Half String Ward Identity

Full String 3-Vertex Creation Only

I have:

$$|0_L\rangle\,|0_R\rangle = e^{\frac{-1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\alpha_n^{+}\varphi_{n,m}\alpha_m^{+}}\;|0\rangle$$

And for my 3-Vertex, I have

$$|V_3\rangle = e^{\frac{-1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{r=0}^{1}\sum_{s=0}^{1}b_{n,i,r}^{+}H_{n,m}^{i,j,r,s}b_{m,j,s}^{+}}\prod_{i=1}^{3}\prod_{r=\{L,R\}}|0_{i,r}\rangle$$

Where

$$H_{n,m}^{i,j,s,t} = -\delta_{m,n}\left(\delta_{j,((i+1)Mod\,3)+1}\delta_{s,L}\delta_{t,R}+\delta_{i,((j+1)Mod\,3)+1}\delta_{s,R}\delta_{t,L}\right)$$

So that (recalling the psi and A matrices from the earlier part of my work)

$$|V_3\rangle\prod_{i=1}^{3}\prod_{r=\{L,R\}}|0_{i,r}\rangle = e^{\frac{-1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{r=0}^{1}\sum_{s=0}^{1}b_{n,i,r}^{+}H_{n,m}^{i,j,r,s}b_{m,j,s}^{+}}\;e^{\frac{-1}{2}\sum_{n=0}^{\infty}\sum_{i=1}^{3}\sum_{m=0}^{\infty}\alpha_{i,n}^{+}\varphi_{n,m}\alpha_{i,m}^{+}}\prod_{i=1}^{3}|0_i\rangle$$

I recall that

$$b^{+}_{r,n(odd)} = \frac{1}{2}\frac{1-(-1)^n}{2\sqrt{n}}\left(\sum_{k=0}^{\infty}((-1)^{r+1}\delta_{n,k} + \frac{1+(-1)^k}{2}A_{n,k})\alpha^{+}_{k} + \sum_{k=0}^{\infty}((-1)^{r+1}\delta_{n,-k} + \frac{1+(-1)^k}{2}A_{n,-k})\alpha_{k}\right)$$

$$b^{+}_{r,n(even)} = 0$$

$$\left[\alpha_{i,n}, \alpha^{+}_{j,m}\right] = n\delta_{m,n}\delta_{i,j} + \delta_{m,0}\delta_{n,0}\delta_{i,j}$$

$$\left[b^{+}_{n,i,r}, \alpha^{+}_{j,m}\right] = \delta_{i,j}\delta_{n(odd)}\delta_{m(even)}\frac{m}{2\sqrt{n}}A_{n,-m}$$

Recalling the Gaussian identity,

$$e^{wA^{-1}w} = \prod_{i=1}^{N}\pi^{\frac{N}{2}}\det[A]^{\frac{1}{2}}\int e^{-xAx}e^{\sqrt{-1}wx}\partial x_i$$

Expanding my gaussians,

$$e^{\frac{-1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{r=0}^{1}\sum_{s=0}^{1} b^{+}_{n,i,r} H^{i,j,r,s}_{n,m} b^{+}_{m,j,s}} e^{\frac{-1}{2}\sum_{n=0}^{\infty}\sum_{i=1}^{3}\sum_{m=0}^{\infty} \alpha^{+}_{i,n}\varphi_{n,m}\alpha^{+}_{i,m}}$$

I should have for the b gaussian

NB for the equations below, I need to assume that the determinant of the inverse of the zero matrix is 1. This is a result of my "unrolling" – I need these coefficients to multiply out to the proper product, but if H happens to be the zero matrix, I really couldn't have expanded as a gaussian in the first place – so call determinant of the inverse of the zero matrix one for now.

$$\prod_{u=1}^{N}\pi^{\frac{N}{2}}\left(\prod_{i}\prod_{j}\prod_{r}\prod_{s}\det\left(\left(H^{i,j,r,s}\right)^{-1}\right)\right)e^{-\frac{1}{2}\sum_{r}\sum_{s}\sum_{i}\sum_{j}\sum_{m}\sum_{n} x_{n,i,r}\,(H^{i,j,r,s})^{-1}_{n,m}\,x_{m,j,s}}\; e^{\sqrt{-1}\sum_{n}\sum_{i}\sum_{r} b^{+}_{n,i,r}x_{n,i,r}}\, dx$$

