The Ward Identities and the Singularity of the Gauss 2F1 Function

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The Full-String Ward Identities are a widely accepted result in string theory. However, actually verifying them proves cumbersome. This paper will show that the difficulty lies, in fact, in the Gauss 2F1 function and that by accepting the result, a statement can be made on the singularity of the 2F1 function.

After running the necessary commutations on the Full-String 3-Vertex, one is left with two sets of coefficients, all of which must be zero. I will write them as such:

$$F_m^{r,s} = F(1)_m^{r,s} + F(2)_m^{r,s} + F(3)_m^{r,s} + F(4)_m^{r,s} = 0$$

Where

$$F(1)_m^{r,s} = \sum_{k=1}^{m-1} (m-k) N_{m-k,0}^{r,s} Cos\left(\frac{k\pi}{2}\right)$$

$$F(2)_m^{r,s} = \frac{1}{2}(1-3m) m N_{m,0}^{r,s}$$

$$F(3)_m^{r,s} = \frac{1}{2} m \tilde{N}_{m,0}^{r,s}$$

$$F(4)_m^{r,s} = \delta_{r,s} Cos\left(\frac{m\pi}{2}\right)$$

And

$$S_{m,k}^{r,s} = S(1)_{m,k}^{r,s} + S(2)_{m,k}^{r,s} + S(3)_{m,k}^{r,s} + S(4)_{m,k}^{r,s} + S(5)_{m,k}^{r,s} + S(6)_{m,k}^{r,s} = 0$$

Where

$$S(1)_{m,k}^{r,s} = \sum_{n=1}^{k-1} m \tilde{N}_{m,n}^{r,s} Cos\left(\frac{(k-n)\pi}{2}\right)$$

$$S(2)_{m,k}^{r,s} = \sum_{n=1}^{m-1} n N_{n,k}^{r,s} Cos\left(\frac{(m-n)\pi}{2}\right)$$

$$S(3)_{m,k}^{r,s} = \delta_{r,s} Cos\left(\frac{(k+m)\pi}{2}\right)$$

$$S(4)_{m,k}^{r,s} = \frac{1}{2} m(1+3k)\tilde{N}_{m,k}^{r,s}$$

$$S(5)_{m,k}^{r,s} = \frac{1}{2} m(1-3m) N_{m,k}^{r,s}$$

$$S(6)_{m,k}^{r,s} = m\tilde{N}_{m,0}^{r,s} Cos\left(\frac{k\pi}{2}\right)$$

[From Hlousek and Jevicki / Interacting String Field Theory]

To fully evaluate these, however, I need to know the values of $N_{m,n}^{r,s}$ and $\tilde{N}_{m,n}^{r,s}$.

By definition,

$$N_{n,m}^{r,s} = \delta_{n\neq m}\left(\frac{M_{n,m}^{r,s}}{\alpha_r n + \alpha_s m} + \frac{M_{n,m}^{r,-s}}{\alpha_r n + \alpha_{-s} m}\right)$$
$$+\delta_{n,m}\left(\frac{\delta_{n\in even}}{n} + \frac{3\delta_{n\in odd}}{n} - \frac{4}{3}a_n b_n + \frac{3}{2}b_n a_{n+1} - \frac{11}{6}b_n a_{n-1} - \frac{1}{12}a_n b_{n-1} + \frac{3}{4}b_{n+1}a_n - \frac{3}{2}b_n\right.$$

Where

$$C_{1,n}^{c} = \sum_{m=1}^{n-1}(-1)^m \frac{c_m c_{n-m-1}}{m+1}$$

$$C_{2,n}^{c} = \sum_{m=1}^{n-1}(-1)^m \frac{c_m c_{n-m+1}}{m-1}$$

[C2 cannot possibly be right.]

