, and it is legal at each y_i by the definition of miss. Let the colour on xy_m be miss(y_m), and we have a $\Delta+1$ colouring of the edges of G.

If miss(y_m) is on an edge incident with x then by maximality of m, miss(y_m) is on xy_j for some $1 < j < m$. For $i = 2, \dots, j$, shift the colour from xy_i to the edge xy_{i-1}. Now the colour miss(y_m) is missing from y_j. Let miss(x) $= s$ and miss(y_m) $=$ miss(y_j) $= t$. Let H be the subgraph of G induced by edges that are now the colours s or t. s is still missing from x, and the maximum degree of H is 2, so $d_H(x) \le 1$. Also, y_j and y_m both have degree at most 1 in H. H is a disjoint union of paths and cycles, so x, y_j, and y_m are endpoints of paths, and hence cannot all be in the same connected component. Suppose first that y_j is not in the same component as x. On the path containing y_j in H, swap all the s's and t's. This induces a legal colouring of $G - xy_j$, and we may colour xy_j with s to get a $\Delta + 1$ colouring of the edges of G. If y_j is in the same component as x then they are the two endpoints of a path and y_m is in a different connected component. Swap s and t in y_m's component to get another legal colouring of $G - xy_j$. Now in G, shift the colour on xy_i to xy_{i-1} for $i = j + 1, \dots, m$ (swapping s and t does not affect these colours). Colour xy_m with s, and we are done. □

1.4 Vertex List Colouring

1.18 Definition. Let $L(v)$ be a set of colours for each $v \in V$. G is said to be *list coloured with respect to* L if $\varphi : V \to \{1, 2, \dots\}$ is a proper colouring of G and $\varphi(v) \in L(v)$ for all $v \in V$.

1.19 Definition. G is *k-list-colourable* (or *k-choosable*) if for every $L : V \to \mathscr{P}(\{1, 2, \dots\})$ with $|L(v)| \ge k$ then G can be list coloured with respect to L. The minimum such k is called the *list chromatic number* (or *choice number*) of G, denoted $\chi_L(G)$ (or ch(G)).

If $L(v) = \{1, \dots, k\}$ for all v then G can be list-coloured with respect to L if and only if $\chi(G) \le k$. Whence $\chi_L(G) \ge \chi(G)$, (this is also seen by taking $L(v) = \{1, \dots, \chi(G) - 1\}$ for all $v \in V$). Is there G such that this inequality is strict? Yes, for example $K_{3,3}$.

1.20 Theorem (Alon, 1993). *There is a function* $f : \mathbb{N} \to \mathbb{N}$ *with* $\lim_{k\to\infty} f(k) = \infty$ *such that every graph with average degree at least* $f(k)$ *has list chromatic number at least* k.

1.21 Definition. A *plane graph* is a graph embedded in the plane properly. A *plane near-triangulation* is a plane graph in which the boundry of every face is a simple closed curve and all faces except the unbounded one are triangles (*i.e.* 3-cycles). If the external face is also a 3-cycle then this is known as a *plane triangulation*.

1.22 Lemma. *Every planar graph G with at least 3 vertices can be embedded in a plane triangulation.*

PROOF: Take a plane graph embedding of G. Add diagonals greedily in each face with boundry of edge-length at least 4. We must avoid adding any loops or multiple edges. Take any face with boundry of edge-length at least 4. If a loop can be created then instead add a diagonal between a point inside the loop and a point outside the loop. This diagonal will not be a loop or multiple edge. If a multiple edge were to be created then instead add a diagonal from a point inside of the multiple edges and a point outside. The result is a plane triangulation. □

1.23 Proposition. *Let* G *be a plane near-triangulation. Let* v_1 *and* v_2 *be two consecutive external vertices (i.e. vertices on the external face that are consecutive on the exterior boundry). Suppose that* $|L(v_1)| = 1 = |L(V_2)|$ *and* $L(v_1) \ne L(v_2)$ *and* $|L(v)|$ *for all other external vertices* v, *and* $|L(v)| = 5$ *for all internal* v. *Then* G *has a list colouring with respect to* L.

PROOF: By induction on the number of vertices $|V(G)| = n$. If $n = 3$ then clearly G is colourable with respect to L. Assume that the proposition hold for all graphs with fewer that n vertices, where $n \ge 4$. Let G be a plane near-triangulation on n vertices and L is as described. There are a number of cases.

If G has a diagonal joining non-consecutive external vertices, then $G = G_1 \cup G_2$, both of which are near-triangulations with less than n vertices each. Either v_1 and v_2 are both in G_1 or both in G_2. Without loss of generality, both are in G_1. By induction, G_1 can be list coloured with respect to L. If u_1 and u_2 are the vertices on the diagonal, then they now have colours, and let the lists on all of the remaining vertices of G_2 agree with L. By induction G_2 can be list coloured with respect to this listing, which gives a list colouring of G with repsect to L.

Now suppose that G has no diagonal. Let v_3 and v_4 be the next two consecutive vertices around the external face after v_1 and v_2 (it may be the case that $v_4 = v_1$). Let $u_1, \dots, u_k$ be the internal neighbours of v_3. Then $G \setminus \{v_3\}$ is a plane near-triangulation since G has no diagonals. Let $x, y \in L(v_3) \setminus L(v_2)$. For $1 \le i \le k$, put $L'(u_i) = L(u_i) \setminus \{x, y\}$, minus whatever else you want to make $|L'(u_i)| = 3$. For all other vertices let $L' = L$. But induction we can list colour $G \setminus \{v_3\}$ with respect to L. But now we can colour G according to this colouring, colouring v_3 either x or y, whichever is not on v_4. □

1.24 Corollary (Thomassen). *Every planar graph is 5-choosable.*

PROOF: Every planar graph can be embedded in a plane triangulation by 1.22, and any list L such that $|L(v)| \ge 5|$ for all $v \in V(G)$ may be restricted to a list L' that satisfies the conditions of 1.23. A colouring with respect to L' is a colouring with respect to L. □

1.5 Edge List Colouring

1.25 Definition. Analogously, $\chi'_L(G)$ (or $\mathrm{ch}'(G)$) is defined to be $\chi_L(L(G))$, the *edge-list-chromatic number* (or *list-chromatic index*). It is the minimum k such that for all $L : E(G) \to \mathscr{P}(\{1, 2, \dots\})$ with $|L(e)| \ge k$ for all $e \in E(G)$, there is a proper edge colouring of G with respect to L.

Applying the Greedy Algorithm to G implies that $\chi_L(G) \le \Delta(G) + 1$. Hence

$$\chi'_L(G) = \chi_L(L(G)) \le \Delta(L(G)) + 1 \le 2\Delta(G) - 1$$

Vizing thinks we can do better.

1.26 Conjecture (Vizing, 1975). $\chi_L(G) = \chi'(G)$ *for all* G.

1.27 Theorem (Kahn, 1993). $\lim_{\Delta \to \infty} \max_{\Delta(G) = \Delta} \dfrac{\chi'_L(G)}{\chi'(G)} = 1.$

Dinitz later conjectured that $\chi'_L(K_{n,n}) = \chi'(K_{n,n})$ for all n, even though his conjecture is implied by Vizing's conjecture. Galvin eventually proved Vizing's conjecture holds for bipartite graphs, but no one has yet proven the general conjecture.

Recall that for a directed graph D, for any $v \in V(D)$, $d^+(v)$ is the number of edges leaving v and $d^-(v)$ is the number of edges entering v. A *kernel* of a digraph is an indepedent set $U \subseteq V(D)$ such that for each $v \in V(D) \setminus U$, there is $u \in U$ such that $vu \in E(D)$.

1.28 Lemma. *Let G be a graph and D an orientation of G such that every induced subdigraph of D has a kernel. Assume that L is given such that, for all $v \in V(G)$, $|L(v)| \ge d^+_D(v) + 1$. Then G can be vertex-list-coloured with respect to L.*

PROOF: By induction on $n = |V(G)|$. When $n = 1$ the lemma is trivial. Suppose that $n > 1$ and the lemma holds for all graphs with fewer vertices than G. Pick a colour c appearing in the lists and let $W = \{v \in V(G) \mid c \in L(v)\}$. Let U be a kernel of the subgraph induced by W (guaranteed by hypothesis). Colour each vertex of U with c. Let $G' = G - U$, $L'(v) = L(v) \setminus \{c\}$ for all $v \in V(G')$, and $D' = D - U$. D' is an orientation of G' and every induced subdigraph of D' has a kernel since each subdigraph of D' is a subdigraph of D. $|L'(v)| = |L(v)|$ if $v \notin W$, and if $v \in W$ then $|L'(v)| = |L(v)| - 1$ and $d^+_{D'}(v) \le d^+_D(v) - 1$ since v sends an edge to U. Therefore $|L'(v)| \ge d^+_{D'}(v) + 1$ for all $v \in V(D')$. By induction G' can be list coloured with respect to L'. Such a colouring does not use the colour c, so this induces a colouring of G with respect to L. □

1.29 Theorem (Galvin, 1995). $\chi'_L(G) = \chi'(G)$ *if G is bipartite.*

PROOF: Let G be bipartite. We know $\chi'_L(G) \geq \chi'(G)$, so we need only prove the reverse inequality. Take a k-edge-colouring φ, where $k = \chi'(G)$ and let L be such that $|L(x)| \geq k$ for all $x \in E(G)$. We want to edge list colour G with respect to L.

