Matrix Inversion
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Continuing from the last paper, [HSMatrix1.doc], I would like to invert the matrix

$$S1+\frac{1}{2}S2 \qquad \textbf{(A-1)}$$

Using the definitions from [Day1.nb] and the work file [Sinverse.nb], I have found
That this inversion amounts to

$$\left[\left(\left(\sqrt{\frac{2m}{2n+1}}\right)^{-1}\left(M^{1}_{m,n+1}+\frac{1}{2}M^{2}_{m,n+1}\right)\right)_{m,n}\right]^{-1} \qquad \textbf{(A-2)}$$

Note now that the paper <u>N-String Vertices in String Field Theory</u> gives a general solution for matrices of the form

$$\alpha M_1^T - \beta M_2^T \quad \textbf{(A-3)}$$

In order to evaluate the inversion (A-2), I'll certainly first want to verify the general inversion of (A-3) for clues as to how to adapt it to the additional factor in (A-2).

The proposed form of the inverse is

$$\alpha' \frac{v_{2m}^{\frac{1}{p}} u_{2n-1}^{\frac{1}{p}} + u_{2m}^{\frac{1}{p}} v_{2n-1}^{\frac{1}{p}}}{2m-2n+1} + \beta' \frac{v_{2m}^{\frac{1}{p}} u_{2n-1}^{\frac{1}{p}} - u_{2m}^{\frac{1}{p}} v_{2n-1}^{\frac{1}{p}}}{2m+2n-1} \quad \textbf{(A-4)}$$

Where the index gives the coefficient in the Taylor series

$$u^{\frac{1}{p}} = \left(\frac{1+x}{1-x}\right)^{\frac{1}{p}}\Bigg|_{x=0} \qquad \textbf{(A-5)}$$

$$v^{\frac{1}{p}} = \left(\frac{1+x}{1-x}\right)^{1-\frac{1}{p}}\Bigg|_{x=0} \qquad \textbf{(A-6)}$$

Now I will multiply (A-3) by the proposed inverse (A-4) and attempt to fix alpha' and beta'. This gives

$$\sum_{m=0}^{\infty}\left(\alpha M_1^T-\beta M_2^T\right)_{n,m}\left(\alpha'\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}+u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m-2k+1}+\beta'\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}-u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m+2k-1}\right)_{m,k}=\delta_{n,k}\quad\mathbf{(W\text{-}1)}$$

Making a replacement from [Day1.nb], I have

$$\frac{1}{\pi\sqrt{2n-1}}\sum_{m=0}^{\infty}\left(\alpha\left(\sqrt{2m}\,\frac{(-1)^{n+m}}{m-\left(n-\frac{1}{2}\right)}\right)-\beta\left(\sqrt{2m}\,\frac{(-1)^{n+m}}{m+\left(n-\frac{1}{2}\right)}\right)\right)\left(\alpha'\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}+u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m-2k+1}+\beta'\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}-u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m+2k-1}\right)$$

$$=\delta_{n,k}$$

Now multiplying this out, I have

$$\frac{1}{\pi\sqrt{2n-1}}\sum_{m=0}^{\infty}\left(\begin{array}{l}\alpha\left(\sqrt{2m}\,\frac{(-1)^{n+m}}{m-\left(n-\frac{1}{2}\right)}\right)\alpha'\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}+u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m-2k+1}-\beta\left(\sqrt{2m}\,\frac{(-1)^{n+m}}{m+\left(n-\frac{1}{2}\right)}\right)\alpha'\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}+u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m-2k+1}+\\ \alpha\left(\sqrt{2m}\,\frac{(-1)^{n+m}}{m-\left(n-\frac{1}{2}\right)}\right)\beta'\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}-u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m+2k-1}-\beta\left(\sqrt{2m}\,\frac{(-1)^{n+m}}{m+\left(n-\frac{1}{2}\right)}\right)\beta'\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}-u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m+2k-1}\end{array}\right)$$

$$=\delta_{n,k}$$

Life would be extraordinarily easy if I could collapse this sum.

So I need a function with singularities indexed from 0 to infinity, with values

$$\sum_{m=0}^{\infty}\left(\sqrt{2m}\,\frac{(-1)^{n+m}}{m-\left(n-\frac{1}{2}\right)}\right)\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}+u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m-2k+1}$$

Or in other words, f(z) such that at poles indexed from m = 0 to infinity, with order n have the property that

$$\left(\sqrt{2m}\,\frac{(-1)^{n+m}}{m-\left(n-\frac{1}{2}\right)}\right)\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}+u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m-2k+1}=\frac{1}{(n+1)!}\lim_{z\to m}\frac{d^{n-1}}{dz^{n-1}}f(z)(z-c)^n$$

This might be easier if I knew v and u explicitly in terms of m. This, unfortunately, doesn't look very plausible.

