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The Toeplitz form matrix derived in the last progress report may well be useful, despite the difficulty in finding its determinant. Here, I'll try an expansion in small separation and see if a more useful form emerges. Taking my form from March 21, I have:

$$\begin{bmatrix} \{0\} & \{D\} & \{2D\} & \{3D\} & \{4D\} & \{5D\} & \{6D\} & \{7D\} & \{8D\} & & \\ \{D\} & \{0\} & \{D\} & \{2D\} & \{3D\} & \{4D\} & \{5D\} & \{6D\} & \{7D\} & & \\ \{2D\} & \{D\} & \{0\} & \{D\} & \{2D\} & \{3D\} & \{4D\} & \{5D\} & \{6D\} & & \\ \{3D\} & \{2D\} & \{D\} & \{0\} & \{D\} & \{2D\} & \{3D\} & \{4D\} & \{5D\} & & \\ \{4D\} & \{3D\} & \{2D\} & \{D\} & \{0\} & \{D\} & \{2D\} & \{3D\} & \{4D\} & \dots & \\ \{5D\} & \{4D\} & \{3D\} & \{2D\} & \{D\} & \{0\} & \{D\} & \{2D\} & \{3D\} & & \\ \{6D\} & \{5D\} & \{4D\} & \{3D\} & \{2D\} & \{D\} & \{0\} & \{D\} & \{2D\} & & \\ \{7D\} & \{6D\} & \{5D\} & \{4D\} & \{3D\} & \{2D\} & \{D\} & \{0\} & \{D\} & & \\ \{8D\} & \{7D\} & \{6D\} & \{5D\} & \{4D\} & \{3D\} & \{2D\} & \{D\} & \{0\} & & \\ & & & & \dots & & & & & & \dots \end{bmatrix}$$

$P$  odd  $\alpha = \frac{1}{2} \frac{1}{D} \approx 0$   $D$  even  $N = DP$   $P \times P$  matrix

$$\{X\} = \binom{N}{\alpha N + X} \text{ in this case } \{X\} = \{ |X| \}$$

**Attempt 1: Stirling**

Now suppose I took a Stirling approximation and then let  $\frac{1}{D}$  small.

$n! \approx \sqrt{2\pi n} n^{n+\frac{1}{2}} e^{-n}$ , and certainly here my exponentials will be large.

$$\begin{aligned} \binom{N}{\alpha N + cD} &= \frac{N!}{(\alpha N + cD)! (\alpha N - cD)!} \approx \frac{1}{\sqrt{2\pi}} \frac{N^{N+\frac{1}{2}} e^{-N}}{(\alpha N + cD)^{(\alpha N + cD)+\frac{1}{2}} e^{-\alpha N - cD} (\alpha N - cD)^{(\alpha N - cD)+\frac{1}{2}} e^{-\alpha N + cD}} \\ &\approx \frac{1}{\sqrt{2\pi}} \frac{N^{N+\frac{1}{2}}}{(\alpha N + cD)^{(\alpha N + cD)+\frac{1}{2}} (\alpha N - cD)^{(\alpha N - cD)+\frac{1}{2}}} \end{aligned}$$

Now it's clear that I could factor the  $N^{N+\frac{1}{2}}$  out of the matrix entirely. Dealing with what remains, however, looks challenging. Now the results are going to be extremely sensitive to changes in  $\frac{1}{D}$ : in fact, zero is an essential singularity of this function. Let me try this another way.

### Attempt 2: Gamma/Mathematica

Using Mathematica, expanding  $\{cD\} = \binom{N}{\alpha N + cD}$   $D \rightarrow \infty$  gives

$$\alpha = \frac{1}{2}$$

$$\begin{aligned} \{cD\} &= \binom{N}{\alpha N + cD} \approx \frac{\frac{\Gamma(1+N)}{\alpha N} \left[ \frac{\alpha N}{cD} - \left(\frac{\alpha N}{cD}\right)^2 + \left(\frac{\alpha N}{cD}\right)^3 - \dots \right]}{\Gamma(-cD+1+\alpha N)\Gamma(cD+\alpha N)} \\ &= \frac{\frac{\Gamma(1+N)}{\alpha N} \frac{\alpha N}{cD} \left[ \frac{1}{1+\frac{\alpha N}{cD}} \right]}{\Gamma(-cD+1+\alpha N)\Gamma(cD+\alpha N)} = \frac{\Gamma(1+N)}{cD \left(1+\frac{\alpha N}{cD}\right) \Gamma(-cD+1+\alpha N)\Gamma(cD+\alpha N)} \end{aligned}$$