Define D as follows. Let $V = A \cup B$ be the bipartition of G. $V(D) = E(G)$, and for edges x and y incident at a vertex in A, let $xy \in E(D)$ if and only if $\varphi(x) > \varphi(y)$. If x meets y at a vertex in B then let $xy \in E(D)$ if and only if $\varphi(x) < \varphi(y)$. We show that D is an orientation of $L(G)$ that satisfies the properties of the lemma. For $x \in V(D)$, $d^+_D(x) \leq k-1 \leq |L(x)|-1$ since there are only k edge colours in φ—lower ones appear on the A side of x, higher ones appear on the B side, and x has some colour.

We finally prove that D has the kernel property by induction on $|E(G)| = |V(D)|$. If $V(D)$ is empty then there is nothing to prove. Assume $|V(D)| \geq 1$ and the induction hypothesis holds for all smaller digraphs defined as above from an edge colouring of a bipartite graph. Every proper induced subdigraph of D is a digraph formed in the same way from a proper subset of the edges of G, and so has a kernel. We only need to show that D has a kernel. For $v \in A$, let $m(v)$ be the edge of lowest colour incident with v. Let $U = \{m(v) \mid v \in A\}$. If this is a matching then U is an independent set in D, and indeed a kernel, since for any $ab \in U$, $abm(a)$ is an edge of D since the colour on ab is greater than the colour on $v(a)$. Assume that U is not a matching. Then there are edges $x, x' \in U$ that are both incident to a point $v \in B$. Without loss of generality $xx' \in E(D)$. By induction, $D - \{x\}$ has a kernel U'. We claim that U' is a kernel of D. If $x' \in U'$ then since $xx' \in E(D)$, U' is immediately a kernel of D. If $x' \notin U'$ then since U' is a kernel there is $x'' \in U$ such that $x'x'' \in E(D) - \{x\}$. Then x' and x'' are both incident at B, since they cannot be incident at A. (x' has a lowest colour adjacent to its endpoint in A, by definition of U.) In fact, they are incident at v, since x and x' are incident at v. But x' has a lower colour than x'' and x has a lower colour than x', so by definition of D, $xx'' \in E(D)$, and again U' is a kernel of D. $\square$

1.6 Total Colourings

1.30 Definition. Let $G = (V, E)$ and $\varphi : V \cup E \to \mathbb{N}$, a *total colouring* of G. φ is *proper* if $\varphi(u) \neq \varphi(v)$ whenever $uv \in E$, or u and v are adjacent edges, or u is an edge incident with a vertex v. The *total chromatic number* of G is $\chi''(G) = \min\{k \mid \varphi : V \cup E \to \{1, \ldots, k\} \text{ is proper}\}$

Clearly $\chi''(G) \geq \Delta(G) + 1$ and $\chi''(G) \leq \chi(G) + \chi'(G)$.

1.31 Conjecture (Behzad, Vizing, 1964–1968). $\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2$

1.32 Theorem (Molloy & Reed, 1991). *Let G be a graph and k be such that $k! \geq |V(G)|$. Then $\chi''(G) \leq \Delta(G) + k + 2$.*

PROOF: We have seen that there exists a proper vertex colouring $\varphi : V(G) \to \{1, \ldots, \Delta(G)+1\}$ and, by Vizing's Theorem there is a proper edge colouring $\theta : E(G) \to \{1, \ldots, \Delta(G)+1\}$ For any permutation σ of $\{1, \ldots, \Delta(G)+1\}$, $\sigma \circ \theta : E(G) \to \{1, \ldots, \Delta(G)+1\}$ is a proper edge colouring. For a vertex v, call an edge e incident with v *poor at v with respect to $\sigma \circ \theta$* if $\theta(e) = \varphi(u)$ for some $u \sim v$. For a fixed vertex v, how many permutations σ have at least k edges poor at v?

The idea is to pick k edges incident with v and count the number of σ's for which those particular edges are poor. We can choose k edges incident with v in $\binom{d(v)}{k}$ ways, which is certainly less than $\binom{\Delta+1}{k}$. Choosing k edges incident with v to be poor determines the k colours of those edges, though sometimes some of the colours of the corresponding k neighbours of v will be the same. If k values of $\sigma \circ \theta$ are determined then there are $(\Delta + 1 - k)!$ ways of choosing the remaining values. Therefore there are less than $\binom{\Delta+1}{k}(\Delta + 1 - k)! \leq \frac{(\Delta+1)!}{n}$ different σ's with at least k edges poor at v (and this is an overcounting if there could be greater than k poor at v). Let $V(G) = \{v_1, \ldots, v_n\}$ and let $A_i = \{\sigma \mid \sigma \circ \theta \text{ has at least } k \text{ edges poor at } v_i\}$. Then $|A_i| < \frac{(\Delta+1)!}{n}$, so $|A_1 \cup \cdots \cup A_n| \leq |A_1| + \cdots + |A_n| < (\Delta + 1)!$. There are $(\Delta + 1)!$ permutations of the set $\{1, \ldots, \Delta + 1\}$, so at least one is not in $\bigcup_i A_i$. This σ has the property that the number of edges poor at v with repsect to $\sigma \circ \theta$ is less than k, for all $v \in V$. Use φ to colour V and $\sigma \circ \theta$ to colour all of the non-poor edges. So far this is a "proper" total colouring. Let H be the subgraph of G induced by the edges that are poor at any vertex. Colour $E(H)$ using fresh colours. $d_H(v)$ is the number of poor edges incident with v, which is the number of edges poor at v together with possibly one more edge that is poor at a neighbour of v. Therefore $d_H(v) \leq k$, so by Vizing's Theorem we can edge colour H with at most $k+1$ new colours. Therefore we can totally colour G with $\Delta(G)+k+2$ colours. □

1.33 Theorem (Molloy & Reed, 1998). *There is a constant c such that $\chi''(G) \leq \Delta(G)+c$.*

They proved $c = 10^{26}$ will do, and they claim that $c = 500$ can be proved with the same technique. They believe the proof could be tightened to as low as $c = 100$ with a lot of work. Notice that the 1991 theorem is better than the 1998 theorem (where $c = 10^{26}$) for all graphs G with less than $(10^{26}-2)!$ vertices.

2 Counting

2.1 Little Tricks

Recall that $\binom{n}{k}$ is the number of k-subsets of an n-set. $[n]_k$ denotes the *falling factorial* $n(n-1)\cdots(n-k+1)$. The number of k-subsets of an n-set that contain a given s-subset is $\binom{n-s}{k-s} = \frac{[k]_s}{[n]_s}\frac{n}{k}$.

2.1 Definition. A *labelling* of a graph G is a bijection $V(G) \to \{1,\ldots,|V|\}$. Let $G(n)$ be the set of all labelled graphs on the vertices $[n]$. $G(n,m)$ is the subset of $G(n)$ consisting of graphs with m edges.

For the remainder of this section, let $N = \binom{n}{2}$. Then $|G| = 2^N$ and $|G(n,m)| = \binom{N}{m}$. There are $\frac{1}{2}(n-1)!$ labelled n-cycles.

2.2 Definition. An *automorphism* of G is a bijection $\sigma : V(G) \to V(G)$ such that $v \sim w$ if and only if $\sigma(v) \sim \sigma(w)$ for all $v, w \in V(G)$.

Two labellings f_1 and f_2 give the same element of $G(n)$ if and only if $f_1 \circ \sigma = f_2$ for some non-trivial automorphism σ. Such labellings are called *equivalent*, and this is an equivalence relation. The size of each equivalence class is equal to the number of automorphisms of G, defined to be a_G. The total number of labellings is $n!$, so the number of equivalence classes of labellings is $\frac{n!}{a_G}$, the number of "inequivalent" labellings of G, or the number of elements of $G(n)$ isomorphic to G. For example, $a_{C_n} = 2n$, so there are $\frac{n!}{2n} = \frac{1}{2}(n-1)!$ labelled C_n's.

