Microeconomics I
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1 Demand Theory

1.1 Preference relations

Let X be the set of feasible alternatives in some economic situation. Let $B \subseteq X \times X$ be a binary relation and write xBy if $(x,y) \in B$. Let $\succsim$ be such a *preference relation*. We say

1. a is *weakly preferred* to b if $a \succsim b$.
2. a is *indifferent* to b if $a \sim b$.
3. a is *strictly preferred* to b if $a \succ b$.

1.1.1 Definition. $\succsim$ is *rational* (or *regular*) if it is complete and transitive.

Reflexivity follows from completeness, so a rational preference is simply a total order on X.

A preference relation is *negative transitive* if for all $x, y, z \in X$, if $\neg(x \succsim y)$ and $\neg(y \succsim z)$ then $\neg(x \succsim z)$. A preference relation is *acyclic* if for all $n \in \mathbb{N}$ and for all $x_1, \dots, x_n \in X$, if $x_i \succsim x_{i+1}$ for all $i = 1, \dots, n-1$ then $x_1 \succsim x_n$.

1.1.2 Exercises.

1. $\succ$ is irreflexive and transitive.
2. $\sim$ is reflexive and transitive.
3. If $x \succ y \succsim z$ then $x \succ z$.
4. $\succsim$ is acyclic.

1.1.3 Example. Suppose we have a family with a mother, father, and child that makes decisions on what to do for fun by majority rule. Suppose they choose between opera, skating, and concert and the preferences are as follows: mother likes $o \succ s \succ c$, father likes $c \succ o \succ s$, and the child likes $s \succ c \succ o$. But then $o \succ s \succ c \succ o$ by the family, so democratic preference is not rational. This is known as the Condecet paradox.

1.2 Utility functions

1.2.1 Definition. A function $u : X \to \mathbb{R}$ is a *utility representation* for a preference relation $\succsim$ if for all $x, y \in X$ we have $u(x) \geq u(y)$ if and only if $x \succsim y$.

Clearly a preference has a utility representation if and only if it is order isomorphic to a subset of $(\mathbb{R}, \geq)$.

1.2.2 Proposition. *Let $\succsim$ be a preference relation. If it has a utility representation then it is rational.*

1.2.3 Proposition. *Suppose u represents $\succsim$. For any strictly increasing function $f:\mathbb{R}\to\mathbb{R}$ the function $f\circ u$ represents $\succsim$ as well.*

It follows that (in all nontrivial cases) there is no unique representation.

1.2.4 Proposition. *Suppose that X is finite. Then a rational preference relation $\succsim$ has a utility representation.*

PROOF: A representation is defined by $u(x) := \#\{x' \mid x \succsim x'\}$. □

1.2.5 Proposition. *Suppose that X is countable. Then a rational preference relation $\succsim$ has a utility representation.*

PROOF: Construct a representation u as follows. Write $X = \{x_1, x_2, \dots\}$. Set $u(x_1) = 0$, and for $n > 1$, if $x_n \sim x_k$ for some $1 \le k < n$ then set $u(x_n) := u(x_k)$. If there is no such k then the maximum element of $\{u(x_k) \mid x_n \succ x_k\} \cup \{-1\}$ is less than the minimum element of $\{u(x_k) \mid x_k \succ x_n\} \cup \{1\}$, so set $u(x_n)$ to be any real number strictly between them. u is a utility representation by the transitivity and completeness of $\succsim$. □

1.2.6 Example (Lexicographic preference). Take $X = \mathbb{R}^2_+$. Define the *lexicographic preference* $\succsim_L$ on X by $(x,y) \succsim_L (x',y')$ if $[x > x']$ or $[x = x'$ and $y \ge y']$. It is easy to see that $\succsim_L$ is rational.

1.2.7 Proposition. *Lexicographic preference $\succsim_L$ on $[0,1]\times[0,1]$ cannot be represented by a utility function.*

PROOF: Suppose that $u : X = [0,1]\times[0,1] \to \mathbb{R}$ represents $\succsim_L$. For any $a \in [0,1]$, $(a,1) \succ_L (a,0)$, so $u(a,1) > u(a,0)$. Let $q : [0,1] \to \mathbb{Q}$ be a map such that $u(a,1) > q(a) > u(a,0)$. If $a' > a$ then $q(a') > q(a)$ by definition of lexicographic preference. But then q is one-to-one since it is increasing, which contradicts that $[0,1]$ is not countable. □

From now on we assume that there is a (sufficiently nice) topology on X. We give two definitions for a preference to be continuous, and prove they are equivalent.

1.2.8 Definition. We say that $\succsim$ is *continuous* if whenever $a \succ b$ then there are neighbourhoods $B_a, B_b \subseteq X$ such that for all $x \in B_a$ and all $y \in B_b$ we have $x \succ y$.

1.2.9 Definition. Let $R^+(x) := \{x' \mid x' \succsim x\}$ and $R^-(x) := \{x' \mid x \succsim x'\}$ be the *upper contour set* and *lower contour set*, respectively. We also say that $\succsim$ is *continuous* if both $R^+(x)$ and $R^-(x)$ are closed for all x. (This is clearly equivalent to the requirement that if (a_n, b_n) is a sequence in $X \times X$ satisfying $a_n \succsim b_n$ for all n and $a_n \to a$ and $b_n \to b$, then $a \succsim b$.)

1.2.10 Proposition. *The two definitions of continuity are equivalent.*

PROOF: Suppose that $a_n \succsim b_n$ for all n, $a_n \to a$, $b_n \to b$, but that $b \succ a$. By the first definition, if $\succsim$ is continuous, then there are neighbourhoods B_a and B_b such that for all $x \in B_a$ and $y \in B_b$, $y \succ x$. But for sufficiently large n we have $a_n \in B_a$ and $b_n \in B_b$, so $a_n \succ b_n$ for sufficiently large n, a contradiction.

Let $a \succ b$, and assume for contradiction that there are $a_n \in B_a(\frac{1}{n})$ and $b_n \in B_b(\frac{1}{n})$ such that $b_n \succsim a_n$. But $a_n \to a$ and $b_n \to b$, so by the second definition we have $b \succsim a$. □

1.2.11 Definition. We say that $\succsim$ (a preference on $\mathbb{R}^n$) is *monotone* if for all $x, y \in \mathbb{R}^n_+$ if $x \geq y$ then $x \succsim y$ and if $x \gg y$ then $x \succ y$. We say $\succsim$ is *strongly monotone* if $x \geq y$ and $x \neq y$ imply $x \succ y$.

1.2.12 Theorem. *Suppose that $\succsim$ is rational and continuous. Then there exists a continuous utility function that represents this preference relation.*

PROOF: For this proof we will also suppose that $\succsim$ is monotone. Let e be the unit vector in the direction of the vector of all ones. For every $x \in \mathbb{R}^n_+$, monotonicity implies that $x \succsim 0$. There is some $\bar{\alpha}$ such that $\bar{\alpha}e \gg x$, so $\bar{\alpha}e \succ x$. Continuity implies that there is some $\alpha(x)$ such that $\alpha(x)e \sim x$. We will show that $u(x) = \alpha(x)$ is a utility representation. Now $x \succsim y$ if and only if $\alpha(x)e \succsim \alpha(y)e$, if and only if $\alpha(x) \geq \alpha(y)$ again by monotonicity since these points are on the diagonal ray. That α is continuous is in Problem Set 1 (the proof is also in MWG). □

1.3 Choice functions

Now consider a set $\mathscr{B} \subseteq 2^X \setminus \{\varnothing\}$.

1.3.1 Definition. A *choice function* is a function $c : \mathscr{B} \to 2^X \setminus \{\varnothing\}$ such that for all $B \in \mathscr{B}$, $c(B) \subseteq B$.

1.3.2 Definition. The choice function c satisfies the *weak axiom of revealed preference* (or *WARP*) if the following holds. If there is $B \in \mathscr{B}$ with $x, y \in B$ such that $x \in c(B)$ then for any $B' \in \mathscr{B}$ with $x, y \in B'$, if $y \in c(B')$ then $x \in c(B')$. In words, "if x is ever chosen when y is available then there can be no feasible set for which y is chosen but x is not."

1.3.3 Definition. Given a choice structure $(\mathscr{B}, c)$, the *revealed preference relation* $\succsim^*$ is defined by $x \succsim^* y$ if and only if there is $B \in \mathscr{B}$ such that $x, y \in B$ and $x \in c(B)$. The strict preference becomes $x \succ^* y$ if and only if there is $B \in \mathscr{B}$ such that $x, y \in B$ and $x \in c(B)$ and $y \notin c(B)$.

WARP is equivalent to $[x \succsim^* y \implies \neg(y \succ^* x)]$.

1.3.4 Definition. Given a preference $\succsim$, the *preference maximizing choice* $c^*(\cdot, \succsim)$ is defined by $c^*(B, \succsim) := \{x \in B \mid x \succsim y \text{ for all } y \in B\}$. A rational preference relation $\succsim$ *rationalizes* $(\mathscr{B}, c)$ if $c(B) = c^*(B, \succsim)$ for all $B \in \mathscr{B}$. We also say that $\succsim$ is *consistent with* c.

1.3.5 Proposition. *Let $\succsim$ be a rational preference relation. Then $c^*(\cdot, \succsim)$ satisfies WARP.*

The converse is not true.

1.3.6 Example. Let $X = \{x, y, z\}$, $\mathscr{B} = \{\{x, y\}, \{x, z\}, \{y, z\}\}$. Define

$$c(\{x, y\}) = \{x\}, \quad c(\{x, z\}) = \{z\}, \quad c(\{y, z\}) = \{y\}.$$

If there were a preference relation rationalizing $(\mathscr{B}, c)$ then we would have $x \succ y$ and $y \succ z$, but $z \succ x$ so this cannot be the case.

1.3.7 Proposition. *If c satisfies WARP on $\mathscr{B}$ and $\mathscr{B}$ includes all subsets of X of up to three elements then there is a rational preference relation rationalizing $(\mathscr{B}, c)$.*

PROOF: In this case the revealed preference relation $\succsim^*$ rationalizes $(\mathscr{B}, c)$. □

It follows that if $\mathscr{B} = 2^X \setminus \{\varnothing\}$ and $|X| \geq 3$ then WARP is a necessary and sufficient condition for rationalizability.