Let me consider a different approach here by considering some properties of Taylor Series.

In order to sum all of the Taylor Coefficients of a series, I can simply evaluate the parent function at 1.

In order to sum all of the coefficients multiplied by a constant, I can evaluate that constant times the parent function evaluated at 1.

In order to sum the Taylor Coefficients of a series plus the coefficients of another, I can evaluate the sum of the parent functions evaluated at 1.

In order to multiply each coefficient by 1/n+1, where indexing starts at zero, I should take the integral of the parent function.

In order to multiply each coefficient by n, where indexing starts at zero, I should take the derivative of the parent function.

Now suppose that I want to sum all of the Taylor Coefficients of a series, where it has been element-wise multiplied such that the nth element (starting with zero) is multiplied by x^n , I instead evaluate the parent function at x.

Now let me reconsider, as earlier, the identity

$$\frac{1}{\pi\sqrt{2n-1}}\sum_{m=0}^{\infty}\left(\begin{array}{l} \alpha\left(\sqrt{2m}\,\frac{(-1)^{n+m}}{m-\left(n-\frac{1}{2}\right)}\right)\alpha'\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}+u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m-2k+1}-\beta\left(\sqrt{2m}\,\frac{(-1)^{n+m}}{m+\left(n-\frac{1}{2}\right)}\right)\alpha'\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}+u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m-2k+1}+ \\ \alpha\left(\sqrt{2m}\,\frac{(-1)^{n+m}}{m-\left(n-\frac{1}{2}\right)}\right)\beta'\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}-u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m+2k-1}-\beta\left(\sqrt{2m}\,\frac{(-1)^{n+m}}{m+\left(n-\frac{1}{2}\right)}\right)\beta'\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}-u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m+2k-1} \end{array}\right)$$

$$=\delta_{n,k}$$

(W-3)

Of course, if this is indeed the inverse, then it must work for any arbitrary choice of n and k, and I should hopefully be able to fix alpha' and beta'. Then, let me select n=k=1.

$$\frac{1}{\pi}\sum_{m=0}^{\infty}\left(\begin{array}{l} \alpha\left(\sqrt{2m}\,\frac{(-1)^{1+m}}{m-\left(\frac{1}{2}\right)}\right)\alpha'\frac{v_{2m}^{\frac{1}{p}}u_{1}^{\frac{1}{p}}+u_{2m}^{\frac{1}{p}}v_{1}^{\frac{1}{p}}}{2m-3}-\beta\left(\sqrt{2m}\,\frac{(-1)^{n+m}}{m+\left(\frac{1}{2}\right)}\right)\alpha'\frac{v_{2m}^{\frac{1}{p}}u_{1}^{\frac{1}{p}}+u_{2m}^{\frac{1}{p}}v_{1}^{\frac{1}{p}}}{2m-3}+ \\ \alpha\left(\sqrt{2m}\,\frac{(-1)^{1+m}}{m-\left(\frac{1}{2}\right)}\right)\beta'\frac{v_{2m}^{\frac{1}{p}}u_{1}^{\frac{1}{p}}-u_{2m}^{\frac{1}{p}}v_{1}^{\frac{1}{p}}}{2m+1}-\beta\left(\sqrt{2m}\,\frac{(-1)^{n+m}}{m+\left(\frac{1}{2}\right)}\right)\beta'\frac{v_{2m}^{\frac{1}{p}}u_{1}^{\frac{1}{p}}-u_{2m}^{\frac{1}{p}}v_{1}^{\frac{1}{p}}}{2m+1} \end{array}\right)$$

$$=1$$

$$
\frac{-1}{2\pi}\sum_{m=0}^{\infty}\left(\begin{array}{l}
\alpha\left(\sqrt{2m}\,\frac{(-1)^m}{2m-1}\right)\alpha'\frac{v_{2m}^{\frac{1}{p}}u_1^{\frac{1}{p}}+u_{2m}^{\frac{1}{p}}v_1^{\frac{1}{p}}}{2m-3}-\beta\left(\sqrt{2m}\,\frac{(-1)^m}{2m+1}\right)\alpha'\frac{v_{2m}^{\frac{1}{p}}u_1^{\frac{1}{p}}+u_{2m}^{\frac{1}{p}}v_1^{\frac{1}{p}}}{2m-3}+\\
\alpha\left(\sqrt{2m}\,\frac{(-1)^m}{2m-1}\right)\beta'\frac{v_{2m}^{\frac{1}{p}}u_1^{\frac{1}{p}}-u_{2m}^{\frac{1}{p}}v_1^{\frac{1}{p}}}{2m+1}-\beta\left(\sqrt{2m}\,\frac{(-1)^m}{2m+1}\right)\beta'\frac{v_{2m}^{\frac{1}{p}}u_1^{\frac{1}{p}}-u_{2m}^{\frac{1}{p}}v_1^{\frac{1}{p}}}{2m+1}
\end{array}\right)
$$

$$
=1
$$

Now in order to collapse the sum in terms of n, I need to be able to make sense of these series: for instance,

$$
C\sum_{m=0}^{\infty}\left(\sqrt{2m}\,\frac{(-1)^m}{(2m-1)(2m-3)}\right)v_{2m}^{\frac{1}{p}}
$$