How many copies of G occur in K_n? Suppose that $|V(G)| = s$. The number is $\binom{n}{s}\frac{s!}{a_G} = \frac{[n]_s}{a_G}$. The idea is to choose s of the n vertices and then label those vertices. What is the average number of copies of G in $G(n,m)$? If G is a triangle then $a_G = 6$, so the number of copies of G in K_n is $\frac{n(n-1)(n-2)}{6}$. The number of graph in $G(n,m)$ that contain a given copy of C_3 is $\frac{m(m-1)(m-2)}{N(N-1)(N-2)}\binom{N}{m}$, so the total number of triangles that occur in all of $G(n,m)$ is $\frac{n(n-1)(n-2)}{6} \cdot \frac{m(m-1)(m-2)}{N(N-1)(N-2)}\binom{N}{m}$. Therefore the average number of triangles in a graph in $G(n,m)$ is $\frac{n(n-1)(n-2)}{6} \cdot \frac{m(m-1)(m-2)}{N(N-1)(N-2)}$. Call this number $\overline{t}$.

2.3 Corollary. *Less than half the graphs in $G(n,m)$ have more than $2\overline{t}$ triangles.*

For example, show that there is a graph G on 9 vertices such that G and $\overline{G}$ each contain at most 19 triangles. Consider $G \in G(9,18)$. Since $N = \binom{9}{2} = 36$, $\overline{G} \in G(9,18)$ as well. The average number of triangles in $G(9,18)$ is $\frac{48}{5}$. Let $S_1, S_2 \subseteq G(9,18)$ be the set of graphs with more than $2\frac{48}{5}$ (*i.e.* more than 19) triangles and whose complement have more than 19 triangles, respectively. Then $|S_1|, |S_2| < \frac{1}{2}G(9,18)$, so there is $G \in G(9,18) \setminus (S_1 \cup S_2)$, which is the required G.

2.4 Example (Tournaments). An *n-tournament* is a labelled, oriented copy of K_n. There are $2^N = 2^{\binom{n}{2}}$ n-tournaments. The number of directed Hamilton cycles on n vertices is $\frac{n!}{n} = (n-1)!$. The number of n-tournaments containing a particular one is 2^{N-n}, so the total number of directed Hamilton cycles in all tournaments is $(n-1)!2^{N-n}$. Therefore the average number is $\frac{(n-1)!}{2^n} =: f_n$. This is interesting because no construction is known for an n-tournament with at least f_n Hamilton cycles. It has been shown that the maximum number of Hamilton cycles is at most $cn^{\frac{3}{2}} f_n$.

2.2 Asymtotics

Notation. Let $f, g : \mathbb{N} \to \mathbb{C}$ be functions. We write $f(n) \prec g(n)$ to mean $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$. Landau (1909) instead used $f(n) = o(g(n))$ ("little-o notation") to mean the same thing. Here $o(g(n))$ is to be interpreted as the collection $\{h(n) \mid \lim_{n\to\infty} \frac{h(n)}{g(n)} = 0\}$, and "=" is to be interpreted as "$\in$".

2.5 Example. $1 \prec \log n \prec (\log n)^2 \prec \sqrt{n} \prec n \prec n^2 \prec (\log n)^{\log n} \prec n^{\log n} \prec e^{\sqrt{n}} \prec e^n \prec n! \prec n^n \prec 2^{\binom{n}{2}}$

Notation. We write $f(n) \sim g(n)$ to mean $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$. This is equivalent to $f(n) = (1 + o(1))g(n)$.

Stirling's formula asserts that $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, which implies $\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}}$.

Notation. Bachmann (1894) introduced the notation $f(n) = O(g(n))$ ("big-O notation") to denote that $f(n) \le cg(n)$ for some constant c and n sufficiently large. Here $O(g(n))$ is to be intepreted as the collection of functions $\{h(n) \mid \exists c > 0, N \in \mathbb{N}\ \forall n \ge N\ |h(n)| \le cg(n)\}$, and "=" is to be interpreted as "$\in$".

2.6 Theorem (Taylor). *Suppose that $f^{(k+1)}(x)$ is bounded for all $|x| \le a$. Then for all such x,*

$$f(x) = \sum_{i=0}^{k} f^{(i)}(0)\frac{x^i}{i!} + O(|x|^{k+1})$$

2.7 Example. If $g(n) = o(1)$ then $\frac{1}{1-g(n)} = 1+O(g(n))$. We have the following estimates on binomial coefficients. For $k = o(\sqrt{n})$, $\binom{n}{k} \sim \frac{n^k}{k!}$. For $1 \le k \le n$, $\binom{n}{k} \le \frac{n^k}{k!}$ and $\binom{n}{k} \le (\frac{en}{k})^k$.

Recall $\log(1 + x) = x - \frac{1}{2}x^2 + O(|x|^3)$ and $e^x = 1 + x + O(|x|^2)$.

2.3 Properties of Almost All Graphs

2.8 Definition. We say that *almost all graphs have property P* if the proportion of graphs in $G(n)$ with property P goes to 1 as $n \to \infty$. We may also look only at graphs in a specific class.

2.9 Definition. The *diameter* of a graph G is $\max\{d(u, v) \mid u, v \in V(G)\}$. The diameter of a disconnected graph is ∞.

2.10 Theorem. *Almost all graphs have diameter 2.*

PROOF: The complete graph K_n is the only graph in $G(n)$ with diameter 1, so the proportion of graphs with diameter 1 is $\frac{1}{N}$. Let $S_{i,j} \subseteq G(n)$ be those graphs with no 2-path from i to j. Then $\bigcup S_{i,j}$ contains all graphs with diameter at least 3 in $G(n)$. In $S_{i,j}$, for any vertex $w \ne i, j$, either wi or wj may be edges, or neither, but not both. Therefore there are 3^{n-2} possibilities for such edges incident with i or j in graphs in $S_{i,j}$. The remaining

$\binom{n}{2} - 2(n-2)$ possible edges can be placed in $2^{\binom{n}{2}-2(n-2)}$ ways, so $|S_{i,j}| = 3^{n-2}2^{\binom{n}{2}-2(n-2)}$. There are $\binom{n}{2}$ different $S_{i,j}$, so the size of the union is at most

$$\binom{n}{2}3^{n-2}2^{\binom{n}{2}-2(n-2)} = \binom{n}{2}\left(\frac{3}{4}\right)^{n-2}|G(n)| = o(1)|G(n)| \qquad \square$$

2.11 Corollary. *Almost all graphs are connected.*

Show that almost all graphs are k-connected, for any fixed k, using the same method. In fact, k can be an unbounded function of n.

2.12 Corollary. *Almost all graphs have every edge in a triangle.*

2.13 Theorem. *Almost all graphs have no non-trival automorphisms.*

Proof: The number of graphs with no non-trivial automorphism is at most $\sum_{\sigma \in \mathfrak{S}_n \setminus \{id\}} \text{fix}(\sigma)$, where fix($\sigma$) is the number of graphs in $G(n)$ fixed by σ. Given σ, define $\sigma^*(\{u,v\}) = \{\sigma(u), \sigma(v)\}$ for all $u, v \in V(G)$ (which we will identify with $\{1,\dots,n\}$). Then σ^* is a permutation of the unordered pairs of elements of $\{1,\dots,n\}$. The *support* of a permutation is the set of elements not fixed by the permutation. Given a permutation σ, let R be the support of σ and S be the support of σ^*, and $r = |R|$. We want a bound on $|S|$ in terms of r. If $u \in R$ and $v \in V \setminus R$ then $\{u,v\} \in S$, and there are $r(n-r)$ such pairs. If $u, v \in R$ and $\{u,v\} \notin S$ then $\sigma(u) = v$ and $\sigma(v) = u$, and there are at most $\frac{r}{2}$ such pairs. Hence S contains at least $\binom{r}{2} - \frac{r}{2}$ pairs $\{u,v\}$ with $u, v \in R$.

$$\begin{aligned}
|S| &\geq r(n-r) + \binom{r}{2} - \frac{r}{2} \\
&= nr - r^2 + \frac{r^2 - r}{2} - \frac{r}{2} \\
&= \frac{r}{2}(2n - 2 - r) \\
&\geq \frac{r}{2}(n-2) && \text{since } r \leq n
\end{aligned}$$