We can define the *strong axiom of revealed preference* (or *SARP*) as a kind of "recursive closure" of WARP—SARP requires that $\succsim^*$ to be acyclic.

1.4 Consumer preferences

A consumer is a rational agent making choices between available commodities. From now on we take $X = \mathbb{R}^L_+$ to be amounts of L different commodities. An element of X is called a *bundle*.

1.4.1 Definition. $\succsim$ is *convex* if $x \succsim y$ and $\alpha \in (0, 1)$ implies that $\alpha x + (1 - \alpha)y \succsim y$. We also say that $\succsim$ is convex if $R^+(x)$ is a convex set (and indeed these definitions are equivalent).

1.4.2 Definition. $\succsim$ is *strictly convex* if for every $a, b \succsim y$ with $a \neq b$ then $\alpha a + (1 - \alpha)b \succ y$ for all $\alpha \in (0, 1)$.

1.4.3 Examples.

1. $u(x) = \sqrt{x_1} + \sqrt{x_2}$ is strictly convex.
2. $u(x) = \min\{x_1, x_2\}$ is convex but not strictly convex.
3. $u(x) = x_1 + x_2$ is convex but not strictly convex.
4. $u(x) = x_1^2 + x_2^2$ is not convex.

1.4.4 Definition. A function u is *quasi-concave* if for all $x \in X$ the set $\{x' \mid u(x') \geq u(x)\}$ is convex. Equivalently, u is quasi-concave if $u(x') \geq u(x)$ implies $u(\alpha x' + (1 - \alpha)x) \geq u(x)$ for all $\alpha \in (0, 1)$.

1.4.5 Exercise. Suppose that $\succsim$ is represented by u. Show that $\succsim$ is convex if and only if u is quasi-concave.

1.4.6 Definition. A function u is *strictly quasi-concave* if $u(x) \geq u(y)$ implies $u(\alpha x + (1-\alpha)y) > u(y)$ for all $\alpha \in (0,1)$.

Recall that u is *concave* if for all $x, y \in X$ and $\alpha \in (0,1)$ then

$$u(\alpha x + (1-\alpha)y) \geq \alpha u(x) + (1-\alpha)u(y).$$

1.4.7 Exercises.

1. Prove u is quasi-concave if it is concave.
2. Find a convex preference relation and a utility function representing it which is not concave.
3. Prove that quasi-concavity is preserved for any monotonic transformation, but concavity is not.

Let $p \gg 0$ be a vector representing the prices of the L commodities in X. *We assume that commodity prices are constant.* A consumption bundle $x \in \mathbb{R}^L_+$ is affordable if it does not exceed the consumer's wealth level.

1.4.8 Definition. The *Walrasian budget set* for prices $p \gg 0$ and wealth $w \in \mathbb{R}$ is the set of *affordable* bundles $B(p,w) := \{x \in \mathbb{R}^L_+ \mid p \cdot x \leq w\}$. The set $\{x \mid p \cdot x = w\}$ is called the *budget hyperplane*.

The *consumer choice problem* is to find the set $c^*(B(p,w), \succsim)$ of bundles from $B(p,w)$ that are maximal for a given preference relation $\succsim$.

1.4.9 Proposition. *If $\succsim$ is rational and continuous then the consumer choice problem has a solution (i.e. $c^*(B(p,w), \succsim) \neq \varnothing$).*

PROOF: Obviously $B(p,w)$ is convex (it is a half-space intersected with $\mathbb{R}^L_+$) and compact. $\succsim$ can be represented by a continuous utility function u, so $u(B(p,w))$ attains a (finite) maximum value. □

1.4.10 Proposition. *If $\succsim$ is convex then $c^*(B(p,w), \succsim)$ is convex.*

PROOF: Recall that $\succsim$ is convex if for all $x \succsim y$ we have $\alpha x + (1-\alpha)y \succsim y$ for all $\alpha \in (0,1)$. From this it is clear that $c^*(B(p,w), \succsim)$ is convex. □

1.4.11 Proposition. *If $\succsim$ is strictly convex then there is at most one solution to the consumer choice problem.*

PROOF: Recall that if $\succsim$ is strictly convex and $x \sim y$ then $\alpha x + (1-\alpha)y \succ x, y$ for all $\alpha \in (0,1)$. □

In particular, if $\succsim$ is rational, continuous, and strictly convex then there is a unique solution to the consumer choice problem.

From now on we assume that consumer preferences are such that for any $B(p,w)$ the consumer choice function has a unique solution. We denote this solution by $x(p,w)$, the *Walrasian demand function*.

1.4.12 Definition. We say that $x(p,w)$ is *homogeneous of degree 0* (or *HD0*) if for any $\alpha > 0$ we have $x(\alpha p, \alpha w) = x(p,w)$ for all p, w, and $x(p,w)$ is said to satisfy *Walras' Law* if for every price $p \gg 0$ we have $p \cdot x(p,w) = w$.

Note that Walras' Law is always satisfied if the preference is monotone.

1.5 Comparative statistics

We will now think about how consumer choice changes with wealth and prices. We assume that $x(p,w)$ is a function and is differentiable. Fix prices $\bar{p}$ and consider what happens as wealth varies. Let $E_{\bar{p}} = \{x(\bar{p},w) \mid w > 0\}$. This set is a curve in $\mathbb{R}^L_+$ (for sufficiently nice preferences) and is called the *Engel curve* or *wealth expansion path*.

1.5.1 Definition. $\frac{\partial x_\ell}{\partial w}(p,w)$ is called the *wealth effects* for good ℓ at wealth level w. A commodity ℓ is called *normal* (resp. *inferior*) at wealth level w if the wealth effects at that level is non-negative (resp. strictly negative).

Similarly, we may consider the curve $x(\hat{p},\bar{w})$ as one price p_k varies and $\bar{w}$ is held fixed.

1.5.2 Definition. The *price effects* of p_k on commodity ℓ is $\frac{\partial x_\ell}{\partial p_k}(p,w)$. Usually the price effects are strictly negative. A commodity for which the price effects are sometimes strictly positive is called a *Giffen good*, otherwise the commodity is an *ordinary good*.

We have some relationships between the wealth and price effects. Since $x(p,w)$ is HD0, for all $\alpha > 0$ we have $x(\alpha p, \alpha w) - x(p,w) = 0$. Taking the derivative with respect to α and setting $\alpha = 1$, we get

$$\begin{aligned} 0 &= \frac{d}{d\alpha}(x_\ell(\alpha p, \alpha w) - x_\ell(p,w))\Big|_{\alpha=1} \\ &= \sum_{k=1}^{L} \frac{\partial}{\partial p_k} x_\ell(\alpha p, \alpha w) p_k + \frac{\partial}{\partial w} x_\ell(\alpha p, \alpha w) w\Big|_{\alpha=1} \\ &= \sum_{k=1}^{L} \frac{\partial}{\partial p_k} x_\ell(p,w) p_k + \frac{\partial}{\partial w} x_\ell(p,w) w \end{aligned}$$

In vector notation, $D_p x(p,w)p = -D_w x(p,w)w$.

1.5.3 Definition. The *elasticity* of x_ℓ with respect to p_k and w are

$$\varepsilon_{\ell k}(p,\omega) = \frac{\partial x_\ell(p,w)}{\partial p_k} \cdot \frac{p_k}{x_\ell(p,w)} \quad \text{and} \quad \varepsilon_{\ell w}(p,\omega) = \frac{\partial x_\ell(p,w)}{\partial w} \cdot \frac{w}{x_\ell(p,w)}.$$

Dividing by $x_\ell(p,w)$ we get equations relating the elasticities

$$\sum_{k=1}^{L} \varepsilon_{\ell k}(p,w) = -\varepsilon_{\ell w}(p,w).$$

If the demand function satisfies Walras' Law ($p \cdot x(p,w) = w$) then (by differentiating with respect to p_k) we get

$$\sum_{\ell=1} p_\ell \frac{\partial}{\partial p_k} x_\ell(p,w) + x_k(p,w) = 0.$$

In vector notation, $p \cdot D_p x(p,w) + x(p,w)^T = 0^T$. This is known as *Cournot aggregation*, that the total expenditure does not change when only prices change. Differentiating Walras' Law with respect to w yields *Engel aggregation*,

$$\sum_{\ell=1} p_\ell \frac{\partial}{\partial w} x_\ell(p,w) = 1,$$

that total expenditure must change by the size of the wealth changes.

1.6 Weak axiom of revealed preference

1.6.1 Definition. We say that a demand function $x(p,w)$ satisfies *WARP* if $p \cdot x(p',w') \le w$ and $x(p',w') \ne x(p,w)$ together imply $p' \cdot x(p,w) > w'$ for any (p,w) and (p',w').

The idea is that if we ever choose $x(p,w)$ when $x(p',w')$ is feasible then choosing $x(p',w')$ implies $x(p,w)$ is not feasible. This is exactly the same WARP as before but adapted to the specific choice framework we are now dealing with (specifically, $(\{B(p,w) \mid p \gg 0, w \ge 0\}, x(p,w))$).

Given an old price and wealth level (p,w) and a new price p', the *compensated price change* in wealth is $w' = w + (p'-p) \cdot x(p,w)$. The idea is that the new wealth is chosen so that $x(p,w)$ is still available on the boundary, i.e. $p' \cdot x(p,w) = w'$. Compensated price changes preserve a consumer's real wealth and allow us to study the effects of changes in the relative costs of commodities.

1.6.2 Definition. $x(p,w)$ satisfies the *compensated law of demand* (or *CLD*) if for any compensated price change from (p,w) to $(p',w') = (p', p' \cdot x(p,w))$ we have

$$(p'-p) \cdot (x(p',w') - x(p,w)) \le 0$$

and the inequality is strict if $x(p,w) \ne x(p',w')$.

The CLD says that "demand moves opposite to prices."