By evaluating v’s parent function at -1, I could take care of the -1 factor. By strategically taking integrals, and multiplying the function by the parameter overall, I could take care of the denominator. The square-root factor, however, is an issue.

Note that in the actual inversion I want, there is no such factor!

With that in mind, let me return to (A-2).

$$
\left[\left(\left(\sqrt{\frac{2m}{2n+1}}\right)^{-1}\left(M_{m,n+1}^{1}+\frac{1}{2}M_{m,n+1}^{2}\right)\right)_{m,n}\right]^{-1}
$$

Let me generalize this to

$$
\left[\left(\left(\sqrt{\frac{2m}{2n+1}}\right)^{-1}\left(\frac{1}{\pi}\sqrt{\frac{2m}{2n+1}}\left(\alpha\left(\frac{(-1)^{n+1+m}}{m-\left(n+\frac{1}{2}\right)}\right)+\beta\left(\frac{(-1)^{n+1+m}}{m+\left(n+\frac{1}{2}\right)}\right)\right)\right)\right)_{m,n}\right]^{-1}
$$

note that the above constrains that the indexing of m must start at 1.

and absorbing a lot of constants into alpha and beta,

$$
\left[\left(\alpha\left(\frac{(-1)^{n+m}}{2m-(2n+1)}\right)+\beta\left(\frac{(-1)^{n+m}}{2m+(2n+1)}\right)\right)_{m,n}\right]^{-1} \qquad \textbf{(W-4)}
$$

As an inverse, I will propose

$$\sum_{m=1}^{\infty}\left(\alpha\left(\frac{(-1)^{n+m}}{2m-(2n+1)}\right)+\beta\left(\frac{(-1)^{n+m}}{2m+(2n+1)}\right)\right)\left(\alpha'\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}+u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m-2k+1}+\beta'\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}-u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m+2k-1}\right)=\delta_{n,k}$$

$$\sum_{m=1}^{\infty}\left(\begin{array}{l}\alpha\left(\frac{(-1)^{n+m}}{2m-(2n+1)}\right)\alpha'\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}+u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m-2k+1}-\beta\left(\frac{(-1)^{n+m}}{2m+(2n+1)}\right)\alpha'\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}+u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m-2k+1}+\\ \alpha\left(\frac{(-1)^{n+m}}{2m-(2n+1)}\right)\beta'\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}-u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m+2k-1}-\beta\left(\frac{(-1)^{n+m}}{2m+(2n+1)}\right)\beta'\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}-u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m+2k-1}\end{array}\right)=\delta_{n,k}$$

(W-5)

Now let me consider an arbitrary sum, since these infinite sums separate.

$$\sum_{m=1}^{\infty}\left(\alpha\left(\frac{(-1)^{n+m}}{2m-(2n+1)}\right)\alpha'\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}}{2m-2k+1}\right)$$

$$\alpha\alpha'(-1)^{n}u_{2k-1}^{\frac{1}{p}}\sum_{m=1}^{\infty}\left(\left(\frac{(-1)^{m}}{2m-(2n+1)}\right)\frac{v_{2m}^{\frac{1}{p}}}{2m-2k+1}\right) \qquad \textbf{(W-6)}$$

Suppose that I want to use the Taylor Series rules I came up with earlier to try to determine this sum. The constants outside certainly constitute a constant factor on my hypothetical parent function. In fact, it should be clear that

$$\alpha\alpha'(-1)^{n}u_{2k-1}^{\frac{1}{p}}\sum_{m=1}^{\infty}\left((-1)^{m}v_{2m}^{\frac{1}{p}}\right)=\alpha\alpha'(-1)^{n}u_{2k-1}^{\frac{1}{p}}\left(\operatorname{Re}\left(v^{\frac{1}{p}}(i)\right)-v^{\frac{1}{p}}(0)\right) \qquad \textbf{(W-7)}$$

Now, perhaps by multiplying v by its parameter and strategically taking integrals, I could add the desired m-related parameters from **(W-6)**

It should be clear from the form of the fractions above that it is not possible to divide by zero in such an integral. The issue at hand is that I need to bring two separate factors: one equal to 2m-(2n+1), and one equal to 2m-2k+1.