Consider the digraph D defined by $V(D) = \{u,v\}$, $u \neq v \in V$, and containing every arc $\{u,v\}\sigma^*(\{u,v\})$. For each cycle C of D, if σ is an automorphism of a graph G then either all vertices of C are edges of G or none are. So if d is the number of cycles of D then fix(σ) $= 2^d$. Note that the number of vertices in D in cycles of length at least 2 is $|S|$. So the number of such cycles is at most $\frac{|S|}{2}$. The number of cycles in D of length 1 is $|V(D)| - |S| = N - |S|$. Hence

$$d \leq N - |S| + \frac{|S|}{2} = N - \frac{|S|}{2} \leq N - \frac{r}{4}(n-2)$$

Therefore $\text{fix}(\sigma) = 2^d \leq 2^{N-\frac{r}{4}(n-2)}$. Therefore the number of graphs with a non-trivial automorphism is at most

$$\begin{aligned}\sum_{\sigma \neq id} \text{fix}(\sigma) &\leq \sum_{r=2}^{n} \sum_{\substack{\sigma \text{ with} \\ |R|=r}} 2^{N-\frac{r}{4}(n-2)} \\ &\leq \sum_{r=2}^{n} r! \frac{n}{r} 2^{N-\frac{r}{4}(n-2)} \\ &\leq \sum_{r=2}^{n} n^r 2^{N-\frac{r}{4}(n-2)} \\ &= |G(n)| \sum_{r=2}^{n} \left(n 2^{-\frac{1}{4}(n-2)}\right)^r \\ &= |G(n)| o(1)\end{aligned}$$

Therefore almost all graphs have no non-trivial automorphisms. □

This theorem is not true for almost all trees, considering that it is likely that some branch of the tree will have 2 or more leaves. Permuting these leaves gives a non-trivial automorphism.

2.4 Obtaining Asymtotic Estimates from Generating Functions

Suppose we have a generating series $f(z) = \sum_{n \geq 0} a_n z^n$. Recall that the radius of convergences of this series is $(\limsup_{n \to \infty} |a_n|^{\frac{1}{n}})^{-1}$. The radius of converge is also the minimal modulus amoung the singularities of f.

The plan is, given an analytic function f, find the radius of convergence in terms of the singularities of f and then use it to estimate the growth of the coefficients of the Taylor series.

2.14 Example. Let $f(z) = \frac{1}{2-e^z}$. The singularities of this function are $\log 2 + 2k\pi i$ for $k \in \mathbb{Z}$. The minimum modulus singularity is $\log 2$, so the radius of convergence is $\log 2$. If $\frac{1}{2-e^z} = \sum_{n \geq 0} a_n z^n$ then $\limsup |a_n|^{\frac{1}{n}} = \frac{1}{\log 2}$. Therefore only finitely many of them have $|a_n| > (\frac{1}{\log 2} + \varepsilon)^n$, and infinitely many have $|a_n| > (\frac{1}{\log 2} - \varepsilon)^n$. for any $\varepsilon > 0$.

2.15 Example. Labelled 2-regular graphs consist of disjoint unions of labelled cycles. The number of labelled n-cycles is $\frac{n!}{2n}$. This implies that the exponential generating series for the set of all cycles $\mathscr{C}$ is

$$\sum_{n \geq 3} \frac{n!}{2n} \frac{x^n}{n!} = \frac{1}{2} \sum_{n \geq 3} \frac{x^n}{n} = -\frac{1}{2} \log(1-x) - \frac{1}{2}x - \frac{1}{4}x^2$$

Recall from C&O 330 that the exponential generating series for the collection of canonical unordered sets $\mathscr{U}$ is e^x. Clearly the collection of all 2-regular graphs decomposes as $\mathscr{U} \star \mathscr{C}$, so by the $\star$-composition lemma the exponential generating function for labelled 2-regular graphs is

$$e^{\frac{1}{2}\log(1+x) - \frac{1}{2}x - \frac{1}{4}x^2} = \frac{e^{-\frac{1}{2}x - \frac{1}{4}x^2}}{\sqrt{1-x}}$$

This function has a singularity only at $x = 1$, so the radius of convergence of the exponential generating function is $R = 1$. Let a_n be the number of labelled 2-regular graphs on n-vertices. Then

$$\frac{a_n}{n!} = [x^n] \frac{e^{-\frac{1}{2}x - \frac{1}{4}x^2}}{\sqrt{1-x}}$$

so $\limsup_{n\to\infty}(\frac{a_n}{n!})^{\frac{1}{n}} = 1$. This is a very weak conclusion, however. Indeed, $\frac{n!}{2n} \le a_n \le n!$ for all $n \ge 3$.

2.16 Example. Consider the set $\mathscr{C}$ of labelled 2-regular graphs in which the cycles have been coloured one of k colours. The generating series for labelled cycles coloured one of k colours is $-\frac{k}{2}\log(1-x) - \frac{k}{2}x - \frac{k}{4}x^2$. It follows that the generating function for $\mathscr{C}$ is

$$\left(\frac{e^{-\frac{1}{2}x-\frac{1}{4}x^2}}{\sqrt{1-x}}\right)^k$$

Therefore, if the number of such graphs on n vertices is a_n, $\limsup_{n\to\infty}(\frac{a_n}{n!})^{\frac{1}{n}} = 1$. This implies, for any k, that $a_n < (1+\varepsilon)^n n!$ except for finitely many n. This gives us information about the number of cycles in 2-regular graphs, on average.

3 Linear Algebra

3.1 Eigenvectors of Graphs

3.1 Definition. Let A be an $n \times n$ matrix. An *eigenvalue* of A is a solution to the equation $\det(\lambda I_n - A) = 0$.

Recall that the determinant is zero if there is a solution to $(\lambda I_n - A)x = 0$, or $Ax = \lambda x$. Such a vector x is called the *eigenvector* of A associated with eigenvalue λ. Recall further that $\det(\lambda I_n - A)$ is a polynomial in λ (the *characteristic polynomial*) which has degree n, so A has n eigenvalues (counted with respect to multiplicity).

3.2 Definition. Given a graph G on vertices $\{1,\dots,n\}$, the *adjacency matrix* of G is $A = A(G)$, where $A_{i,j} = 1$ if $ij \in E$ and 0 otherwise. The *eigenvalues* of G are defined to be the eigenvalues of $A(G)$.

Notice that A is a symmetric $\{0,1\}$-matrix with zero diagonal (since our graphs have no loops or multiple edges). The eigenvalues of G don't depend upon the labelling of the vertices.

3.3 Lemma. *The number of (i,j)-walks in G of length r is $[A(G)^r]_{i,j}$.*

PROOF: By induction on r. If $r = 0$ then $A^0 = I_n$, which is clearly correct. Suppose $r > 0$ and the lemma holds for all smaller r. $A^r = A^{r-1}A$, and the (i,j) entry of this matrix is $\sum_{k=1}^n [A^{r-1}]_{i,k} A_{k,j}$. But $[A^{r-1}]_{i,k}$ is the number of walks from i to k or length $r-1$, and $A_{k,j}$ is the number walks from k to j of length 1 (clearly 1 or 0 depending on whether k and j are neighbours). But any walk from i to j (where $i \neq j$) must pass through a neighbour of j, so this is the number of (i,j)-walks of length r. □

3.4 Corollary. *The number of closed walks of length r in G is* $\mathrm{Tr}(A(G)^r)$.

3.5 Corollary. *Let G be a graph with n vertices, m edges, and t triangles. Then*

(i) $\mathrm{Tr}(A) = 0$
(ii) $\mathrm{Tr}(A^2) = 2m$
(iii) $\mathrm{Tr}(A^3) = 6t$

Recall that for a real symmetric matrix A,

(i) all of the eigenvalues of A are real.
(ii) linearly indepedent eigenvectors of A associated with different eigenvalues of A are orthogonal.
(iii) $\mathbb{R}^n$ and $\mathbb{C}^n$ have an orthonormal bases consisting of eigenvectors of A.

Given a graph $G = (V, E)$, suppose that $x \in \mathbb{R}^V$ is a non-zero labelling of G such that there is λ so that $\sum_{u\sim v} x(u) = \lambda x(v)$, for every $v \in V$. We may consider x to be a vector, and if A is the adjacency matrix of x then $[Ax]_v = \sum_{u\sim v} x_u = \lambda x_v$. But since this works for all $v \in V$, x is an eigenvector of A associated with eigenvalue λ. Conversely, if we have an eigenvector then we can find such a labelling, so this gives us a characterization of the eigenvectors and eigenvalues of a graph independent of the adjacency matrix.

3.6 Example. Let $\tau = e^{\frac{2\pi i}{n}}$ be an n^{th} root of unity and $f_k(u) = \tau^{ku}$ for each $0 \le k < n$ and $u \in V(C_n) = \{1, \dots, n\}$. Then

$$f_k(u-1) + f_k(u+1) = \tau^{(u-1)k} + \tau^{(u+1)k} = \tau^{uk}(\tau^{-k} + \tau^k) = (\tau^{-k} + \tau^k) f_k(u)$$

for every u. Therefore f_k is an eigenvector with associated eigenvalue $\tau^{-k} + \tau^k$, for all k. In fact, $\{f_k \mid 0 \le k < n\}$ is a collection of n linearly independent eigenvectors of C_n, so it is a complete collection and this set forms an orthogonal basis for $\mathbb{C}^n$.