1.6.3 Proposition. *Suppose that $x(p,w)$ is HD0 and satisfies Walras' Law. Then $x(p,w)$ satisfies WARP if and only if $x(p,w)$ satisfies CLD.*

PROOF: Assume WARP. If $x(p,w) = x(p',w')$ then CLD clearly holds. If $x(p,w) \neq x(p',w')$ then

$$\begin{aligned}(p'-p)\cdot(x(p',w')-x(p,w)) \\ &= p'\cdot x(p',w') - p'\cdot x(p,w) - p\cdot x(p',w') + p\cdot x(p,w) \\ &= w' - w' - p\cdot x(p,w) + w \\ &< -w + w = 0\end{aligned}$$

Conversely, suppose that CLD holds but WARP does not. Then there are (p,w) and (p',w') such that $p\cdot x(p',w') \leq w$ and $p'\cdot x(p,w) \leq w'$ and $x(p,w) \neq x(p',w')$. Without loss of generality we may assume that $p\cdot x(p',w') = w$ (see MWG). But then

$$p\cdot(x(p',w')-x(p,w)) = 0 \quad \text{and} \quad p'\cdot(x(p',w')-x(p,w)) \geq 0,$$

contradicting CLD. □

1.6.4 Example. A simple price change is one for which only one component of p changes. Applying the CLD to simple price changes implies that increasing one price (and holding all others fixed) decreases the component of the demand function for that commodity.

Consider (p,w) and a differential price change dp and let $dw = x(p,w)dp$ be the corresponding compensated price change in wealth. The CLD becomes $dp\cdot dx \leq 0$. We have

$$\begin{aligned}dx &= D_p x(p,w)dp + D_w x(p,w)dw \\ &= D_p x(p,w)dp + D_w x(p,w)(x(p,w)dp) \\ &= (D_p x(p,w) + D_w x(p,w)x(p,w)^T)dp\end{aligned}$$

so $dp\cdot(D_p x(p,w) + D_w x(p,w)x(p,w)^T)dp \leq 0$.

$$S(p,w) := D_p x(p,w) + D_w x(p,w)x(p,w)^T$$

is called the *Slutsky matrix*. Specifically, $S_{\ell k} = \frac{\partial x_\ell(p,w)}{\delta p_k} + \frac{\partial x_\ell(p,w)}{\partial w}x_k(p,w)$ and is the effect of p_k on x_ℓ with compensated price change.

1.6.5 Proposition. *Suppose that $x(p,w)$ satisfies WARP, Walras' Law, and is HD0. Then at any (p,w), $S(p,w)$ satisfies $v\cdot S(p,w)v \leq 0$ for all $v \in \mathbb{R}^L$ (i.e. $S(p,w)$ is negative semidefinite, or NSD).*

In particular, $S_{\ell\ell} \leq 0$ for all ℓ, so the effect of the price of a commodity on its consumption is negative. Again,

$$S_{\ell\ell}(p,w) = \frac{\partial x_\ell(p,w)}{\delta p_\ell} + \frac{\partial x_\ell(p,w)}{\partial w}x_\ell(p,w) \leq 0.$$

Recall that when $\frac{\partial x_\ell(p,w)}{\delta p_\ell} > 0$, commodity ℓ is a Giffen good, and this implies that $\frac{\partial x_\ell(p,w)}{\delta w} < 0$, so Giffen goods are inferior.

Now $S(p,w)$ is not symmetric in general (unless $L = 2$). We will see that the symmetry of S is connected with it being associated with a demand function arising from rational preferences.

1.6.6 Proposition. *If $x(p,w)$ is differentiable, satisfies Walras' Law, and is HD0, then $S(p,w)p = 0$.*

PROOF: We have $\sum_{k=1}^{L} \frac{\partial x_\ell}{\partial p_k} p_k + \frac{\partial x_\ell}{\partial w} w = 0$, so noting that $w = \sum_{k=1}^{L} p_k x_k$ by Walras' Law, we get

$$0 = \sum_{k=1}^{L} \frac{\partial x_\ell}{\partial p_k} p_k + \frac{\partial x_\ell}{\partial w} \sum_{k=1}^{L} p_k x_k = \sum_{k=1}^{L} \left(\frac{\partial x_\ell}{\partial p_k} + \frac{\partial x_\ell}{\partial w} x_k \right) p_k = \sum_{k=1}^{L} S_{\ell k} p_k. \qquad \square$$

1.6.7 Example (WARP does not imply rationality).
Take

$$p^1 = (2,1,2), \quad p^2 = (2,2,1), \quad p^3 = (1,2,2)$$

and

$$x^1 = (1,2,2), \quad x^2 = (2,1,2), \quad x^3 = (2,2,1).$$

These data do not violate WARP, but $x^2 \notin B_1$, $x^1 \notin B_2$, and $x^3 \notin B_2$ so $x^1 \succ^* x^3 \succ^* x^2 \succ^* x^1$ and the revealed preference is not rational.

1.6.8 Summary.

1. *WARP (plus HD0 and Walras' Law) is equivalent to CLD.*
2. *CLD implies that $S(p,w)$ is NSD.*
3. *WARP (plus HD0 and Walras' Law) does not imply that $S(p,w)$ is symmetric (except when $L = 2$).*
4. *(We will see that) SARP implies that $S(p,w)$ is symmetric and the underlying (revealed?) preferences are rational.*

1.7 Utility maximization problem

1.7.1 Definition. $\succsim$ is *locally non-satiated* (or *LNS*) if for every $x \in X$ and every $\varepsilon > 0$ there is $y \in X$ such that $\|y - x\| \le \varepsilon$ and $y \succ x$.

1.7.2 Exercise. Show that if $\succsim$ is monotone then it is LNS.

1.7.3 Definition. A monotone preference relation $\succsim$ is *homothetic* if $x \sim y$ implies $\alpha x \sim \alpha y$ for all $\alpha > 0$.

1.7.4 Definition. A preference relation $\succsim$ on $X = \mathbb{R} \times \mathbb{R}_+^{L-1}$ is *quasi-linear* with respect to the first commodity (called the *numeraire*) if

1. $x \sim y$ implies $x + \alpha e_1 \sim y + \alpha e_1$ for all $\alpha \in \mathbb{R}$; and
2. $x + \alpha e_1 \succ x$ for all x and $\alpha > 0$.

1.7.5 Example. Consider the preference $\succsim$ on $\mathbb{R}_+^2$ defined by

$$x \succsim x' \iff \min\{x_1, x_2\} \geq \min\{x_1', x_2'\}.$$

This is called the *Liontief preference*. This preference cannot be represented by a differentiable utility function.

1.7.6 Exercises.

1. A continuous $\succsim$ is homothetic if it admits a utility representation u which is HD1, i.e. $u(\alpha x) = \alpha u(x)$.
2. A continuous $\succsim$ is quasi-linear with respect to x_1 if it admits a utility representation u which is of the form $u(x) = x_1 + \varphi(x_2, \dots, x_n)$.

The *utility maximization problem* is to maximize $u(x)$ subject to the constraint that $p \cdot x \leq w$, where u is a utility representation of a rational, continuous, LNS preference.

1.7.7 Proposition. *If u is a continuous function representing a LNS preference relation $\succsim$ on $\mathbb{R}_+^L$ then $x(p, w)$ satisfies*

1. *HD0: $x(\alpha p, \alpha w) = x(p, w)$ for all $\alpha > 0$*
2. *Walras' Law: $p \cdot x(p, w) = w$*
3. *If $\succsim$ is convex (whence u is quasi-concave) then $x(p, w)$ is a convex set, and if $\succsim$ is strictly convex then $x(p, w)$ consists of a single element.*

Continuity and LNS will henceforth be known as the usual conditions *on a utility function.*

The *Kuhn-Tucker conditions* for the UMP say that if $x^* \in x(p, w)$ then there exists a Lagrange multiplier $\lambda \geq 0$ such that for all $\ell \in \{1, \dots, L\}$ we have

$$\frac{\partial u}{\partial x_\ell}(x^*) \leq \lambda p_\ell$$

and the relation holds with equality if $x_\ell^* > 0$. Equivalently

$$\nabla u(x^*) \leq \lambda p \quad \text{and} \quad x^* \cdot (\nabla u(x^*) - \lambda p) = 0.$$

(This is complementary-slackness from linear programming.) In particular, when $x^* \gg 0$ the Kuhn-Tucker conditions imply that $\nabla u(x^*) = \lambda p$, so we have

$$\frac{\frac{\partial u}{\partial x_\ell}(x^*)}{\frac{\partial u}{\partial x_k}(x^*)} = \frac{p_\ell}{p_k}.$$

The left hand side is known as the *marginal rate of substitution* (or *MRS*) of good ℓ for k at x^*. (There is a bit more to say about this. See MWG.) Changes in u induced by changing w, for $x(p,w) \gg 0$, satisfy

$$\frac{\partial}{\partial w} u(x(p,w)) = \nabla u(x(p,w)) \cdot D_w x(p,w) = \lambda p \cdot D_w x(p,w) = \lambda.$$

The marginal change in utility from a marginal increase in wealth is λ. Note that the K-T conditions are necessary only. If u is quasi-concave and strongly monotone then they are sufficient. If u is not quasi-concave then x^* is a local maximum if u is locally quasi-concave at x^*. See Appendix M in MWG for more equivalent conditions.

1.7.8 Example (Cobb-Douglas). The *Cobb-Douglas* utility function is

$$u(x) = k x_1^{\alpha} x_2^{1-\alpha},$$

where $\alpha \in (0,1)$. We wish to maximize $\alpha \log x_1 + (1-\alpha) \log x_2$ subject to $x \cdot p \leq w$. Since $\log 0 = -\infty$ the maximizer is in the interior of $\mathbb{R}^L_+$. Therefore by the Kuhn-Tucker conditions we have

$$\frac{\partial u}{\partial x_1} = \frac{\alpha}{x_1} = \lambda p_1 \quad \text{and} \quad \frac{\partial u}{\partial x_2} = \frac{1-\alpha}{x_1} = \lambda p_2.$$

We get $p_1 x_1 = \frac{\alpha}{1-\alpha} p_2 x_2$, so a maximizer is $x^* = (\frac{\alpha w}{p_1}, \frac{(1-\alpha)w}{p_2})$.

1.7.9 Definition. For $(p,w) \gg 0$, the utility value for the utility maximization problem is denoted $v(p,w)$, i.e. $v(p,w) = u(x^*)$ for $x^* \in x(p,w)$. Then $v(p,w)$ is called the *indirect utility function*. It is a utility function on price-wealth pairs. See Rubinstein for examples and explanation.