And for the alpha gaussian,

$$\prod_{v=1}^{M}\pi^{\frac{M}{2}}\left(\prod_{i}\det\left((\varphi)^{-1}\right)\right)e^{-\frac{1}{2}\sum_{i}\sum_{m}\sum_{n} y_{i,n}\,(\varphi)^{-1}_{n,m}\,y_{i,m}}\; e^{\sqrt{-1}\sum_{i}\sum_{n}\alpha^{+}_{i,n}y_{i,n}}\, dy$$

As I work my way through this commutation, I will hide the irrelevant parts.

$$e^{\sqrt{-1}\sum_{n}\sum_{i}\sum_{r} b^{+}_{n,i,r}x_{n,i,r}}\; e^{\sqrt{-1}\sum_{j}\sum_{m}\alpha^{+}_{j,m}y_{j,m}}$$

$$e^{\sqrt{-1}\sum_j\sum_m \alpha^+_{j,m} y_{j,m}} e^{\sqrt{-1}\sum_n\sum_i\sum_r b^+_{n,i,r} x_{n,i,r}} e^{-\sum_n\sum_m\sum_i\sum_j\sum_r x_{n,i,r} y_{j,m}\left[b^+_{n,i,r},\alpha^+_{j,m}\right]}$$

I'll prepare to recombine my alpha gaussian first, since I'd like to annihilate part of my b expression.

$$e^{\sqrt{-1}\sum_j\sum_m y_{j,m}\left(\alpha^+_{j,m}+\sqrt{-1}\sum_n\sum_i\sum_r x_{n,i,r}\left[b^+_{n,i,r},\alpha^+_{j,m}\right]\right)} e^{\sqrt{-1}\sum_n\sum_i\sum_r b^+_{n,i,r} x_{n,i,r}}$$

Now I'd like to take the opportunity to kill off my alpha-annihilators in my b expression.

I recall that

$$e^{A+B} = e^{B+A} = e^B e^A e^{\frac{-1}{2}[B,A]}$$

So I'll split my B into two parts:

$$b^{++}_{n,i,r} = \frac{1}{2}\frac{1-(-1)^n}{2\sqrt{n}}\left(\sum_{k=0}^{\infty}((-1)^{r+1}\delta_{n,k} + \delta_{(k\in even)}A_{n,k})\alpha^{+}_{i,k}\right)(n \in odd)$$

$$b^{++}_{n,i,r} = 0\,(n \in even)$$

$$b^{+-}_{n,i,r(odd)} = \frac{1}{2}\frac{1-(-1)^n}{2\sqrt{n}}\left(\sum_{k=0}^{\infty}((-1)^{r+1}\delta_{n,-k} + \delta_{(k\in even)}A_{n,-k})\alpha_{i,k}\right)(n \in odd)$$

$$b^{+-}_{n,i,r(odd)} = 0(n \in even)$$

Now evaluating my commutator, I have

$$\left[b^{++}_{n,i,r}, b^{+-}_{m,j,s}\right]$$

$$\delta_{n\in odd}\delta_{m\in odd}\frac{1}{4\sqrt{n}\sqrt{m}}\left[\left(\sum_{k=0}^{\infty}((-1)^{r+1}\delta_{n,k} + \delta_{(k\in even)}A_{n,k})\alpha^{+}_{i,k}\right), \left(\sum_{k=0}^{\infty}((-1)^{s+1}\delta_{m,-k} + \delta_{(k\in even)}A_{m,-k})\alpha_{j,k}\right)\right]$$

$$\delta_{n\in odd}\delta_{m\in odd}\frac{-1}{4\sqrt{n}\sqrt{m}}\sum_{k=0}^{\infty}\left((-1)^{r+1}\delta_{n,k} + \delta_{(k\in even)}A_{n,k}\right)\left((-1)^{s+1}\delta_{m,-k} + \delta_{(k\in even)}A_{m,-k}\right)k\delta_{i,j}$$