If G is k-regular then k is an eigenvalue of G with eigenvector $(1, \dots, 1)^T = \mathbf{1}$. Indeed, $[A\mathbf{1}]_i = d_G(i)$, so $\mathbf{1}$ is an eigenvector if and only if G is k-regular. If J is the matrix of all ones, the i^{th} row of AJ is $d_G(i)\mathbf{1}^T$, while the j^{th} column of JA is $d_G(j)\mathbf{1}$. Therefore $AJ = JA$ if and only if G is k-regular.

For any graph, $A(\overline{G}) = J - I - A(G)$, and if G is k-regular then each vertex of $\overline{G}$ has degree $n - 1 - k$. Suppose $\{\mathbf{1}^T, x_2, \dots, x_n\}$ be an orthogonal basis of $\mathbb{C}^n$ consisting of eigenvectors of G, associated with eigenvectors $\{k, \lambda_2, \dots, \lambda_n\}$. Then for $1 < i \le n$,

$$A(\overline{G})x_i = Jx_i - Ix_i + A(G)x_i = 0 - x_i - \lambda_i x_i = -(1 + \lambda_i)x_i$$

Therefore the $\overline{G}$ also has eigenvectors $\{\mathbf{1}^T, x_2, \dots, x_n\}$, and $\{n-1-k, -(1+\lambda_2), \dots, -(1+\lambda_n)\}$ are the associated eigenvalues.

3.7 Theorem. *If G is k-regular and $A = A(G)$ then*

(i) if θ is an eigenvalue of G then $|\theta| \le k$.

(ii) if G is connected then k has multiplicity one, and $-d$ is an eigenvalue if and only if G is bipartite.

PROOF: Let $x = (x_1, \dots, x_n)^T$ be an eigenvector of associated with θ. Assume that v maximizes $|x_v|$. Then

$$|\theta||x_v| = |\theta x_v| = \left|\sum_{u\sim v} x_u\right| \le \sum_{u\sim v} |x_u| \le k|x_v|$$

Therefore $|\theta| \le k$.

If $|\theta| = k$ then the inequalities above become equations, so $|x_u| = |x_v|$ for every $u \sim v$. Furthermore, the direction (*a.k.a.* the complex sgn) of x_u is the same for every $u \sim v$. If we assume that G is connected then it follows that $|x_i| = |x_j|$ for all $i, j \in V(G)$. Therefore $x_i = \pm x_v$ for all $i \in V(G)$. If $\theta = k$ then $\mathbf{1}$ is the only eigenvector, so the eigenspace associated with k has dimension 1, so k has multiplicity one. If $\theta = -k$ then $x_u = -x_v$ for all $u \sim v$, so there are two types of vertices in G, $\{i \in V \mid x_i = x_v\}$ and $\{i \in V \mid x_i = -x_v\}$. In fact, this is a bipartition and G is birpartite. Conversely, the vector x defined by $x_i = 1$ if $i \in A$ and $x_i = -1$ if $i \in B$ (where $V(G) = A \cup B$) then the eigenvalue associated with x is $-k$. □

3.2 Moore Graphs

Recall that the length of the shortest cycle in G is the *girth* of G, and the *diameter* of G is $\max_{u,v}\{d(u, v)\}$. If G is a non-tree graph with diameter d then the maximum length that a shortest cycle in G can have is $2d + 1$, *i.e.* the girth of G is at most $2d + 1$.

3.8 Definition. A *Moore graph* is a graph with diameter d and girth $2d + 1$.

3.9 Lemma. *In a graph with girth $2d+1$, any two vertices are joined by at most one path of length at most d.*

PROOF: Two paths of length at most d between two vertices would combine to form a cycle of length at most $2d$, which is impossible. □

3.10 Theorem. *Every Moore graph is regular.*

PROOF: Let G be a Moore graph with diamter $d \geq 1$. Let u and v be vertices such that $d(u,v)=d$, and let P be the path from u to v of length d. Let w and x be the neighbours of u and v on P, respectively. Let $w' \neq w$ be another neighbour of u (if one exists). Then there is a path P' of length at most d from w' to v. On this path there is an neighbour x' of v. But $x \neq x'$, since otherwise there would be two paths of length at most d from w' to x. We can do this for each neighbour of u, so the number of neighbours of u is at most the number of neighbours of v. By symmetry, $d(u)=d(v)$.

Now the girth of G is $2d+1$, so let C be a $(2d+1)$-cycle. Let u and v be distance d apart around the cycle. By 3.9, $d(u,v)=d$, so $d(u)=d(v)$. Continuing around the cycle, all vertices on C have the same degree. Let w be a vertice not in the cycle (if one exists), and let P be shortest path from w to the cycle. P has length at most d, so extend P in C to length d. Again by 3.9, if x is the other endpoint of the extended P, $d(w,x)=d$, so $d(w)=d(x)$, so every vertex in G has the same degree. □

3.11 Theorem. *There are no Moore graphs with $d \geq 3$ except C_{2d+1}.*

PROOF: Uses algebraic methods. We will be using similar methods when looking at the case $d=2$. □

When $d=2$ the girth is 5, and suppose that G is k-regular. Then G has $1+k+k(k-1)=k^2+1$ vertices. This leads to an alternative definition of a Moore graph: k-regular, diameter 2, and k^2+1 vertices. The cycle C_5 has $k=2$, and the Peterson graph has $k=3$. For which other k do Moore graphs exist (with $d=2$)?

3.3 Strongly Regular Graphs

3.12 Definition. G is *strongly regular* with *parameters* $(n,k;a,c)$ if G has n vertices and

$$|N(u)\cap N(v)| = \begin{cases} k & \text{if } u=v \text{ (i.e. } G \text{ is } k\text{-regular)} \\ a & \text{if } uv \in E(G) \\ c & \text{if } uv \notin E(G) \text{ and } u \neq v \end{cases}$$

For example, a Moore graph of diameter 2 is strongly regular with parameters $(k^2+1,k;0,1)$. Let G be strongly regular with parameters $(n,k;a,c)$. If $c=0$ then $G=\frac{n}{k+1}K_{k+1}$ and $a=k-1$, we will neglect this case. Let $A=A(G)$. Recall that $[A^2]_{i,j}$ is the number of (i,j)-walks of length 2. Hence

$$[A^2]_{i,j} = \begin{cases} k & \text{if } i=j \\ a & \text{if } ij \in E(G) \\ c & \text{if } ij \notin E(G) \text{ and } i \neq j \end{cases}$$

Therefore $A^2 = aA + c(J-I-A) + kI = (a-c)A + (k-c)I + cJ$. We know that k is an eigenvalue with eignevector $\mathbf{1}$. Let θ be another eigenvalue. Then any eigenvector z associated with θ is orthogonal to $\mathbf{1}$. Therefore we can apply the above equation to z to get

$$0 = cJz = A^2z-(a-c)Az-(k-c)Iz = \theta^2z-(a-c)\theta z-(k-c)z$$

Therefore the matrix $(\theta^2-(a-c)\theta-(k-c))I$ is not invertible (since $z\neq 0$), which implies that $\theta^2-(a-c)\theta-(k-c)=0$. Therefore

$$\theta=\frac{(a-c)\pm\sqrt{(a-c)^2+4(k-c)}}{2}$$

so G has eigenvalues k,θ_1,θ_2, where θ_1 and θ_2 are the $+$ and $-$ solutions, respectively.

Let m_i be the multiplicity of θ_i (we already know that k has multiplicity 1). Then $m_1+m_2+1=n$ and $\theta_1 m_1+\theta_2 m_2+k=0$ since the sum of the eigenvalues is $\mathrm{Tr}(A)=0$. Moore graphs have $a=0$ and $c=1$, so the eigenvalues are

$$\theta_1=\frac{-1+\sqrt{1+4(k-1)}}{2}\qquad\text{and}\qquad\theta_2=\frac{-1-\sqrt{1+4(k-1)}}{2}$$

Solving, we get that for Moore graphs

$$m_1=\frac{\theta_2(n-1)+k}{\theta_2-\theta_1}\qquad\text{and}\qquad m_2=\frac{\theta_1(n-1)+k}{\theta_1-\theta_2}=\frac{k-\frac{n-1}{2}}{\sqrt{4k-3}}+\frac{n-1}{2}$$

For Moore graphs $n=k^2+1$, so $m_2=\frac{k-\frac{k^2}{2}}{\sqrt{4k-3}}+\frac{k^2}{2}$. If $k-\frac{k^2}{2}=0$ then $k=0$ or 2, so it must be the case that $G=C_5$ since $k=0$ is invalid. Otherwise $\sqrt{4k-3}$ must be rational, so $4k-3$ is a perfect square, say $4k-3=w^2$. Then $4k=w^2+3$, so

$$m_2=\frac{1}{32w}(8(w^2+3)-(w^2+3)^2)+\frac{(w^2+3)^2}{32}$$

In particular, $\frac{1}{w}(-w^4+2w+15)$ is an integer, so $w\mid 15$. Therefore $w=1,3,5,15$, so $k=1,3,7,57$. Of these, $k=1$ is invalid, $k=3$ is the Peterson graph, $k=7$ is the so-called Hoffman-Singleton graph, and $k=57$ is *unknown*.