1.7.10 Proposition. *Let u satisfy the usual conditions. Then $v(p,w)$ is*

1. *HD0;*
2. *strictly increasing in w and non-increasing in prices;*
3. *quasi-convex, in that $\{(p,w) \mid v(p,w) < C\}$ is convex for all constants C;*
4. *continuous in p and w.*

PROOF:

1. The solution to the UMP does not change when p and w are multiplied by the same scalar, so the value of v does not change.
2. $\frac{\partial v(p,w)}{\partial w} = \lambda > 0$ by Kuhn-Tucker, so v is strictly increasing in w, and $\frac{\partial v(p,w)}{\partial p_\ell} \leq 0$ since $B(p,w)$ gets smaller when p_ℓ increases.

3. Suppose that $v(p,w), v(p',w') \leq C$ and consider

$$(p'',w'') = \alpha(p,w) + (1-\alpha)(p',w'),$$

where $\alpha \in (0,1)$. It suffices to show for all x such that $p'' \cdot x \leq w''$ that $u(x) \leq C$ (since then it will certainly be true for the maximizing x). Now

$$\alpha p \cdot x + (1-\alpha)p' \cdot x \leq \alpha w + (1-\alpha)w',$$

so either $p \cdot x \leq w$ or $p' \cdot x \leq w'$. In either case $u(x) \leq C$.

4. When $x(p,w)$ is a function then $v(p,w)$ is composition of continuous functions and so is continuous. The assertion holds true in general. □

1.7.11 Example. Take $u(x) = \alpha \log x_1 + (1-\alpha)\log x_2$. In the example above we have seen $x(p,w) = (\frac{\alpha w}{p_1}, \frac{\alpha w}{p_2})$, so

$$v(p,w) = \alpha \log \alpha + (1-\alpha)\log(1-\alpha) + \log w - \alpha \log p_1 - (1-\alpha)\log p_2.$$

1.8 Expenditure minimization problem

The *expenditure minimization problem* (or *EMP*) is to minimize the level of wealth required to reach a given utility level. Formally, for $p \gg 0$ and $u > u(0)$, the problem is to minimize $p \cdot x$ subject to the constraint $u(x) \geq u$. The EMP is the dual problem to the UMP.

1.8.1 Proposition. *Let $p \gg 0$ and u satisfy the usual conditions.*

1. *If x^* is optimal in the UMP when wealth is w then x^* is optimal in the EMP when the required wealth level is $u(x^*)$.*

2. *If x^* is optimal in the EMP when the required utility level is u then x^* is optimal in the UMP when wealth is $p \cdot x^*$.*

PROOF:

1. Suppose that x^* is not optimal in the EMP. Then there is some feasible x' such that $u(x') \geq u(x^*)$ and $p \cdot x' < p \cdot x^*$. By LNS there is some x'' very close to x' such that $u(x'') > u(x^*)$ and $p \cdot x'' < p \cdot x^*$. This contradicts the optimality of x^* in the UMP.

2. Similarly, suppose that x^* is not optimal in the UMP. Then there is some feasible x' such that $u(x') > u(x^*)$ and $p \cdot x' \leq p \cdot x^*$. Take $\alpha \in (0,1)$ and consider $\alpha x'$. By continuity, for α close to 1, $u(\alpha x') > u(x^*)$ and $p \cdot (\alpha x') < p \cdot x' \leq p \cdot x^*$, which contradicts the fact that x^* is optimal for the EMP. □

1.8.2 Definition. Given $p \gg 0$ and $u > u(0)$, the value attained in the EMP is denoted $e(p,u)$ and is called the *expenditure function*. $e(p,u) = p \cdot x^*$ for an optimal solution x^* to the EMP.

1.8.3 Proposition. *For u continuous and monotonic, $e(p,u)$ is*

1. *HD1 in p;*
2. *strictly increasing in u and non-decreasing in p;*
3. *concave in p;*
4. *continuous in p and u.*

PROOF: The proofs are in MWG. The intuition for 3. is as follows. For fixed prices $\bar{p}$ and optimal $\bar{x}$, if we change prices to p then the expenditure level for $\bar{x}$ is $p \cdot \bar{x}$, which is linear in p, and probably less than $e(p,u)$. Hence $e(p,u)$ lies below the line and is concave. □

Note these very important facts: $e(p, v(p,w)) = w$ and $v(p, e(p,u)) = u$.

Denote the set of optimal commodity bundles for the EMP by $h(p,u)$, the *Hicksian demand function* (or *Hicksian compensated demand function*).

1.8.4 Proposition. *Under the usual assumptions, $h(p,u)$ satisfies*

1. *HD0 in p;*
2. *no excess utility, i.e. $u(x) = u$ for any $x \in h(p,u)$;*
3. *if the preference relation is convex then $h(p,u)$ is a convex set and if the preference relation is strictly convex then $h(p,u)$ is single-valued;*

1.8.5 Exercises.

1. Assume that u is differentiable. Prove the *Kuhn-Tucker conditions* for the EMP (or *first order conditions*)

$$p \geq \lambda \nabla u(x^*) \quad \text{and} \quad x^* \cdot (p - \lambda \nabla u(x^*)) = 0.$$

2. $h(p,u) = x(p, e(p,u))$
3. $x(p,w) = h(p, v(p,w))$

As prices vary $h(p,u)$ gives the demand if consumer wealth is also adjusted to keep the utility level constant. This is why $h(p,u)$ is the called a *compensated* demand function.

1.8.6 Proposition. *Under the usual assumptions and the assumption that $h(p,u)$ is single-valued, $h(p,u)$ satisfies the CLD, i.e.*

$$(p - p') \cdot (h(p,u) - h(p',u)) \leq 0.$$

PROOF: For $p \gg 0$, $h(p,u)$ achieves a lower expenditure level than any available consumption vector, so $p'' \cdot h(p'',u) \leq p'' \cdot h(p',u)$, and the reverse. Adding these completes the proof. □

An implication of this is that the own price effect is non-positive,

$$(p''_\ell - p'_\ell) \cdot (h_\ell(p'',u) - h_\ell(p',u)) \leq 0.$$

1.8.7 Exercise. Find $h(p,u)$ and $e(p,u)$ for the Cobb-Douglas utility function.

1.8.8 Proposition (Shepard's Lemma). *Assume that u is continuous, differentiable, $\succsim$ is LNS and strictly convex, h and x are single valued, and $p \gg 0$. Then $h(p,u) = D_p e(p,u)$ or equivalently, $h_\ell(p,u) = \frac{\partial e}{\partial p_\ell}(p,u)$.*

PROOF: Recall the Envelope Theorem, that for the minimization problem

$$\min_x f(x,\alpha) \quad \text{subject to} \quad g(x,\alpha) = 0$$

the minimum value ϕ as a function of the parameters α satisfies

$$\frac{\partial \phi}{\partial \alpha_m}(\alpha) = \frac{\partial f}{\partial \alpha_m}(x^*(\alpha),\alpha) - \lambda \frac{\partial g}{\partial \alpha}(x^*(\alpha),\alpha).$$

Since $e(p,u) = \min_x p \cdot x$ subject to $u(x) = u$, we get

$$\frac{\partial}{\partial p_\ell} e(p,u) = x^*_\ell - \lambda 0 = x^*_\ell = h_\ell(p,u).$$

□

It follows that the expenditure depends only on the consumption level.(?)

1.8.9 Proposition. *Under the usual assumptions plus differentiability, we have the following.*

1. $D_p h(p,u) = D_p^2 e(p,u)$
2. $D_p h(h,u)$ *is NSD*
3. $D_p h(p,u)$ *is symmetric*
4. $D_p h(p,u) \cdot p = 0$.

PROOF:

1. 1.8.8.
2. The expenditure function is concave in p.
3. As above.
4. h is HD0 in p. □

Note that the NSDness of $D_p h(p,u)$ is an analog of the CLD.

1.8.10 Definition. Commodities ℓ and k are *complements* if

$$\frac{\partial h_\ell(p,u)}{\partial p_k} = \frac{\partial h_k(p,u)}{\partial p_\ell} \le 0$$

and *substitutes* if this quantity is at least zero.

Recall that $\frac{\partial h_\ell(p,u)}{\partial p_\ell} \le 0$, so $D_p h(p,u)\cdot p = 0$ implies that there will be some commodity k for which ℓ and k are substitutes.

1.8.11 Proposition (Slutsky Equation). *For all (p,w) and $u = v(p,w)$ we have*

$$\frac{\partial h_\ell}{\partial p_k}(p,u) = \frac{\partial x_\ell}{\partial p_k}(p,w) + \frac{\partial x_\ell}{\partial w}(p,w)x_k(p,w)$$

for all ℓ and k. In matrix form, $D_p h(p,u) = D_p x(p,w) + D_w x(p,w)x(p,w)^T$.

PROOF: Consider a consumer facing $(\bar{p},\bar{w})$ and attaining $\bar{u}$ (note that $\bar{w} = e(\bar{p},\bar{u})$ and $h_k(\bar{p},\bar{w}) = x_k(\bar{p},\bar{w})$). Recall $h_\ell(p,u) = x_\ell(p,e(p,u))$, so differentiating this with respect to p_k and evaluating at $(\bar{p},\bar{w})$ we get the result. □

If $\frac{\partial x_\ell}{\partial w} > 0$ (i.e. good ℓ is normal) then the slope of the graph of h_ℓ with respect to p_ℓ is steeper at (p,w) than the graph of x_ℓ with respect to p_ℓ, and *visa versa* for inferior goods.

Another implication of 1.8.11 is that $D_p h(p,u) = S(p,w)$, the Slutsky matrix, where $w = e(p,u)$ is the minimized expenditure. We proved in 1.8.9 that $D_p h(p,u)$ is NDS and symmetric, so we obtain that the Slutsky matrix is symmetric when it comes from the maximization of a utility function. See the discussion in MWG about the intuition surrounding this result. From before, WARP implied that the Slutsky matrix was NSD, but that it was not necessarily symmetric for $L > 2$. In particular, WARP is not as strong of an assumption as that of preference maximization.