Constraining that m and n are greater than zero, I have for my commutator

$$\left[b^{++}_{n,i,r}, b^{+-}_{m,j,s}\right] = \delta_{n\in odd}\delta_{m\in odd}\frac{-1}{4\sqrt{n}\sqrt{m}}\sum_{k=0}^{\infty}\left((-1)^{r+1}\delta_{n,k} + \delta_{(k\in even)}A_{n,k}\right)\left(\delta_{(k\in even)}A_{m,-k}\right)k\delta_{i,j}$$

Now looking at the relevant portions of my overall expression, I will annihilate part of my b expression.

$$e^{\sqrt{-1}\sum_n\sum_i\sum_r b^{+}_{n,i,r}x_{n,i,r}}\,|\,0\rangle$$

$$e^{\sqrt{-1}\sum_n\sum_i\sum_r \left(b^{++}_{n,i,r}+b^{+-}_{n,i,r}\right)x_{n,i,r}}\,|\,0\rangle$$

$$e^{\sqrt{-1}\sum_n\sum_i\sum_r b^{++}_{n,i,r}x_{n,i,r}}\,e^{\sqrt{-1}\sum_n\sum_i\sum_r b^{+-}_{n,i,r}x_{n,i,r}}\,e^{\frac{1}{2}\sum_n\sum_m\sum_i\sum_j\sum_r\sum_s x_{n,i,r}x_{m,j,s}\left[b^{++}_{n,i,r},b^{+-}_{m,j,s}\right]}\,|\,0\rangle$$

Now I can allow my b^{+-} to annihilate the vacuum, leaving me with just

$$e^{\sqrt{-1}\sum_n\sum_i\sum_r b^{++}_{n,i,r}x_{n,i,r}}\,e^{\frac{1}{2}\sum_n\sum_m\sum_i\sum_j\sum_r\sum_s x_{n,i,r}x_{m,j,s}\left[b^{++}_{n,i,r},b^{+-}_{m,j,s}\right]}\,|\,0\rangle$$

Overall, then, the terms I've modified so far read:

$$e^{\sqrt{-1}\sum_j\sum_m y_{j,m}\left(\alpha^{+}_{j,m}+\sqrt{-1}\sum_n\sum_i\sum_r x_{n,i,r}\left[b^{+}_{n,i,r},\alpha^{+}_{j,m}\right]\right)}e^{\sqrt{-1}\sum_n\sum_i\sum_r b^{++}_{n,i,r}x_{n,i,r}}\,e^{\frac{1}{2}\sum_n\sum_m\sum_i\sum_j\sum_r\sum_s x_{n,i,r}x_{m,j,s}\left[b^{++}_{n,i,r},b^{+-}_{m,j,s}\right]}\,|\,0\rangle$$

Now I'd like to start recombining gaussians. First I'll combine the alpha gaussian.

$$e^{\frac{-1}{2}\sum_j\sum_n\sum_m\left(\alpha_{j,n}^{+}+\sqrt{-1}\sum_{n'}\sum_{i'}\sum_{r'}x_{n',i',r'}\left[b_{n',i',r'}^{+},\alpha_{j,n}^{+}\right]\right)\varphi_{n,m}\left(\alpha_{j,m}^{+}+\sqrt{-1}\sum_{n''}\sum_{i''}\sum_{r''}x_{n'',i'',r''}\left[b_{n'',i'',r''}^{+},\alpha_{j,m}^{+}\right]\right)}$$

$$e^{\sqrt{-1}\sum_n\sum_i\sum_r b_{n,i,r}^{++}x_{n,i,r}}\;e^{\frac{1}{2}\sum_n\sum_m\sum_i\sum_j\sum_r\sum_s x_{n,i,r}x_{m,j,s}\left[b_{n,i,r}^{++},b_{m,j,s}^{+-}\right]}|0\rangle$$