3.4 Friendship Graphs

3.13 Definition. A *friendship graph* G is a graph in which every pair of distinct vertices have a single common neighbour.

3.14 Lemma. *If G is regular then $n=1$ or 3.*

PROOF: Indeed, if G is regular then it is strongly regular with parameters $(n,k;1,1)$. G has one eigenvalue k and the others are $\theta_1,\theta_2=\pm\sqrt{k-1}$. Then $k=(m_2-m_1)\sqrt{k-1}$, so since m_2-m_1 is integer, $\frac{k}{\sqrt{k-1}}$ is an integer. Therefore

$$\frac{k^2}{k-1}(k-1)=k^2=(k-1)(k+1)+1$$

so $k-1\mid 1$ and $k=0,2$. But in these cases $n=1$ or 3. □

3.15 Lemma. *G has no 4-cycles.*

PROOF: Each pair of vertices has a *unique* neighbour in common. □

3.16 Lemma. *If $d(u,v)=2$ then $d(u)=d(v)$.*

PROOF: Let $c(u,v)$ denote the common neighbour of u and v. Suppose that x is the vertex in the middle of the path of length 2 between u and v (this path is unique because G has no 4-cycles by 3.15). Let $w=c(u,x)$ and $w'=c(x,v)$. Then $w\neq w'$ since G has no 4-cycles. Let $u'\in N(u)\setminus\{w,x\}$ and $v'=c(u',v)$. Then $v'\neq x$ and $v'\neq w'$ (again, no 4-cycles), so v' is a new neighbour of v. For any other neighbour u'' of u, by similar arguments there is another new neighbour $v''=c(u'',v)$ of v. Therefore $d(u)\leq d(v)$, and by symmetry $d(u)=d(v)$. □

3.17 Lemma. *If there is v such that $2 < d(v) < n-1$ then G is regular.*

PROOF: Let v be such that $2 < d(v) < n-1$. Then there is u such that $uv \notin E(G)$ and $d(u,v) = 2$. Then $d(u) = d(v)$. Let $w = c(v,x)$, (where x is the vertex in the middle of the path of length 2 from v to u). Notice that $wu \notin E(G)$, so $d(w,u) = 2$, which implies that $d(w) = d(u) = d(v)$. $d(v) \geq 3$, so let v' be any other neighbour of v. Then $v'w \notin E(G)$, so $d(v',w) = 2$, which implies that $d(v') = d(w) = d(v)$. Finally, $d(x) = d(v') = d(v)$, so G is regular. □

3.18 Lemma. *$d(v) \neq 1$ for any v.*

PROOF: If $n = 1$ then the only vertex has degree 0, and if $n > 1$ then $n \geq 3$, and each vertex has degree at least 2. □

3.19 Lemma. *If all vertex degrees are 2 and $n-1$ then G is a windmill (i.e. there is v with $d(v) = n-1$ and $G \setminus \{v\}$ is 1-regular).*

PROOF: If u and v both have degree $n-1$ and if there is more than one other vertex then let w be a neighbour of u. Then v is adjacent to $c(u,w)$, and this contradicts 3.15

We may assume that there is at most one u with degree $n-1$. But then $G \setminus \{u\}$ is 1-regular. If there are no vertices of degree $n-1$ then G is regular. □

3.20 Theorem. *All friendship graphs are windmills.*

4 Directed Graphs

4.1 Definition. A *digraph* D is an ordered pair (V,E), where V is a set of *nodes* and $E \subseteq V \times V \setminus \text{diag}(V)$ is a set of *arcs* between the nodes. If $v,w \in V$ and $vw \in E$ then v is the *tail* and w is the *head*.

For any node v, we have $N^+(v) = \{w \mid vw \in E\}$ and $N^-(v) = \{w \mid wv \in E\}$, the *out*- and *in*- neighbours of v, respectively. From these we can define the out- and in- degrees of v. A *directed walk* is an alternating sequence of nodes and arcs such that each arc is directed from the node preceding it to the node succeeding it. The *underlying* graph of D is the graph G such that $V(G) = V(D)$ and $E(G) = \{\{u,v\} \mid uv \in E(D)\}$ (sometimes it is convienient to permit G to have muliple edges).

4.1 Connectedness

4.2 Definition. A digraph D is *weakly connected* if its underlying graph is connected. D is *strongly connected* if for any $v,w \in V$ there is a directed path from v to w.

4.3 Theorem. *Every tournament has a Hamilton path.*

PROOF: Order the vertices of a tournament T as $v_1,\dots,v_n$ in order to maximize the number $k(v_1,\dots,v_n)$ of edges v_iv_j, for $i < j$, that go forward. If $v_{i+1}v_i \in E(T)$ then swapping v_i and v_{i+1} increases the number of forward arcs. Since k was maximal, $v_1 \dots v_n$ is a Hamilton path. □

4.4 Theorem (Camion). *If T is strongly connected tournament with at least 3 vertices then T has a Hamilton cycle.*

4.5 Theorem (Moon). *If $T = (V,E)$ is strongly connected tournament and $v \in V$ then for all $k = 3,\dots,|V|$ there is a k-cycle through v.*

PROOF: Note that for any vertex $v \in V$, $\{v\} \cup N^+(v) \cup N^-(v) = V$. Fix T and v, and suppose that $|V| \geq 3$. The proof is by induction on k. If $k = 3$ then, since T is strongly connected, there is a path from v to each other vertex, so $N^+(v) \neq \varnothing$. Similarily, $N^-(v) \neq \varnothing$, and there is a path from v to a vertex in $N^-(v)$. Therefore there is an edge from $N^+(v)$ to $N^-(v)$, which gives a 3-cycle containing v.

Suppose that $|V| \geq k > 3$ and there is a $(k-1)$-cycle C containing v. Assume there is no k-cycle containing v. Let $w \in V \setminus V(C)$. If $vw \in E$ then let v' be the vertex of C that follows v in C. If $wv' \in E$ then there is a k-cycle containing v (namely, replace vv' in C with the two edges $vw\ wv'$), so we may assume that $v'w \in E$. Therefore we may assume that $uw \in E(T)$ for every $u \in V(C)$. On the other hand, if $wv \in E$ then by analogous reasoning we may suppose that $wu \in E(T)$ for every $u \in V(C)$.

Let R be the collection of all vertices that dominate C and S the collection of vertices that are dominated by C. T is strongly connected, so R and S are either both empty (but this is impossible since C is not a Hamilton cycle) or they are both non-empty. Therefore there is an edge between R and S. Again by strongness, there must be an edge that goes from S to R (otherwise there would be no edges leaving S) and from this is it clear how to get a k-cycle. □

4.2 Governmental Issues

4.6 Definition. A *king* in a digraph D is a node $v \in V(D)$ such that $\{v\} \cup N^+(v) \cup N^+(N^+(v)) = V(D)$.

4.7 Theorem (Landau). *Every tournament has a king.*

PROOF: Recall that for any vertex $v \in V(T)$, $N^-(v) \cup \{v\} \cup N^+(v) = V(T)$. Let v be such that $d^+(v)$ is maximal. For any $u \in N^-(v)$, if there is no $w \in N^+(v)$ such that $wu \in E(D)$ then $uw \in E(D)$ for all $w \in N^+(v)$ (there is an undirected edge between any two nodes, so the direction must be uw since it is not wu), so that $N^+(u) = \{v\} \cup N^+(v)$, a contradiction. Whence v reaches every vertex in $N^-(v)$ via a path of length 2, so v is a king. □

4.8 Definition. A *council* in a digraph D is an independant set S such that

$$S \cup N^+(S) \cup N^+(N^+(S)) = V(D)$$

4.9 Theorem. *Every digraph has a council.*

PROOF (LOVASZ): Clearly the empty graph has a council. Let $v \in V(D)$, and consider $D \setminus (\{v\} \cup N^+(v))$ By induction this digraph has a council S. If there is no arc from S to v then $S \cup \{v\}$ is a council. If there is an arc from S to v then S is a council for D. □

The preceding results are nothing like what would appear in undirected graphs. The following results are analogous to results in undirected graphs.