1.8.12 Proposition (Roy's Identity). *Make the usual assumptions on the demand function plus differentiability and assume $\succsim$ is strictly convex and differentiable. Then for any $(p,w) \gg 0$ we have*

$$x_\ell(p,w) = -\frac{\partial v(p,w)/\partial p_\ell}{\partial v(p,w)/\partial w}$$

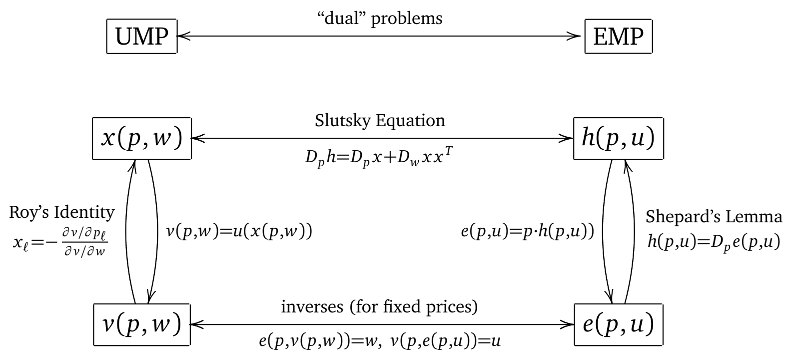
PROOF: Recall that $v(p,w)$ is the optimal value of

$$\max u(x) \quad \text{subject to} \quad p\cdot x = w.$$

Whence by the Envelope Theorem

$$\frac{\partial v(p,w)}{\partial p_\ell} = \frac{\partial u(x^*)}{\partial p_\ell} - \lambda\frac{\partial(p\cdot x^* - w)}{\partial p_\ell} = -\lambda x^*_\ell.$$

and $\lambda = \frac{\partial v(p,w)}{\partial w}$, again by the Envelope Theorem, so we are done. □



We have seen that rational preferences imply Walras' Law, HD0, and that $S(p,w)$ is NSD and symmetric. Do Walras' Law, HD0, and $S(p,w)$ being NSD and symmetric imply that the implied preferences are rational? (It turns out to be "yes.") To recover $\succsim$ from $x(p,w)$, we precede in two steps. First we recover $e(p,u)$ from $x(p,w)$ and then recover $\succsim$ from $e(p,u)$.

To do the second of these, given $e(p,u)$ we need to find an at-least-as-good set $V_u \subseteq \mathbb{R}^L_+$ such that $e(p,u) = \min_{x \in V_u} p \cdot x$. The V_u give the indifference curves for $\succsim$ since we have $x \succ x'$ if $x \in V_u$ but $x' \notin V_u$ for some u. Proposition 3.H.1 in MWG shows that $V_u := \{x \in \mathbb{R}^L_+ \mid p \cdot x \geq e(p,u)\}$ works.

For the second part, in MWG they show that for a fixed utility level u and wealth level w we have

$$\frac{\partial e}{\partial p_\ell}(p) = x_\ell(p, e(p))$$

for all $\ell = 1, \dots, L$. This is a system of PDE. Existence of a solution is possible only when the Hessian matrix $D^2_p e(p)$ is symmetric. A result called Frobenius' Theorem implies that this condition is sufficient. Since $D^2_p e(p) = S(p,w)$, a solution exists when the Slutsky matrix is symmetric. Therefore we obtain (along with Walras' Law, HD0, and NSDness) that $\succsim$ exists if and only if $S(p,w)$ is symmetric.

1.9 Strong axiom of revealed preference

We have seen that WARP does not imply the existence of a rational preference. What is a necessary and sufficient condition? The answer was found by Houthakker in 1950, as SARP.

Remember that the (direct) revealed preference relation is defined by $x \succ^* x'$ if there are (p,w) such that $x = x(p,w)$ and $p \cdot x' \leq w$ (assuming, as we do this entire section, that $x(p,w)$ is single-valued).

1.9.1 Definition. x' is *indirectly revealed preferred* to x, written $x' \succ^{**} x$ if either $x' \succ^* x$ or there exist $x_1, \dots, x_n$ such that

$$x' \succ^* x_1, x_1 \succ^* x_2, \dots, x_{n-1} \succ^* x_n, x_n \succ^* x.$$

1.9.2 Definition. A demand function $x(p,w)$ satisfies the *strong axiom of revealed preference* (or *SARP*) if it is never the case that $x \succ^{**} x$.

SARP is equivalent to saying that there do not exist $x_1, \dots, x_n$ such that

$$x \succ^{*} x_1 \succ^{*} \dots \succ^{*} x_n \succ^{*} x,$$

or equivalently that $\succ^{*}$ is acyclic, or equivalently that $\succ^{**}$ is irreflexive.

1.9.3 Proposition.

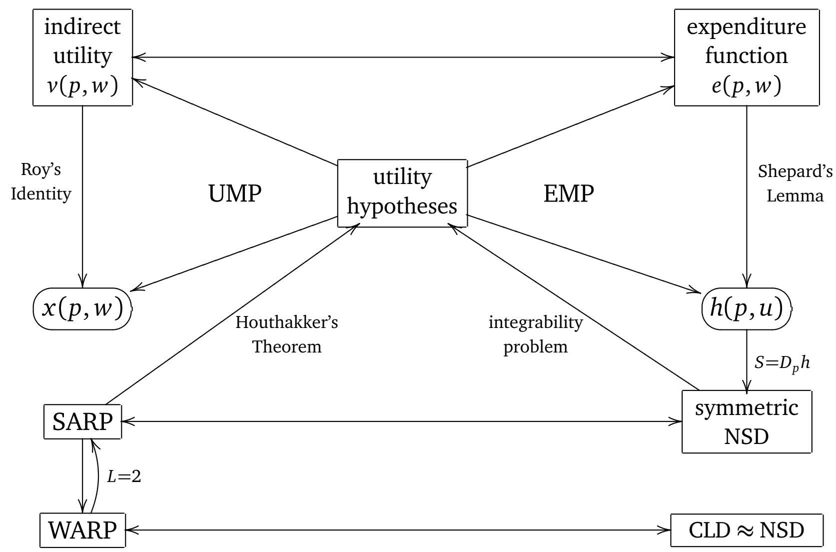
1. *The demand function resulting from UMP satisfies SARP.*
2. *SARP implies WARP.*
3. *WARP implies SARP when $L = 2$.*

PROOF:

1. If $x' \succ^{*} x''$ then $u(x') > u(x'')$. If SARP did not hold then there would be $x, x_1, \dots, x_n$ such that $u(x) > u(x_1) > \cdots > u(x_n) > u(x)$, a contradiction.
2. This is in the definition.
3. Exercise. □

1.9.4 Proposition. *If $x(p,w)$ satisfies SARP then there is $\succsim$ that rationalizes $x(p,w)$, i.e. $x(p,w) \succ y$ for all $y \in B(p,w)$, $y \neq x$.*

PROOF: Define $\succ^{*}$ and $\succ^{**}$ as above. Observe that by construction $\succ^{**}$ is transitive. Zorn's Lemma tells us that every partial order has a total extension (?) so there is $\succ^{***}$ such that $x \succ^{**} y$ implies $x \succ^{***} y$ and $\succ^{***}$ is complete. Define indifference via $x \succsim y$ if $x \succ^{***} y$ or $x = y$. Confirm that $\succsim$ is rational and $x(p,w) \succ y$ for all $v \in B(p,w)$, $y \neq x$. □



2 Aggregate Demand

The sum of the demands arising from all of the economy's consumers is the *aggregate demand*. What kind of structure is imposed on aggregate demand when individual demand is utility driven? Can we find a utility function u such that it generates aggregate demand? For a *representative consumer*, would u generate aggregate demands?

2.0.5 Definition. Suppose there are I consumers with rational preference relations $\succsim_i$, wealths $w = (w^1, \dots, w^I)$, and Walrasian demand functions $x^i(p, w^i)$. Given prices $p \in \mathbb{R}^L_+$, *aggregate demand* is $x^*(p, w) = \sum_{i=1}^I x^i(p, w^i)$.

2.0.6 Example. Individual WARP does not imply aggregate demand WARP.

1. Consider two individuals with the same income under two price sets p and q. They demand x^1, y^1 and x^2, y^2 respectively. See diagrams.
2. Consider two consumers with utility functions $u_i(x) = x_1 x_2 + x_2$ and wealth levels w^i at $p = (1, 1)$. Solving we get $x_1^i = \frac{w^i-1}{2}$ and $w_2^i = \frac{w^i+1}{2}$ if $w^i \geq 1$, otherwise x_1^i=0 and $x_2^1 = w^i$. At $w = (1, 1)$ we have $x^1 + x^2 = (0, 2)$, whereas at $w = (2, 0)$ we have $x^1 + x^2 = (\frac{1}{2}, \frac{3}{2})$. Finish this.

2.0.7 Definition. Aggregate demand is *independent of income distribution* if $x^*(p, w) = x^*(p, \bar{w})$ whenever $\sum_i w^i = \sum_i \bar{w}^i$.

2.0.8 Proposition. *Aggregate demand is independent of the distribution of income if and only if all consumers have the same homothetic preferences.*

PROOF: Suppose all the consumers have the same homothetic preferences.

$$x^*(p,w) = \sum_i x^i(p,w^i) = \sum_i x(p,w^i) = \sum_i x(p,w)w^i = x(p,1)\sum_i w^i,$$

so the aggregate demand is independent of the distribution of income.

Conversely, suppose $x^*(p,w)$ is independent of the distribution of income. Consider the specific distribution of wealth $w^j = w\delta_{ij}$. Then $x^i(p,w) = x^*(p,w)$ in this case, so the demand functions for the consumers are the same. Since they are the same, $x^*(p,w)$ is additive in wealth. The only increasing functions satisfying this property are linear, so $x^*(p,w) = wx^*(p,1)$. □

Suppose that the aggregate demand is independent of the distribution of wealth. For every small chance in the wealths with $\sum_i dw^i = 0$ we would like $\sum_i \frac{\partial x_\ell^i}{\partial w^i} dw_i = 0$ for all ℓ. This (somehow) implies that $\frac{\partial x_\ell^i}{\partial w^i} = \frac{\partial x_\ell^j}{\partial w^j}$ for all ℓ and i,j, so the wealth expansion paths are linear.