Now expanding it, I have (assuming my psi matrix is symmetric),

$$e^{\frac{-1}{2}\sum_j\sum_n\sum_m\left(\alpha_{j,n}^{+}\right)\varphi_{n,m}\left(\alpha_{j,m}^{+}\right)}\;e^{-\sqrt{-1}\sum_j\sum_n\sum_m\left(\alpha_{j,n}^{+}\right)\varphi_{n,m}\left(\sum_{n''}\sum_{i''}\sum_{r''}x_{n'',i'',r''}\left[b_{n'',i'',r''}^{+},\alpha_{j,m}^{+}\right]\right)}$$

$$e^{\frac{1}{2}\sum_j\sum_n\sum_m\left(\sum_{n'}\sum_{i'}\sum_{r'}x_{n',i',r'}\left[b_{n',i',r'}^{+},\alpha_{j,n}^{+}\right]\right)\varphi_{n,m}\left(\sum_{n''}\sum_{i''}\sum_{r''}x_{n'',i'',r''}\left[b_{n'',i'',r''}^{+},\alpha_{j,m}^{+}\right]\right)}\;e^{\sqrt{-1}\sum_n\sum_i\sum_r b_{n,i,r}^{++}x_{n,i,r}}\;e^{\frac{1}{2}\sum_n\sum_m\sum_i\sum_j\sum_r\sum_s x_{n,i,r}x_{m,j,s}\left[b_{n,i,r}^{++},b_{m,j,s}^{+-}\right]}|0\rangle$$

Rewriting these to be more convenient, and including the other relevant x term,

$$
e^{\frac{-1}{2}\sum_{j}\sum_{n}\sum_{m}\left(\alpha^{+}_{j,n}\right)\varphi_{n,m}\left(\alpha^{+}_{j,m}\right)}\, e^{-\sqrt{-1}\sum_{n}\sum_{i}\sum_{r}x_{n,i,r}\sum_{j}\sum_{m}\sum_{k}\alpha^{+}_{j,k}\varphi_{k,m}\left[b^{+}_{n,i,r},\alpha^{+}_{j,m}\right]}\, e^{-\frac{1}{2}\sum_{r}\sum_{s}\sum_{i}\sum_{j}\sum_{m}\sum_{n}x_{n,i,r}(H^{i,j,r,s})^{-1}_{n,m}x_{m,j,s}}
$$

$$
e^{\frac{1}{2}\sum_{n}\sum_{m}\sum_{i}\sum_{j}\sum_{r}\sum_{s}x_{n,i,r}\left(\sum_{i'}\sum_{n'}\sum_{m'}\left[b^{+}_{n,i,r},\alpha^{+}_{i',n'}\right]\varphi_{n',m'}\left[b^{+}_{m,j,s},\alpha^{+}_{i',m'}\right]\right)x_{m,j,s}}\, e^{\sqrt{-1}\sum_{n}\sum_{i}\sum_{r}b^{++}_{n,i,r}x_{n,i,r}}\, e^{\frac{1}{2}\sum_{n}\sum_{m}\sum_{i}\sum_{j}\sum_{r}\sum_{s}x_{n,i,r}\left[b^{++}_{n,i,r},b^{+-}_{m,j,s}\right]x_{m,j,s}}\,|0\rangle
$$

And combining, I have

$$
e^{\frac{-1}{2}\sum_{j}\sum_{n}\sum_{m}\left(\alpha^{+}_{j,n}\right)\varphi_{n,m}\left(\alpha^{+}_{j,m}\right)}\, e^{-\sqrt{-1}\sum_{n}\sum_{i}\sum_{r}x_{n,i,r}\left(\sum_{j}\sum_{m}\sum_{k}\alpha^{+}_{j,k}\varphi_{k,m}\left[b^{+}_{n,i,r},\alpha^{+}_{j,m}\right]+b^{++}_{n,i,r}\right)}
$$

$$
e^{-\frac{1}{2}\sum_{r}\sum_{s}\sum_{i}\sum_{j}\sum_{m}\sum_{n}x_{n,i,r}\left((H^{i,j,r,s})^{-1}_{n,m}-\left(\sum_{i'}\sum_{n'}\sum_{m'}\left[b^{+}_{n,i,r},\alpha^{+}_{i',n'}\right]\varphi_{n',m'}\left[b^{+}_{m,j,s},\alpha^{+}_{i',m'}\right]\right)+\left[b^{++}_{n,i,r},b^{+-}_{m,j,s}\right]\right)x_{m,j,s}}\,|0\rangle
$$