4.10 Lemma (Handshaking di-Lemma). $\sum_{v \in V} d^+(v) = \sum_{v \in V} d^-(v)$ *for every digraph* $D = (V, E)$.

4.11 Theorem (Euler Tours). *If D is weakly connected then D has a closed walk using each edge exactly once if and only if $d^+(v) = d^-(v)$ for all $v \in V(D)$. (In this case D is strongly connected.)*

4.12 Definition. The *converse* of a digraph D is the result reversing the direction of all the edges of D. *i.e.* $V(D') = V(D)$ and $E(D') = \{uv \mid vu \in E(D)\}$. A *source* is a node of in-degree zero, and a *sink* is a node with out-degree zero.

4.13 Theorem. *If D is acyclic then at least one source and at least one sink.*

PROOF: Let P be a longest path P in D, with initial vertex u and final vertex v. Then there is no vertex dominating u that does not lie outside of P by maximality of P. There is also no vertex on the path that dominates u since D has no cycles. Therefore u is a source. The converse of D is also acyclic, so it has a source, which means that D has a sink. □

4.3 Branchings

4.14 Definition. A *branching* in a digraph D is a set of edges B of D such that the underlying graph of $(V(D), B)$ is a forest and each component of the forest has a unique source.

Note that a directed tree with just one source has a dipath from the source to every other node. The sources in the components are called the *roots* of the branching. Define for $U \subseteq V(D)$,

$$d^-(U) = |\{vu \in E(D) \mid v \notin U, u \in U\}|$$

4.15 Theorem (Edmonds, 1973). *Let D be any digraph and let $R_1, \dots, R_k$ be non-empty subsets of $V(D)$. Then there exist pairwise disjoint branchings $B_1, \dots, B_k$ such that the set of roots of B_i is R_i if and only if for all non-empty $U \subseteq V(D)$,*

$$d^-(U) \geq |\{i \mid R_i \cap U = \varnothing\}|$$

For fixed $R_1, \dots, R_k$, define $\text{miss}(U) = |\{i \mid R_i \cap U = \varnothing\}|$.

4.16 Lemma. *For all $A, B \subseteq V(D)$, $d^-(A \cup B) \leq d^-(A) + d^-(B) - d^-(A \cap B)$.*

4.17 Lemma. *For all $A, B \subseteq V(D)$, $\text{miss}(A \cup B) \geq \text{miss}(A) + \text{miss}(B) - \text{miss}(A \cap B)$.*

4.18 Lemma. *If $\text{miss}(A \cup B) > \text{miss}(A) + \text{miss}(B) - \text{miss}(A \cap B)$ then there is R_i such that $R_i \cap A \neq \varnothing$ and $R_i \cap B \neq \varnothing$ but $R_i \cap (A \cap B) = \varnothing$.*

Proof: The cases for the proofs of the first two lemmas are (removing symmetric cases): R_i meets neither A nor B, R_i meets A and not B, R_i meets $A \cap B$, and R_i meets A and B but not $A \cap B$. Of these cases, the only time equality doesn't hold in 4.17 is the last case. □

Proof (of the Branching Theorem, Lovasz): Existence of the branching implies the inequality.

To complete the proof, assume $d^-(U) \geq \text{miss}(U)$ for all $\varnothing \neq U \subseteq V(D)$. Fix $|V(D)| = n$ and prove downwards by induction on $|R_1| + \cdots + |R_k|$. If $R_i = V(D)$ for all i then take B_i with no edges for all i. Inductively assume true for larger sets R_i, and without loss of generality assume $R_1 \neq V(D)$. Let W be minimal such that:

- $W \cap R_1 \neq \varnothing$
- $W \not\subseteq R_1$
- $d^-(W) = \text{miss}(W)$

Such a W exists since $V(D)$ satisfies these three properties. Then

$$d^-(W \setminus R_i) \geq \text{miss}(W \setminus R_1) > \text{miss}(W) = d^-(W)$$

by the theorem hypothesis. Therefore there must exist an edge uv with $u \in R_1 \cap W$ and $v \in W \setminus R_1$. Let $D' = D \setminus \{uv\}$ and $R_1' = R_1 \cup \{v\}$. Then in D' with respect to $R_1', R_2, \dots, R_k$, $d^-(U) \geq \text{miss}(U)$ for all $\varnothing \neq U \subseteq V(D')$. Indeed, assume there is $U \neq \varnothing$ such that $d^-(U) < \text{miss}(U)$ in D'. But $d^-(U) \geq \text{miss}(U)$ in D, and in going from D to D' we delete an edge and add a vertex to R_1. Upon deletion of an edge, d^- either stays the same or goes down by 1, and upon adding a vertex to R_1, miss either stays the same or goes down by 1. In this case miss must have stayed the same and d^- went down by 1. Whence $v \in U$, $u \notin U$, and $U \cap R_1 \neq \varnothing$. By Lemmas 4.16 and 4.17

$$d^-(U \cap W) \leq d^-(U) + d^-(W) - d^-(U \cup W) \leq \text{miss}(U) + \text{miss}(W) - \text{miss}(U \cup W) \leq \text{miss}(U \cap W)$$

But $v \in U \cap W$, so $U \cap W \neq \varnothing$ and $d^-(U \cap W) \geq \text{miss}(U \cap W)$, which implies that the above inequality is equality throughout and $d^-(U \cap W) = \text{miss}(U \cap W)$. Now $R_1 \cap U \neq \varnothing$ and $R_1 \cap W \neq \varnothing$, so $R_1 \cap (U \cap W) \neq \varnothing$ by 4.18. Now $v \in (U \cap W) \setminus R_1$, and $|U \cap W| < |W|$ since $u \in U$ but $u \notin W$. This is a contradiction because W was supposed to be minimal with its properties.

By the induction hypothesis, there are disjoint branchings $B_1', \dots, B_k'$ in D' with root sets $R_1', R_2, \dots, R_k$. Put $B_1 = B_1' + uv$ and $B_i = B_i'$ otherwise. Then $B_1, \dots, B_k$ are disjoint branchings in D, as required. □

4.19 Definition. A *stong component* of a digraph D is a maximal strongly connected subgraph.

4.20 Theorem. *Let D be a digraph. Define D' to be the digraph whose vertices are the strong components of D, obtained by contracting each strong component and replacing multiple edges by single edges and deleting loops. Then D' is acyclic.*

PROOF: Exercise.

4.4 The Luccesi-Younger Theorem

4.21 Definition. Let D be a digraph. For $W \subseteq V(D)$, the *coboundry* of W is the set of edges in $E(D)$ with exactly one end in W, and is denoted $\delta(W)$.

Notice that $\delta(W) = \delta(V(D) \setminus W)$. $\delta(W)$ is *directed* if all arcs have their head in W (in-directed with respect to W) or all have their tail in W (out-directed).

4.22 Definition. A *directed cut* is an inclusion-wise minimal non-empty directed coboundry.

4.23 Lemma. *D is strongly connected if and only if D is weakly connected and has no directed cut.*

PROOF: Assignment. □

4.24 Definition. Let $\mathscr{L}$ be the set of cuts in D. A *transversal* in D is a set $t \subseteq E(D)$ such that $T \cap \gamma \neq \emptyset$ for all $\gamma \in \mathscr{L}$.

Clearly the size of t is at least the maximum number of pairwise disjoint cuts in $\mathscr{L}$.

4.25 Theorem (Luccesi-Younger). *The minimum size of a transversal in a digraph equals the maximum number of pairwise disjoint cuts in D.*

4.26 Definition. For a cut γ, the *positive arugment* of γ is $p^+(\gamma)$ is the set of arcs with at least one end in W, where $\gamma = \delta(W)$ and γ is out-directed from W, and W is minimal with these properties. $p^-(\gamma)$ is similar, but with in-directed W. A *separator* is $S \subseteq E(D)$ such that $\gamma \subseteq S$ or $\gamma \subseteq E(D) \setminus S$, for every cut γ.

4.27 Lemma (Separation).

(i) *For all cuts γ, $p^+(\gamma) \cap p^-(\gamma) = \gamma$.*
(ii) *For all cuts γ, $p^+(\gamma) \cup p^-(\gamma)$ is a separator.*

4.28 Definition. For cuts α and β, the *doubly positive part* is $pp^+(\alpha, \beta) = (p^+(\alpha) \cap p^+(\beta)) \cap (\alpha \cup \beta)$. pp^- is defined similarily.