2.0.9 Proposition. *Consumers exhibit linear, parallel wealth expansion paths if and only if indirect utility functions are of Gorman normal form, i.e.*

$$v^i(p,w^i) = a^i(p) + b(p)\cdot w^i.$$

PROOF: For the forward direction see Deaton and Muellbouer, 1980. For the reverse direction notice that

$$x_\ell^i = -\frac{\partial v^i(p,w)/\partial p_\ell}{\partial v^i(p,w)/\partial w^i} = -\frac{\frac{\partial a^i}{\partial p_\ell}(p) + \frac{\partial b}{\partial p_\ell}(p)w^i}{b(p)},$$

so $\frac{\partial x_\ell^i}{\partial w_i} = -\frac{\partial b}{\partial p_\ell}(p)/b(p)$. □

2.0.10 Exercise. Show that this condition holds when all consumers have identical preferences that are homothetic or when all consumers have quasi-linear preferences with respect to some good.

Even supposing that $w^i = \alpha_i w$, where α_i is the *fixed* share of agent i in the total wealth w (so $\alpha_i \geq 0$ and $\sum_i \alpha_i = 1$), it is not the case that individual WARP implies aggregate WARP. The CLD is not preserved under aggregation since a necessary condition for the CLD is the WARP.

2.0.11 Definition. A demand function $x(p,w)$ satisfies the *uncompensated law of demand* (or *ULD*) if

$$(p-p')\cdot(x(p,w)-x(p',w)) \leq 0$$

for any p,p',w with strict inequality when $x(p,w) \neq x(p',w)$. An analogous definition applies for aggregate demand x^*.

2.0.12 Exercise. $x(p,w)$ satisfies ULD if and only if $D_p x(p,w)$ is NSD.

2.0.13 Proposition. *If $x^i(p,w^i)$ satisfy the ULD then so does the aggregate demand x^*. As a consequence, aggregate demand satisfies WARP.*

PROOF: Consider (p,w) and (p',w) with $x^*(p,w) \neq x^*(p',w)$. Then we must have $x^i(p,w^i) \neq x^i(p',w^i)$ for some i. For all i we have the ULD

$$(p-p')\cdot(x^i(p,w^i)-x^i(p',w^i)) \leq 0,$$

(with at least one of the inequalities strict) so adding these up,

$$(p-p')\cdot(x^*(p,w^i)-x^*(p',w^i)) < 0$$

so x^* satisfies the ULD.

We now prove that the ULD implies WARP. Take any (p,w) and (p',w') with $x(p,w) \neq x(p',w')$ and $p \cdot x(p',w') \leq w$. Define $p'' = \frac{w}{w'}p'$. Since x is HD0, $x(p'',w) = x(p',w')$, and by the ULD,

$$\begin{aligned}
0 &> (p''-p)\cdot(x(p'',w)-x(p,w)) \\
&= p''\cdot x(p'',w) - p''\cdot x(p,w) - p\cdot x(p'',w) + p\cdot x(p,w) \\
&= w - p''\cdot x(p,w) - p\cdot x(p',w') + w \\
&\geq w - p''x
\end{aligned}$$

□

How restrictive is the ULD?

2.0.14 Proposition. *If $\succsim$ is homothetic, then $x(p,w)$ satisfies the ULD.*

PROOF: $D_p x(p,w) = S(p,w) - D_w x(p,w)x(p,w)^T = S(p,w) - \frac{1}{w}xx^T$ since homothetic preferences have linear wealth expansion paths. Therefore $D_p x$ is NSD since $S(p,w)$ is NSD and $vxx^Tv = (v\cdot x)^2$ for all $v \in \mathbb{R}^L$. This is sufficient. □

2.0.15 Proposition. *Suppose that each agent i has a fixed share α_i of the total wealth, and that each i has and HD1 utility function (homothetic preferences). Then there is a utility function U that generates the aggregate demand.*

PROOF: Skipped. See MWG §4.D for related discussion. In fact,

$$U(x) = \max \prod_{i=1}^{I} (u_1(x^i))^{\alpha_i}$$

where the max is taken over all $\sum_i x^i = x$ for feasible x^i. □

To summarize, homothetic preferences (but not necessarily identical) plus fixed income shares imply that there is a representative consumer.

Suppose that the demand functions $x^i(p,w)$ are utility generated. Then what are the restrictions at the aggregate level? Only Walras' Law and HD0 are reserved – symmetry and NSDness of $S(p,w)$ and SARP are not preserved. Anything goes!

2.0.16 Proposition. *Suppose that all consumers have identical preferences with individual demands $x(p, w)$ and that individual wealth is uniformly distributed over $[0, \bar{w}]$. Then the aggregate demand function*

$$x(p) = \int_0^{\bar{w}} x^*(p, w) dw$$

satisfies the ULD.

PROOF: See MWG. □

Note that since the ULD is additive, we don't need to have identical preferences for all consumers. We need the distribution of wealth conditional on each preference to be uniform over some interval that includes zero.

3 Production

3.1 Properties of production sets

Production theory is the theory of the *firm*, a rational agent aiming to maximize profit. Again we consider L commodities. A *production vector* $y \in \mathbb{R}^L$ describes (net) outputs of L commodities. Positive numbers are outputs, negative numbers are inputs. A set of production vectors that are (technologically) feasible is a *production set*, commonly denoted Y. Production sets $Y \subseteq \mathbb{R}^L$ are primitives of the model.

The following properties may apply to production sets Y.

1. Y is closed and non-empty;
2. $0 \in Y$;
3. there is no $y \in Y$ such that $y \geq 0$ and $y \neq 0$ (*no free lunch*);
4. if $y \in Y$ and $y' \leq y$ then $y' \in Y$ (*free disposal*);
5. if $y \in Y$ and $y \neq 0$ then $-y \notin Y$ (*irreversibility*);
6. Y exhibits *non-increasing returns to scale*, i.e. if $y \in Y$ and $\alpha \in [0, 1]$ then $\alpha y \in Y$;
7. Y exhibits *non-decreasing returns to scale*, i.e. if $y \in Y$ and $\alpha \geq 1$ then $\alpha y \in Y$;
8. Y exhibits *constant returns to scale*, i.e. if $y \in Y$ and $\alpha \geq 0$ then $\alpha y \in Y$ (so Y is a cone);
9. Y is *additive*, i.e. if $y, y' \in Y$ then $y + y' \in Y$;
10. Y is convex;

11. Y is a *convex cone*, i.e. if $y, y' \in Y$ then $\alpha y + \beta y' \in Y$ for all $\alpha, \beta \geq 0$.

Note of course that many of these properties are consequences of others.

3.1.1 Exercises.

1. Find a cone that is not a convex cone.
2. Prove Proposition 5.B.1 in MWG.

3.2 Profit maximization and cost minimization

In this section we investigate the behavior of a firm. We always assume competitive (i.e. constant) prices $p \in \mathbb{R}^L_+$ and that Y is non-empty, closed, and there is free disposal. The *profit maximization problem* (or *PMP*)) is to maximize $p \cdot y$ over Y.

The *profit function* $\pi(p) = \max_{y \in Y} p \cdot y$ is an analog of the indirect utility function. The *supply function* (or *supply correspondence* in general) is the solution set to the PMP, $y(p) = \{y \in Y \mid p \cdot y = \pi(p)\}$.

3.2.1 Exercise. Prove that if Y exhibits NIRS then $\pi(p) = \infty$ or 0 for all p.

3.2.2 Proposition. *Assume the production set Y is closed and has free disposal.*

1. *$\pi(p)$ is HD1 in p.*
2. *$y(p)$ is HD0 in p.*
3. *$\pi(p)$ is convex in p.*
4. *If Y is convex then $Y = \{y \in \mathbb{R}^L \mid p \cdot y \leq \pi(p)$ for all $p \gg 0\}$.*
5. *If Y is convex then $y(p)$ is a convex set. If Y is strictly convex then $y(p)$ is single-valued.*

PROOF:

1. Changing p to αp does not alter Y (as it does not depend upon p at all), but multiplies the objective function by α.
2. As above.
3. $\pi(\alpha p + (1-\alpha)p') \leq \alpha\pi(p) + (1-\alpha)\pi(p')$ since π is a supremum, clearly.
4. A convex set is the intersection of all the hyperplanes containing it.
5. We have seen this before. □

3.2.3 Lemma (Hotelling). *If $y(p)$ is single-valued and $\pi(p)$ is differentiable then $\nabla\pi(p) = y(p)$.*

PROOF: By the envelope theorem $\frac{\partial \pi(p)}{\partial p_\ell} = y_\ell(p)$. □

3.2.4 Definition. If y' is the optimal production at p' and y'' is optimal at p'' then $(p' - p'') \cdot (y' - y'') \geq 0$. This is the *law of supply*.

This is analogous to substitution effects in the consumer theory. Notice that there are no analogous to wealth effects since the production set is fixed and does not depend on prices.

3.2.5 Proposition. *If $y(p)$ is C^1 then the matrix $Dy(p) = D^2\pi(p)$ is symmetric and positive semi-definite with $Dy(p)p = 0$.*

PROOF: Positive semi-definiteness follows from the law of supply. □

3.2.6 Definition. A production vector y is *efficient* if there does not exist $y' \in Y$ such that $y' \geq y$ and $y' \neq y$. Let $Y^e \subseteq Y$ denote the efficient vectors in Y.

3.2.7 Theorem. *Suppose that the production set Y is closed and satisfies free disposal. Then there is a continuous function $F : \mathbb{R}^L \to \mathbb{R}$, the transformation function for Y, such that $y \in Y$ if and only if $F(y) \leq 0$, and $y \in Y^e$ if and only if $F(y) = 0$.*

PROOF: Omitted, but similar to the proof of the existence of a utility function. □

3.3 Single-output firm

Suppose now that there are $L - 1$ inputs and good L is produced by using these inputs. We may write $q = f(z)$ for $(-z, q) \in Y^e$, so q is the biggest quantity that can be produced by using z (here $f : \mathbb{R}_+^{L-1} \to \mathbb{R}$). Suppose that input prices are $w = (w_1, \dots, w_{L-1})$. The *cost minimization problem* (or *CMP*) is to minimize $w \cdot z$ (over $z \geq 0$) subject to $f(z) \geq q$. The problem is to find the most economical way of producing q units of output. Define $c(w, q) = \min_{z \geq 0, f(z) \geq q} w \cdot z$, the *cost function*. The cost function is analogous to the expenditure function in the consumer theory. Optimal input choices are given by $z(w, q) = \{z \geq 0 \mid f(z) \geq q, w \cdot z = c(w, q)\}$, the *conditional factor demand function* or *input demand function*.