Now I have to recombine my b-gaussian. The problem is, this one will use a different central matrix, so I have to take this into account when recombining the gaussian.

$$\frac{\sqrt{\left(\prod_i\prod_j\prod_r\prod_s \det\left(\left(H^{i,j,r,s}\right)^{-1}\right)\right)}}{\sqrt{\left(\prod_i\prod_j\prod_r\prod_s \det\left(\left(\left((H^{i,j,r,s})^{-1}_{n,m}-\left(\sum_{i'}\sum_{n'}\sum_{m'}\left[b^{+}_{n,i,r},\alpha^{+}_{i',n'}\right]\varphi_{n',m'}\left[b^{+}_{m,j,s},\alpha^{+}_{i',m'}\right]\right)+\left[b^{++}_{n,i,r},b^{+-}_{m,j,s}\right]\right)^{i,j,r,s}\right)^{-1}\right)\right)}}$$

$$e^{\frac{-1}{2}\sum_j\sum_n\sum_m\left(\alpha^{+}_{j,n}\right)\varphi_{n,m}\left(\alpha^{+}_{j,m}\right)}$$

$$e^{-\frac{1}{2}\sum_r\sum_s\sum_i\sum_j\sum_m\sum_n\left(\sum_{j'}\sum_{m'}\sum_{k'}\alpha^{+}_{j',k'}\varphi_{k',m'}\left[b^{+}_{n,i,r},\alpha^{+}_{j',m'}\right]+b^{++}_{n,i,r}\right)\left((H^{i,j,r,s})^{-1}_{n,m}-\left(\sum_{i'}\sum_{n'}\sum_{m'}\left[b^{+}_{n,i,r},\alpha^{+}_{i',n'}\right]\varphi_{n',m'}\left[b^{+}_{m,j,s},\alpha^{+}_{i',m'}\right]\right)+\left[b^{++}_{n,i,r},b^{+-}_{m,j,s}\right]\right)^{-1}\left(\sum_{j''}\sum_{m''}\sum_{k''}\alpha^{+}_{j'',k''}\varphi_{k'',m''}\left[b^{+}_{m,j,s},\alpha^{+}_{j'',m''}\right]+b^{++}_{m,j,s}\right)}$$

Recall that in these products, I ignore zero determinants and treat them as if they were one.

Finally, I'd like to take this creation-only b gaussian and combine it into a full-string vertex gaussian in terms of only the alpha creation operators.

$$e^{-\frac{1}{2}\sum_r\sum_s\sum_i\sum_j\sum_m\sum_n\left(\sum_{j'}\sum_{m'}\sum_{k'}\alpha^{+}_{j',k'}\varphi_{k',m'}\left[b^{+}_{n,i,r},\alpha^{+}_{j',m'}\right]+b^{++}_{n,i,r}\right)\left((H^{i,j,r,s})^{-1}_{n,m}-\left(\sum_{i'}\sum_{n'}\sum_{m'}\left[b^{+}_{n,i,r},\alpha^{+}_{i',n'}\right]\varphi_{n',m'}\left[b^{+}_{m,j,s},\alpha^{+}_{i',m'}\right]\right)+\left[b^{++}_{n,i,r},b^{+-}_{m,j,s}\right]\right)^{-1}\left(\sum_{j''}\sum_{m''}\sum_{k''}\alpha^{+}_{j'',k''}\varphi_{k'',m''}\left[b^{+}_{m,j,s},\alpha^{+}_{j'',m''}\right]+b^{++}_{m,j,s}\right)}$$