4.29 Lemma (Meeting & Splitting). *If α meets both $p^+(\beta)$ and $p^-(\beta)$ then $pp^+(\alpha, \beta)$ and $pp^-(\alpha, \beta)$ each contain a cut.*

PROOF: Assignment. □

Notation. $\text{top}(\gamma) = p^+(\gamma) \setminus \gamma$ and $\text{bot}(\gamma) = p^-(\gamma) \setminus \gamma$.

4.30 Lemma (Mass meeting). *If α and β are cuts and α meets bot(β) then each cut in $pp^+(\alpha, \beta)$ meets $\beta \cap$ top(α).*

PROOF: Let $\gamma \subseteq pp^+(\alpha,\beta) \subseteq p^+(\beta)$. Then $\gamma \neq \alpha$, so γ meets

$$pp^+(\alpha,\beta) \setminus \alpha = p^+(\alpha) \cap p^+(\beta) \cap (\alpha \cup \beta) \setminus \alpha \subseteq \beta \cap \text{top}(\alpha)$$ □

4.31 Lemma (Meeting Asymmetry). *If α and β are cuts that meet then $\beta \subseteq p^-(\alpha)$ implies that $\alpha \subseteq p^+(\beta)$.*

PROOF: α meets β, so the Separation Lemma implies that α meets $\text{bot}(\beta)$ and $\text{top}(\beta)$. Meeting and Splitting implies that $pp^+(\alpha,\beta)$ contains a cut γ. Given that $\beta \subseteq p^-(\alpha)$, $\beta \cap \text{top}(\alpha) = \emptyset$, so γ cannot meet this. Mass meeting cannot apply, so $\alpha \cap \text{bot}(\beta) = \emptyset$. Therefore $\alpha \subseteq p^+(\beta)$ since $p^+(\beta) \cup p^-(\beta)$ is a separator. □

4.32 Lemma (Meeting Part-Way). *If α and β are cuts that meet then $\alpha \subseteq p^+(\beta)$ implies that top$(\alpha) \subseteq$ top(β).*

4.33 Definition. Cuts α and β are said to *cross* if α meets both $\text{top}(\beta)$ and $\text{bot}(\beta)$.

The definition is not actually asymmetrical, since if α and β cross then β meets both $\text{top}(\alpha)$ and $\text{bot}(\alpha)$. Indeed, α meeting $\text{top}(\beta)$ and $\text{bot}(\beta)$ implies α meets $p^+(\beta)$ and $p^-(\beta)$, so there are cuts γ, γ' in $pp^+(\alpha,\beta)$ and $pp^-(\alpha,\beta)$, respectively. Mass meeting implies that γ meets $\beta \cap \text{top}(\alpha)$, so $\beta \cap \text{top}(\alpha)$ is non-empty. The converse argument implies $\beta \cap \text{bot}(\alpha) \neq \emptyset$.

4.34 Lemma (Cross Products). *If cuts α and β cross then there are cuts $\sigma \subset pp^+(\alpha,\beta)$ and $\tau \subset pp^-(\alpha,\beta)$ with top(σ) properly contained in top(β).*

PROOF: α meets $\text{top}(\beta)$ and $\text{bot}(\beta)$, so Meeting and Splitting implies that there are cuts in pp^+ and pp^-. Mass Meeting implies that σ meets $\beta \cap \text{top}(\alpha)$, so σ meets β. Also, β meets $\text{bot}(\alpha)$, so σ meets $\alpha \cap \text{top}(\beta)$, so σ meets $\text{top}(\beta)$. Meeting Part-Way gives $\text{top}(\sigma) \subseteq \text{top}(\beta)$. □

4.35 Definition. A list $\mathscr{L}$ of cuts is *laminar* if no two cuts in $\mathscr{L}$ cross. For each $e \in E(D)$, e is *used* by a cut γ if $e \in \gamma$, and e is used by $\mathscr{L}$ if e is used by some cut in $\mathscr{L}$.

The following theorems are essential ingredients in the proof the Luccesi-Younger Theorem, but I will not prove them here. The proof of the Partition Theorem may be found in the original paper of Luccesi and Younger, "A minimax theorem for directed graphs", *J. London Math. Soc.* (2) 17 (1978), 369–374. I am not sure where you can find the proof of the Laminarity Theorem, but you could try Professor Wormald's website.

4.36 Theorem (Laminarity Theorem). *If $\mathscr{L}$ is a list of directed cuts in a digraph D then there is a laminar list $\mathscr{L}'$ of directed cuts of the same length as $\mathscr{L}$ such that every edge of D is used at most as often by $\mathscr{L}'$ as by $\mathscr{L}$.*

4.37 Theorem (Partition Theorem). *Let $\mathscr{L}$ be a laminar list of cuts, and let $k(\mathscr{L})$ be the maximum number of times an edge is used by $\mathscr{L}$. Then there is a partition of $\mathscr{L}$ into $k(\mathscr{L})$ sublists $\mathscr{L}_1, \dots, \mathscr{L}_{k(\mathscr{L})}$, each consisting of pairwise disjoint directed cuts.*

4.38 Definition. An edge in D is *essential* if it is contained in every size-wise maximal set of pairwise disjoint cuts.

4.39 Theorem (Essential Edge Theorem). *Every cut contains an essential edge*

PROOF: Let γ be any cut. For each edge $e \in \gamma$, let $\mathscr{L}_e$ be the set of all cuts not containing e. In particular, $\gamma \notin \mathscr{L}_e$ for any e. Let $\mathscr{L}_e^*$ be a size-wise maximal set of pairwise disjoint cuts in $\mathscr{L}_e$. We will prove that there is $e \in \gamma$ such that $|\mathscr{L}_e^*| < |\mathscr{L}^*|$, where $\mathscr{L}^*$ is a size-wise maximal set of pairwise disjoint cuts. Then e is the required essential edge.

Let $\mathscr{L}'$ be the list obtained by concatenating all of the $\mathscr{L}_e^*$'s, plus γ on the end. Each edge occurs no more than once in each $\mathscr{L}_e^*$, and if $e \in \gamma$ then e does not occur in $\mathscr{L}_e^*$ but does occur in e, so each edge of D is used at most $|\gamma|$ times in $\mathscr{L}'$. The the Laminarity Theorem there is a laminar list $\mathscr{L}''$ of cuts of the same length as $\mathscr{L}'$ that uses each edge at most $|\gamma|$ times. By the Partition Theorem there is a partition of $\mathscr{L}''$ into lists $\mathscr{L}_1, \dots, \mathscr{L}_{|\gamma|}$, each consisting of pairwise disjoint cuts. If $\mathscr{L}^*$ is a size-wise maximal set of pairwise disjoint cuts then

$$|\gamma| \cdot |\mathscr{L}^*| \geq \sum_{i=1}^{|\gamma|} |\mathscr{L}_i| = |\mathscr{L}''| = |\mathscr{L}'| = 1 + \sum_{e \in \gamma} |\mathscr{L}_e^*|$$

Whence there is $e \in \gamma$ such that $|\mathscr{L}_e^*| < |\mathscr{L}^*|$, so e is essential. □

PROOF (OF LY): Let $\mathscr{L} = \{\text{cuts in } D\}$ and induct on $|\mathscr{L}|$. If this number is 0 or 1 then we are done, so suppose the theorem holds for all D' with fewer cuts than D. Let $\gamma \in \mathscr{L}$ and e an essential edge in γ (guaranteed by the Essential Edge Theorem). Define D' to be D together with the reverse of e. Then the cuts of D' are $\mathscr{L}' = \mathscr{L} \setminus \{\text{all cuts containing } e\}$. Then $\gamma \notin \mathscr{L}'$, so by induction D' satisfies the theorem. Let $\mathscr{D}$ be a size-wise maximal set cuts in D' and t' be a size-wise minimal transversal for D' such that $|\mathscr{D}| = |t'|$. e is essential, so e is used by every maximal set of pairwise disjoint cuts in D. $\mathscr{D}$ does not use e, so $|\mathscr{D}| < |\mathscr{P}|$ for any maximal set of pairwise disjoint cuts in D. Now $t' \cup \{e\}$ is a transversal for D, so $|t' \cup \{e\}| \geq |\mathscr{P}| > |\mathscr{D}| = |t'|$. Therefore $|t' \cup \{e\}| = |\mathscr{P}|$, and $\mathscr{P}$ is minimal. □

4.40 Definition. A *feedback set* in a digraph D is a set of edges which intersect every cycle.

4.41 Corollary. *If D is planar then the minimum size of a feedback set in D is the maximum number of pairwise edge disjoint cycles.*

PROOF: Let D^* be the dual of D. Cycles in D correspond to cuts in D^*, and feedback sets in D correspond to transversals in D^*. □