3.3.1 Proposition.

1. *$z(w, q)$ is HD0 in w.*
2. *$c(w, q)$ is concave in w if and only if $D_w^2 c(w, q)$ is NSD.*
3. *(Shepard's Lemma) $D_w c(w, q) = z(w, q)$.*
4. *$D_w z(w, q)$ is NSD, z is downward sloping in w.*
5. *$c(w, q)$ is HD1 in w and non-decreasing in q.*

The property of CRS (constant returns to scale, or that of being a cone) corresponds in the single-output case to $\alpha q = f(\alpha z)$ for all $\alpha \geq 0$, i.e. that f is HD1. The property of NIRS (or DRS) corresponds to $c(w, q)$ being convex, and the property of NDRS (IRS) corresponds to $c(w, q)$ being concave.

3.3.2 Definition. For fixed prices, the *average cost* is $AC(q) = \frac{c(q)}{q}$ and the *marginal cost* is $MC(q) = \frac{dc(q)}{dq}$.

3.3.3 Exercise. Show that if $\bar{q}$ minimizes $AC(q)$ then $AC(\bar{q}) = MC(\bar{q})$.

3.4 Aggregation of production

Recall that individual supply is not subject to an analog of wealth effects, so the law of supply holds and is preserved under addition. Consider J firms with production sets $Y_1, \dots, Y_J$, profit functions $\pi_j(p)$, and (single-valued) supply functions $y^j(p)$. The *aggregate supply function* $y^*(p)$ is the sum of the individual supply functions. From the law of supply, $D_p y^j(p)$ is symmetric and PSD, so their sum $y^*(p)$ is symmetric and PSD. Therefore the law of supply holds in aggregate. Does there always exist a representative producer? Yes, use the production set $Y^* := Y_1 + \dots + Y_J$.

3.4.1 Proposition. *For all $p \gg 0$, we have*

1. *The profit function associated with Y^* is the profit function of the aggregate producer.*
2. *$y^*(p)$ is the supply function for Y^*.*

4 Choice Under Uncertainty

4.1 Lotteries

So far we have considered "actions," "decisions," and "choice" to be equivalent and deterministically leading to consequences. In this section we consider where choices lead to uncertain consequences. The idea is that the decision maker is choosing a lottery ticket.

Let $C = \{C_1, \dots, C_N\}$ be a finite set of possible outcomes. We assume the probabilities of the various outcomes to arise are objectively known.

4.1.1 Definition. A *simple lottery* (or simply *lottery*) is a probability measure on C, which in the case that C is finite is simply a non-negative vector p such that $p \cdot e = 1$. Let $\mathscr{L}$ be the set of lotteries over C, so

$$\mathscr{L} = \{p \in \mathbb{R}^N \mid p \geq 0, p \cdot e = 1\}.$$

Our decision maker now chooses (rationally) between lotteries.

4.1.2 Definition. Given K simple lotteries $L_k = \{p_1^k, \dots, p_n^k\}$ and probabilities $\alpha_1, \dots, \alpha_K$, the *compound lottery* $(L_1, \dots, L_K, \alpha_1, \dots, \alpha_K)$ is the risky alternative that yields the simple lottery L_k with probability α_k.

A compound lottery is a lottery whose outcomes are lotteries. For any compound lottery there is a simple lottery L that generates the same ultimate distribution over the outcomes. It is given by $L = \sum_k \alpha_k L_k$. The convention that we use in this class is that only the consequences and their probabilities matter, so the fact that compound lotteries can be reduced to simple lotteries implies that we need only consider preferences on simple lotteries.

We assume that the decision maker has a rational preference relation on $\mathscr{L}$. When can such preferences the preference by a utility function? As before, if the preference is continuous then a utility representation exists. We consider a "simpler" (more structured) utility representation that assigns utilities to each consequence.

4.1.3 Definition. $\succsim$ on $\mathscr{L}$ is *continuous* if for any $L, L', L'' \in \mathscr{L}$ the sets $\{\alpha \in [0,1] \mid \alpha L + (1-\alpha)L' \succsim L''\}$ and $\{\alpha \in [0,1] \mid L'' \succsim \alpha L + (1-\alpha)L'\}$ are both closed.

Continuity means that small changes in probabilities should not change the ordering between lotteries. In other words, preferences are not "overly sensitive" to small changes in probabilities.

4.1.4 Example. Take $C = \{\text{trip by car, death by car accident, stay home}\}$. If the decision maker is a safety fanatic then we would have

$$1(1,0,0) + 0(0,1,0) \succ (0,0,1) \quad \text{but} \quad (0,0,1) \succ (1-\varepsilon)(1,0,0) + \varepsilon(0,1,0)$$

for any $\varepsilon > 0$. In this case the preference relation is not continuous.

4.1.5 Definition. $\succsim$ on $\mathscr{L}$ satisfies the *independence axiom* if for all $L, L', L'' \in \mathscr{L}$ and $\alpha \in (0,1)$ we have

$$L \succsim L' \iff \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''$$

In words, if we mix two lotteries with a third one then the preference ordering should be independent of the third.

4.1.6 Exercise. Show that if $\succsim$ satisfies the independence axiom then $\succ$ and $\sim$ also do. Also show that $L \succ L'$ and $L'' \succ L'''$ then

$$\alpha L + (1-\alpha)L'' \succ \alpha L' + (1-\alpha)L'''.$$

4.1.7 Definition. The utility function $u : \to \mathbb{R}$ has an *expected utility form* if there is an assignment of numbers $u_1, \dots, u_N$ to the N outcomes such that for every simple lottery L we have $u(L) = u \cdot L$. A utility function with an expected utility form is called a *Von Neumann-Morgenstern* (or *VNM*) expected utility function.

Observe that if we let L^n denote the lottery that yields outcome n with probability one, then $u(L^n) = u_n$.

4.1.8 Proposition. *A utility representation u has an expected utility form if and only if it satisfies the property that $u(\sum_k \alpha_k L_k) = \sum_k \alpha_k u(L_k)$ for any compound lottery (L, α) (i.e. linearity).*

PROOF: For the reverse direction, consider for any simple lottery L the compound lottery $(L^1, \dots, L^N, p_1, \dots, p_N)$. Then $u(L) = \sum_i p_i u(L^i)$ so u has an expected utility form. The forward direction is simply linearity of expectation. □

The expected utility form of a preference relation (if it has one) is not unique, as any positive affine transformation also gives an expected utility form. The converse is also true (exercise).

4.1.9 Proposition. *Suppose that u is VNM for $\succsim$. Then $\tilde{u}$ is another VNM if and only if there are two scalars $\beta > 0$ and γ such that $\tilde{u}(L) = \beta u(L) + \gamma$.*

PROOF: Consider the best and worst lotteries $\overline{L}$ and $\underline{L}$ such that for all $L \in \mathscr{L}$ we have $\overline{L} \succsim L \succsim \underline{L}$ and $\overline{L} \succ \underline{L}$. Consider any $L \in \mathscr{L}$ and define $\lambda_L \in [0, 1]$ by

$$u(L) = \lambda_L u(\overline{L}) + (1 - \lambda_L) u(\underline{L}).$$

Since u is linear $\lambda_L \overline{L} + (1 - \lambda_L)\underline{L} \sim L$. If $\tilde{u}$ represents the same preferences and is VNM then it is also linear, so we have

$$\tilde{u}(L) = \lambda_L \tilde{u}(\overline{L}) + (1 - \lambda_L) \tilde{u}(\underline{L}).$$

Equating expressions for λ_L it follows that $\tilde{u}(L) = \beta u(L) + \gamma$ with

$$\beta = \frac{\tilde{u}(\overline{L}) - \tilde{u}(\underline{L})}{u(\overline{L}) - u(\underline{L})} > 0 \quad \text{and} \quad \gamma = \tilde{u}(\underline{L}) - u(\underline{L}) \frac{\tilde{u}(\overline{L}) - \tilde{u}(\underline{L})}{u(\overline{L}) - u(\underline{L})}.$$

□

Now we prove the Expected Utility Theorem.

4.1.10 Theorem. *Suppose that $\succsim$ is a rational preference relation $\mathscr{L}$ that satisfies the independence axiom and is continuous. Then $\succsim$ admits a utility representation of the expected utility form, i.e. we have $L \succsim L'$ if and only if $\sum_{n=1}^N u_n p_n \geq \sum_{n=1}^N u_n p'_n$ for some numbers $u_1, \dots, u_n$.*

Again we consider the best and worst lotteries $\overline{L}$ and $\underline{L}$.

4.1.11 Lemma. *If $0 \leq \alpha < \beta \leq 1$ then $\beta \overline{L} + (1 - \beta)\underline{L} \succ \alpha \overline{L} + (1 - \alpha)\underline{L}$.*

PROOF: Define $\gamma = \frac{\beta - \alpha}{1 - \alpha}$. Then

$$\beta \overline{L} + (1 - \beta)\underline{L} = \gamma \overline{L} + (1 - \gamma)(\alpha \overline{L} + (1 - \alpha)\underline{L})$$

by definition of γ and

$$\overline{L} = \alpha \overline{L} + (1 - \alpha)\overline{L} \succ \alpha \overline{L} + (1 - \alpha)\underline{L}$$

by the independence axiom. Again by independence,

$$\gamma \overline{L} + (1 - \gamma)(\alpha \overline{L} + (1 - \alpha)\underline{L}) \succ \alpha \overline{L} + (1 - \alpha)\underline{L}$$

□

4.1.12 Lemma. *For every $L \in \mathscr{L}$ there is α_L such that $L \sim \alpha_L \overline{L} + (1-\alpha_L)\underline{L}$.*

PROOF: By continuity and completeness of $\succsim$ the sets

$$\{\alpha \mid \alpha\overline{L} + (1-\alpha)\underline{L} \succsim L\} \quad \text{and} \quad \{\alpha \mid L \succsim \alpha\overline{L} + (1-\alpha)\underline{L}\}$$

are both closed. Therefore they intersect, and the intersection point is unique by the last lemma. □

PROOF (OF 4.1.10): Define $u(L) = \alpha_L$. Note that $L \succsim L'$ if and only if $\alpha_L \geq \alpha_{L'}$ by the first lemma. We must show that

$$u(\beta L + (1-\beta)L') = \beta u(L) + (1-\beta)u(L')$$

for any $\beta \in [0,1]$. We have

$$\begin{aligned} \beta L + (1-\beta)L' &\sim \beta(u(L)\overline{L} + (1-u(L))\underline{L}) + (1-\beta)(u(L')\overline{L} + (1-u(L'))\underline{L}) \\ &= (\beta u(L) + (1-\beta)u(L'))\overline{L} + (1-(\beta u(L) + (1-\beta)u(L')))\underline{L} \end{aligned}$$

By the uniqueness in the second lemma,

$$u(\beta L + (1-\beta)L') = \beta u(L) + (1-\beta)u(L').$$ □

If a person's preferences do not satisfy the independence axiom then we can take advantage of them. See the Dutch Book Argument.