For the sake of convenience, I'll call

$$\psi_{n,m}^{i,j,r,s}=\left((H^{i,j,r,s})_{n,m}^{-1}-\left(\sum_{i'}\sum_{n'}\sum_{m'}\left[b_{n,i,r}^{+},\alpha_{i',n'}^{+}\right]\varphi_{n',m'}\left[b_{m,j,s}^{+},\alpha_{i',m'}^{+}\right]\right)+\left[b_{n,i,r}^{++},b_{m,j,s}^{+-}\right]\right)^{-1}$$

$$e^{-\frac{1}{2}\sum_{r}\sum_{s}\sum_{i}\sum_{j}\sum_{m}\sum_{n}\left(\sum_{j'}\sum_{m'}\sum_{k'}\alpha_{j',k'}^{+}\varphi_{k',m'}\left[b_{n,i,r}^{+},\alpha_{j',m'}^{+}\right]+b_{n,i,r}^{++}\right)\psi_{n,m}^{i,j,r,s}\left(\sum_{j''}\sum_{m''}\sum_{k''}\alpha_{j'',k''}^{+}\varphi_{k'',m''}\left[b_{m,j,s}^{+},\alpha_{j'',m''}^{+}\right]+b_{m,j,s}^{++}\right)}$$

It's time to stop hiding behind the b layer of abstraction: I need things in terms of alpha.

$$\left(\sum_{j'}\sum_{m'}\sum_{k'}\alpha_{j',k'}^{+}\varphi_{k',m'}\left[b_{n,i,r}^{+},\alpha_{j',m'}^{+}\right]+b_{n,i,r}^{++}\right)$$

$$\left(\sum_{j'}\sum_{m'}\sum_{k'}\alpha_{j',k'}^{+}\varphi_{k',m'}\left[b_{n,i,r}^{+},\alpha_{j',m'}^{+}\right]+\delta_{(n\in odd)}\frac{1}{2}\frac{1}{\sqrt{n}}\left(\sum_{k=0}^{\infty}((-1)^{r+1}\delta_{n,k}+\delta_{(k\in even)}A_{n,k})\alpha_{i,k}^{+}\right)\right)$$

The j' sum is really irrelevant: The commutator has a delta i-j. So, these terms can really be written as:

$$\sum_{k}\left(\sum_{m'}\varphi_{k,m'}\left[b_{n,i,r}^{+},\alpha_{i,m'}^{+}\right]+\delta_{(n\in odd)}\frac{1}{2}\frac{1}{\sqrt{n}}\left((-1)^{r+1}\delta_{n,k}+\delta_{(k\in even)}A_{n,k}\right)\right)\alpha_{i,k}^{+}$$

However, now I can compact the central matrix even more. Let me create a new one:

$$\xi^{i,j}_{n,m} =$$

$$\sum_r \sum_s \sum_{m"} \sum_{n"} \psi^{i,j,r,s}_{n",m"} \left(\sum_k \varphi_{n,k} \left[b^+_{n",i,r}, \alpha^+_{i,k} \right] + \delta_{(n" \in odd)} \frac{1}{2} \frac{1}{\sqrt{n"}} \left((-1)^{r+1} \delta_{n",n} + \delta_{(k \in even)} A_{n",n} \right) \right) \left(\sum_{k'} \varphi_{m,k'} \left[b^+_{m",i,r}, \alpha^+_{i,k'} \right] + \delta_{(m" \in odd)} \frac{1}{2} \frac{1}{\sqrt{m"}} \left((-1)^{s+1} \delta_{m",m} + \delta_{(k \in even)} A_{m",m} \right) \right)$$

The parenthetical expressions over three lines are multiplied: It is a very long expression.

Finally, using this new squiggle matrix as my central vertex matrix, I have my desired expression for the Witten vertex in the full-string language:

$$e^{\frac{-1}{2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{r=0}^{1} \sum_{s=0}^{1} b^+_{n,i,r} H^{i,j,r,s}_{n,m} b^+_{m,j,s}} \prod_{i=1}^{3} \prod_{r=\{L,R\}} |0_{i,r}\rangle =$$

$$e^{\sum_i \sum_j \sum_n \sum_m \alpha^+_{i,n} \xi^{i,j}_{n,m} \alpha^+_{j,m}} \, e^{\frac{-1}{2} \sum_{n=0}^{\infty} \sum_{i=1}^{3} \sum_{m=0}^{\infty} \alpha^+_{i,n} \varphi_{n,m} \alpha^+_{i,m}} \prod_{i=1}^{3} |0_i\rangle$$