I should be able to obtain such factors by multiplying the desired parent function by x^{a-1}, where a is the number added to 2m in the denominator, integrating with respect to the parameter, then dividing the result by x^{a}, where x again is the parameter. I can repeat this to get additional parameters. This should leave the parameter chosen in (W-7) unaffected.

$$\alpha\alpha'(-1)^n u_{2k-1}^{\frac{1}{p}} \sum_{m=1}^{\infty}\left(\left(\frac{(-1)^m}{2m-(2n+1)}\right)\frac{v_{2m}^{\frac{1}{p}}}{2m-2k+1}\right)=$$

$$\alpha\alpha'(-1)^n u_{2k-1}^{\frac{1}{p}}\left(\operatorname{Re}\left(x^{2k-1}\int x^{-2k+2n+1}\int\left(x^{-2n-2}v^{\frac{1}{p}}(x)-x^{-1}v_{2n+1}^{\frac{1}{p}}\right)dxdx\bigg|_{x=i}\right)-\frac{1}{-(2n+1)}\frac{1}{-2k+1}\right)$$

(W-8)

Where u and v, it turns out, are arbitrary choices of a function. I'm going to try to verify this in [HSMatrix2.nb].

Turns out, Mathematica can't evaluate both integrals. I think the above expression is right, but I could definitely stand to numerically verify my procedure. But there IS a problem! The non-2m terms are not removed until evaluation, but the integrals can allow indeterminate forms in intermediate steps! This needs to be taken care of.

Ben

What follows is not in use.

Now propose for a moment that n does not equal k.

$$\frac{1}{\pi\sqrt{2n-1}}\sum_{m=0}^{\infty}\left(\begin{array}{l}\alpha\left(\sqrt{2m}\,\frac{(-1)^{n+m}}{m-\left(n-\frac{1}{2}\right)}\right)\alpha'\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}+u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m-2k+1}-\beta\left(\sqrt{2m}\,\frac{(-1)^{n+m}}{m+\left(n-\frac{1}{2}\right)}\right)\alpha'\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}+u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m-2k+1}+\\ \alpha\left(\sqrt{2m}\,\frac{(-1)^{n+m}}{m-\left(n-\frac{1}{2}\right)}\right)\beta'\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}-u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m+2k-1}-\beta\left(\sqrt{2m}\,\frac{(-1)^{n+m}}{m+\left(n-\frac{1}{2}\right)}\right)\beta'\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}-u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m+2k-1}\end{array}\right)$$

$$=0$$

I am going to propose that this separates such that

$$\frac{1}{\pi\sqrt{2n-1}}\sum_{m=0}^{\infty}\left(\alpha\left(\sqrt{2m}\,\frac{(-1)^{n+m}}{m-\left(n-\frac{1}{2}\right)}\right)\alpha'\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}+u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m-2k+1}-\beta\left(\sqrt{2m}\,\frac{(-1)^{n+m}}{m+\left(n-\frac{1}{2}\right)}\right)\beta'\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}-u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m+2k-1}\right)$$

$$=0$$

(W-2)
and

$$\frac{1}{\pi\sqrt{2n-1}}\sum_{m=0}^{\infty}\left(\alpha\left(\sqrt{2m}\,\frac{(-1)^{n+m}}{m-\left(n-\frac{1}{2}\right)}\right)\beta'\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}-u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m+2k-1}-\beta\left(\sqrt{2m}\,\frac{(-1)^{n+m}}{m+\left(n-\frac{1}{2}\right)}\right)\alpha'\frac{v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}+u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}}{2m-2k+1}\right)$$

$$=0$$

With n not being equal to k.

Now let me manipulate W-2 for a moment.

$$\frac{1}{\pi\sqrt{2n-1}}\sum_{m=0}^{\infty}\left(\begin{array}{l}\alpha\alpha'\left(\sqrt{2m}\,\dfrac{(-1)^{n+m}}{2m^2-m(2n-1)-m(2k+1)+\dfrac{1}{2}(2n-1)(2k+1)}\right)\left(v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}+u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}\right)\\ -\beta\beta'\left(\sqrt{2m}\,\dfrac{(-1)^{n+m}}{2m^2+m(2n-1)+m(2k-1)+\dfrac{1}{2}(2n-1)(2k-1)}\right)\left(v_{2m}^{\frac{1}{p}}u_{2k-1}^{\frac{1}{p}}-u_{2m}^{\frac{1}{p}}v_{2k-1}^{\frac{1}{p}}\right)\end{array}\right)$$

$$=0$$

Let me try to evaluate this in terms of something I can evaluate.

Trying to find a simpler version of this with Mathematica yields no clues, from HSMatrix2.nb. However, I do know that this must hold true for any arbitrary choice for n and k.