[Gross and Jevicki, Interacting String Field Theory – This may not be totally correct. I'm missing a page corresponding to r=s .]

and

$$\tilde{N}^{r,s}_{n,m} = \delta_{n+m\in even}\Bigg(\delta_{r,s}\delta_{m,n}\frac{(-1)^{n+1}}{n} - \frac{2}{3}Cos\left(\frac{2\pi(r-s)}{3}\right)\Bigg)\Bigg(\tilde{N}_{n,m} - \frac{3(-1)^{m}}{4}\Big(a_n S^{b}_{m} + b_n$$

$$+\delta_{n+m\in odd}\,\frac{-2}{3}Sin\left(\frac{2\pi(r-s)}{3}\right)\tilde{N}_{m,n}$$

[Dr. A's]

Which leaves me in need of another pile of definitions.

$$a_n = \frac{1}{n!}\left.\frac{\partial^n\left(\frac{1+x}{1-x}\right)^{\frac{1}{3}}}{\partial x^n}\right|_{x=0}$$

$$b_n = \frac{1}{n!}\left.\frac{\partial^n\left(\frac{1+x}{1-x}\right)^{\frac{2}{3}}}{\partial x^n}\right|_{x=0}$$

$$S^{c}_{n} = \sum_{k=1}^{\infty}\delta_{k+n\in even}\frac{c_k}{n+k}$$

$$\alpha_r = \delta_{r,-3} - \delta_{r,-2} - \delta_{r,-1} + \delta_{r,1} + \delta_{r,2} - \delta_{r,3}$$

$$M_{n,m}^{r,\pm s}=\frac{1}{3}\delta_{r,s}\delta_{n+m\in even}\left(-1\right)^{n}\left(A_{n}B_{m}\pm B_{n}A_{m}\right)$$

$$-\frac{1}{6}\delta_{s=(r+1Mod\,3)+1}\left(-1\right)^{n}\delta_{n+m\in even}\left(A_{n}B_{m}\pm B_{n}A_{m}\right)$$

$$+\frac{\sqrt{3}}{6}\delta_{s=(r+1Mod\,3)+1}\delta_{n+m\in odd}\left(A_{n}B_{m}\mp B_{n}A_{m}\right)$$

$$-\frac{1}{6}\delta_{s=(r+2Mod\,3)+1}\left(-1\right)^{n}\delta_{n+m\in even}\left(A_{n}B_{m}\mp B_{n}A_{m}\right)$$

$$-\frac{\sqrt{3}}{6}\delta_{s=(r+2Mod\,3)+1}\delta_{n+m\in odd}\left(A_{n}B_{m}\pm B_{n}A_{m}\right)$$

[From Gross and Jevicki, Interacting String Field Theory]

$$\tilde{N}_{m,n}=-\delta_{m,n}\frac{\left(-1\right)^{m}}{m}\left(\left(a_{m}\right)\left(b_{m}\right)-\sum_{k=0}^{m}\left(-1\right)^{k}\left(a_{k}\right)^{2}+\frac{1}{2}\left(-1\right)^{m}\left(a_{m}\right)^{2}-1\right)$$

$$+\frac{1}{2}Sin\left(\frac{\left(m+n\right)\pi}{2}\right)\left(\frac{\left(b_{m}\right)\left(a_{n}\right)+\left(b_{n}\right)\left(a_{m}\right)}{m-n}+\frac{\left(b_{m}\right)\left(a_{n}\right)-\left(b_{n}\right)\left(a_{m}\right)}{m+n}\right)$$

$$+\frac{1}{2}Cos\left(\frac{\left(m+n\right)\pi}{2}\right)\left(\frac{\left(b_{m}\right)\left(a_{n}\right)-\left(b_{n}\right)\left(a_{m}\right)}{m-n}+\frac{\left(b_{m}\right)\left(a_{n}\right)+\left(b_{n}\right)\left(a_{m}\right)}{m+n}\right)$$

$$C_{n}=\left(\delta_{n\in even}(-1)^{\frac{n}{2}}+\delta_{n\in odd}(-1)^{\frac{n-1}{2}}\right)c_{n}$$