On the other hard, *Allais paradox* is as follows. Consider the pairs of lotteries

$$L_1 : \begin{cases} \$3000 & 0.25 \\ \$0 & 0.75 \end{cases} \quad \text{and } L_2 : \begin{cases} \$4000 & 0.2 \\ \$0 & 0.8 \end{cases}$$

and

$$L_3 : \begin{cases} \$3000 & 1 \\ \$0 & 0 \end{cases} \quad \text{and } L_4 : \begin{cases} \$4000 & 0.8 \\ \$0 & 0.2 \end{cases}$$

Most people would prefer L_2 to L_1 and L_3 to L_4, but these choices are not compatible with the independence axiom. Solutions to this paradox can be obtained by expanding the utility to consider things like "regret." There are other issues (see MWG).

4.2 Money Lotteries and Risk Aversion

For this section we look at preferences on risky alternatives whose outcomes are amounts of money. We assume money is a continuous variable, so we have infinitely many outcomes. We take $\mathscr{L}$ to be the collection of lotteries over wealth levels, where a *lottery* is a *distribution function* $F : \mathbb{R} \to [0,1]$, i.e.

1. $F(-\infty) = 0$;

2. $F(+\infty) = 1$;

3. F is non-decreasing; and

4. F is right continuous.

We may also identify a lottery with a random variable with distribution function F, or with its density function with respect to Lebesgue measure. The Expected Utility Theorem in the continuous case is as follows.

4.2.1 Theorem. *Given a continuous preference relation on lotteries, if it satisfies the independence axiom then there exists a function $u : \mathbb{R} \to \mathbb{R}$ (the Bernoulli utility function) such that $U(F) = \int u(x)dF(x)$, the Von Neumann-Morgenstern utility function.*

The idea is that $u(x)$ is the utility of the degenerate lottery that pays x for certain. We may assume that u is continuous and increasing. It is typically assumed that u has an upper bound. If u does not then for every $m \in \mathbb{Z}_+$ there is $x_m \in \mathbb{R}$ such that $u(x_m) > 2^m$. We obtain the *St. Petersburg paradox*. Consider the lottery L that is defined by "toss a coin until tails comes up and receive x_m if the tails occurs on the m^{th} toss." Then the utility of this lottery is $\sum_{m=1}^{\infty} \frac{1}{2^m} u(x_m) = +\infty$.

4.2.2 Definition. An individual with Bernoulli utility function u is *risk averse* if $\int u(x)dF(x) \leq u(\int x\, dF(x))$ for every given lottery F.

4.2.3 Proposition. *An individual is risk averse if and only if u is concave.*

4.2.4 Exercises.

1. An individual is strictly risk averse if and only if u is strictly concave.

2. An individual is risk neutral if and only if u is linear.

4.2.5 Example. Let $u(x) = \sqrt{x}$ and $F(x) = x^2$, so $dF(x) = 2x dx$ and

$$\int_0^1 u(x)dF(x) = \int_0^1 2x^{\frac{3}{2}} dx = \frac{4}{5}$$

but $E(x) = \int_0^1 x\, dF(x) = \frac{2}{3}$, so $u(E(x)) = \sqrt{\frac{2}{3}} > \frac{4}{5}$.

4.2.6 Definition. The *certainty equivalent* of F, denoted by $C(F)$, is the amount of money for which the individual is indifferent between the gamble F and the certainty amount $C(F)$, i.e.

$$u(C(F)) = \int u(x)dF(x).$$

Notice that if the individual is risk averse then $C(F) < E(x) = \int x\, dF(x)$. The quantity $E(x) - C(F)$ is called the *risk premium*.

Measuring risk aversion

When is $\succsim_1$ more risk averse than $\succsim_2$? We might say that $\succsim_1$ is *more risk averse* than $\succsim_2$

1. if the certainty equivalent for $\succsim_1$ is less than the certainty equivalent for $\succsim_2$ for all lotteries.

2. if the utility representation u_1 is *more concave* than u_2, i.e. if there is a concave function φ such that $u_1(x) = \varphi(u_2(x))$.

A proof that the two definitions are equivalent is in Rubinstein.

4.2.7 Definition. If u_1 and u_2 are twice differentiable Bernoulli utility functions representing $\succsim_1$ and $\succsim_2$ then $\succsim_1$ is *more risk averse* than $\succsim_2$ if $r_2(x) \le r_1(x)$ for all x, where $r_i(x) = -\frac{u''(x)}{u_i'(x)}$, the *Arrow-Pratt coefficient of absolute risk aversion*.

4.2.8 Proposition. *This third definition is equivalent to the first two.*

PROOF: $u_1 \circ u_2^{-1}$ is concave if and only if

$$\frac{d}{dt}u_1(u_2^{-1}(t)) = u_1'(u_2^{-1}(t))\frac{1}{u_1'(u_2^{-1}(t))}$$

is non-increasing. This happens if and only if $u_1'(x)\frac{1}{u_1'(x)}$ is non-increasing, which happens if and only if log of it is non-increasing. This happens when the derivative is non-positive. □

4.2.9 Example. Consider $u(x) = -\alpha e^{-ax} + \beta$ with $\alpha, a > 0$. Then $u'(x) = a\alpha e^{-ax}$ and $u''(x) = -a^2\alpha e^{-ax}$, so $r(x) = -\frac{u''(x)}{u'(x)} = a$. This utility function is the *constant risk aversion function*.

Note of course that this ordering on utility representations is a partial order.

Consequentialism and invariance to wealth

Consider the following experiment. You have \$2000 in your bank account and you must choose between the lottery L_1 which involves losing \$500 with certainty and L_2 which involves losing \$1000 with probability $\frac{1}{2}$ or losing nothing. Consider also, with \$1000 in your account, the choice between getting \$500 with certainty or getting \$1000 with probability $\frac{1}{2}$ or getting nothing.

Consider a decision maker who has wealth w. Denote decision makers preferences over lotteries in which the prizes are interpreted as wealth changes by $p \succsim_w q \iff w + p \succsim w + q$. When is $\succsim_w$ independent of wealth? This happens only with the constant absolute risk aversion utility function.

We may compare lotteries (distributions) in terms of return is riskiness. If the probability of returning at least x is greater for one lottery than another for all x then the former should be preferred.

4.2.10 Definition. F is preferred to G with respect to *first-order stochastic dominance* (or *FOSD*) if $\int u(x)dF(x) \geq \int u(x)dG(x)$ for every non-decreasing function $u : \mathbb{R} \to \mathbb{R}$.

4.2.11 Proposition. *F FOSD G if and only if $F(x) \leq G(x)$ for all x*

Proof: Define $H(x) = F(x) - G(x)$ and suppose that $H(\bar{x}) > 0$ for some $\bar{x}$. Then

$$\int u(x)dF(x) - \int u(x)dG(x) = \int \mathbf{1}_{(\bar{x},\infty)}(x)dH(x) = -H(\bar{x}) < 0,$$

contradicting FSOD. The converse is by integration by parts (for u differentiable)

$$\int u(x)dH(x) = u(x)H(x)\Big|_0^1 - \int u'(x)H(x)dx \geq 0.$$

□

F FSOD G implies $\int x\,dF(x) \geq \int x\,dG(x)$, but not conversely. We now restrict our attention to distributions with the same mean. (Why?)

4.2.12 Definition. F is preferred to G with respect to *second-order stochastic dominance* (or *SOSD*) if F and G have the same mean and $\int u(x)dF(x) \geq \int u(x)dG(x)$ for every non-decreasing concave function $u : \mathbb{R} \to \mathbb{R}$.

4.2.13 Definition. G is a *mean preserving spread* of F if G can be obtained from F in the following manner. At the first stage choose x randomly with distribution F and at the second stage choose z with distribution H_x, where H_x has zero mean, and G is the reduced lottery for $x + z$.

4.2.14 Example.

$$\text{If } F\text{: } \begin{cases} 2 & \frac{1}{2} \\ 3 & \frac{1}{2} \end{cases} \text{ and } H_2\text{: } \begin{cases} -1 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{cases} \text{ and } H_3\text{: } \begin{cases} -2 & \frac{1}{2} \\ 2 & \frac{1}{2} \end{cases} \text{ then } G\text{: } \begin{cases} 1 & \frac{1}{4} \\ 3 & \frac{1}{2} \\ 5 & \frac{1}{4} \end{cases}$$

If G is a mean preserving spread of F then for any concave $u : \mathbb{R} \to \mathbb{R}$, by Jensen's inequality,

$$\begin{aligned} \int u(x)dG(x) &= \iint u(x+z)dH_x(z)dF(x) \\ &\leq \int u\left(x + \int z dH_x(z)\right)dF(x) = \int u(x)dF(x) \end{aligned}$$

with equality for $u(x) = x$ (hence "mean preserving").

4.2.15 Definition. *F SOSD G* if for all x

1. $\int_{-\infty}^{x} F(z)dx \leq \int_{-\infty}^{x} G(z)dz$; and

2. $\int_{-\infty}^{\infty} F(z)dz = \int_{-\infty}^{\infty} G(z)dz$.

Integration by parts shows that the second condition is equivalent to F and G having the same means. This definition of SOSD is equivalent to the previous definition.

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