Which should make clear the values of all of the symbols. See the corresponding Ward2F1 Mathematica notebook to see all of these definitions made in Mathematica. Let me start by working with the F identities. Looking at the notebook, I find I can evaluate F numerically, but S4 gives me trouble on diagonal indices. Let me see if I can find the source of the problems of S4. It turns out that the heart of this problem lies in the $\tilde{N}_{m,m}^{r,s}$. The expression that I need to evaluate is, essentially, the S_{n}^{c} defined above:

$$S_n^a = \sum_{k=1}^{\infty} \frac{1+(-1)^{n+k}}{2(n+k)} \frac{1}{k!} \partial_x^k \left(\frac{1+x}{1-x}\right)^{\frac{1}{3}} \Bigg|_{x=0}$$

$$S_n^a = \sum_{k=1}^{\infty} \frac{1+(-1)^{n+k}}{2(n+k)} \frac{1}{2\pi i} \oint_0 \left(\frac{1+x}{1-x}\right)^{\frac{1}{3}} \frac{1}{x^{k+1}} dx$$

$$S_n^a = \frac{1}{2\pi i} \oint_0 \left(\frac{1+x}{1-x}\right)^{\frac{1}{3}} \sum_{k=1}^{\infty} \frac{1+(-1)^{n+k}}{2(n+k)} \frac{1}{x^{k+1}} dx$$

$$S_n^a = \frac{1}{2\pi i} \oint_0 \left(\frac{1+x}{1-x}\right)^{\frac{1}{3}} \frac{1}{2x^2(n+1)} \left({}_2F_1\left(n+1,1,n+2,\frac{1}{x}\right) + (-1)^{n+1}\, {}_2F_1\left(n+1,1,n+\right.\right.$$

Note that at x = 0, from the definition of the Gauss Hypergeometric 2F1 function,

$${}_2F_1(a,b,c,z) = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)\Gamma(c)z^k}{\Gamma(a)\Gamma(b)\Gamma(c+k)}$$

This function is singular with every integer multiplicity greater than zero. In other words, there's no way that I'm going to be able to evaluate this. What if I could solve for this interesting singularity, assuming my ward identities were true? Well, it turns out that that's very plausible. Mathematica can evaluate most of the expression along the diagonal.

$$S(4)_{m,m}^{r,s} = \frac{1}{2} m(1+3m)\tilde{N}_{m,m}^{r,s}$$

$$\tilde{N}_{m,m}^{r,s} = \left(\delta_{r,s}\frac{(-1)^{m+1}}{m} - \frac{2}{3} Cos\left(\frac{2\pi(r-s)}{3}\right)\right)\left(\tilde{N}_{m,m} - \frac{3(-1)^m}{4}\left(a_m S_m^b + b_m S_m^a\right)\right)$$

$$S(4)^{r,s}_{m,m} = \frac{1}{2} m(1+3m)\left(\delta_{r,s}\frac{(-1)^{m+1}}{m} - \frac{2}{3}Cos\left(\frac{2\pi(r-s)}{3}\right)\right)\left(\tilde{N}_{m,m} - \frac{3(-1)^{m}}{4}(a\right.$$

Now, given by the Ward Identities (assuming they're true) that

$$S^{r,s}_{m,m} = S(1)^{r,s}_{m,m} + S(2)^{r,s}_{m,m} + S(3)^{r,s}_{m,m} + S(4)^{r,s}_{m,m} + S(5)^{r,s}_{m,m} + S(6)^{r,s}_{m,m} = 0$$

And noticing from my Mathematica notebook that I can evaluate all of the S expressions except S4, it is clear that I can solve for the one troublesome term:

$$\left(a_m S^{b}_{m} + b_m S^{a}_{m}\right).$$

Leaving me with a cool identity on the value of these infinite singularities using Physics, not Math!