Taking the limit as N becomes infinite now is not an option: this will have to follow collapsing the determinant, as this expression does not converge as N becomes infinite. Let me step back two steps and propose a different direction, in light of D being large.

$$\alpha = \frac{1}{2}$$

$$\begin{aligned} \{cD\} &= \binom{N}{\alpha N + cD} \approx \frac{\frac{\Gamma(1+N)}{\alpha N} \left[ \frac{\alpha N}{cD} - \left(\frac{\alpha N}{cD}\right)^2 + \left(\frac{\alpha N}{cD}\right)^3 - \dots \right]}{\Gamma(-cD+1+\alpha N)\Gamma(cD+\alpha N)} \\ &\approx \frac{\Gamma(1+N)}{cD\Gamma(-cD+1+\alpha N)\Gamma(cD+\alpha N)} \end{aligned}$$

Somehow, I've ended up with the exact binomial expression I started with! The fact is, none of these will converge with large N until the determinant is taken: namely, the result of the log-determinant must be ultimately like N-squared. The issue is trying to expand the terms initially in D, and then seeing if this simplifies. In this case, it certainly has come full circle.

### Attempt 3: Symmetric Toeplitz Determinant

Clearly, the result here will be the determinant of a symmetric Toeplitz matrix. Perhaps by examining the terms in the expansion, I could find a nice expression. I have found some papers addressing this exact topic (in fact, some cite Gessel's work themselves, and also an "H. Widom" who does work with this variety of matrix: are you related?), and here I outline the procedure recommended for finding the determinant:

From <http://www-ee.stanford.edu/~gray/toeplitz.pdf> (Gray) Theorem 5.28, I see that the Toeplitz determinant is given by:

$$\lim_{n \rightarrow \infty} (\det(T_n(f)))^{\frac{1}{n}} = e^{\frac{1}{2\pi} \int_0^{2\pi} \ln(f(\lambda)) d\lambda}$$

if  $T_n(f)$  is a sequence of Hermitian Toeplitz matrices, in the Wiener class such that  $\ln f(\lambda)$  is Riemann integrable and  $f(\lambda) > m_f > 0$ .

My matrix is clearly Hermitian: it is symmetric and all real. The Wiener class is the class of matrices such that the sequence  $\{t_k\}$  is absolutely summable, e.g.

$$f(\lambda) = \sum_{k=-\infty}^{\infty} t_k e^{ik\lambda} \quad t_k = \int_0^{2\pi} f(\lambda) e^{-ik\lambda} d\lambda, \text{ and } T_n(f) = \{t_{k-j}\} \text{ generates my matrix. Certainly,}$$

$\{t_k\}$  is absolutely summable since the combinations would eventually truncate if the matrix were extended to infinity. Next, the natural logarithm must be integrable, which will hopefully be the case since my coefficients and therefore generating function is entirely positive. Finally, I need  $f(\lambda) \geq m_f > 0$ , where  $m_f$  is the lower bound on the generating function. I'll need to check this property once I find it, but I don't expect trouble of that variety.

The first step is to determine my generating function. The problem here is that in the formula given,  $f$  is not a function of  $n$ . My function  $f$  is. While factoring out the  $n$ -factorial in the numerator would not be a problem, dealing with the terms in the denominator is. This is expected, of course, as the "nth-root" in the limit will annihilate one of my  $\frac{1}{N}$  factors from the area, and this behavior will likely annihilate the other.

Certainly, I know that  $t_{N,k} = \frac{\Gamma(N)}{\Gamma(\alpha N + kD)\Gamma(\alpha N - kD)} = \binom{N}{\alpha N + kD}$ , and so that the gamma function involving  $N$  in the numerator could be factored out, the natural logarithm of which would annihilate another  $\frac{1}{N}$  factor. Indeed, these coefficients will be

$$\text{given by } f_N(\lambda) = \sum_{k=-\infty}^{\infty} t_{N,k} e^{ik\lambda} \quad t_{N,k} = \int_0^{2\pi} f_N(\lambda) e^{-ik\lambda} d\lambda.$$

Certainly, however, by symmetry I can write

$$f_N(\lambda) = t_{N,0} + 2 \sum_{k=1}^{\infty} t_{N,k} \cos(k\lambda) \quad t_{N,k} = \int_0^{2\pi} f_N(\lambda) e^{-ik\lambda} d\lambda$$

This is a generating function that Mathematica is willing to collapse. I get (Result in GVLongterm12b.nb)

More generally, one gets

$$f_N(\lambda, D) = t_{N,0} - 4(-1)^{D-1} \left[ \sin(\pi(1+N-N\alpha)) \frac{\Gamma(1+N)\Gamma(D-N+N\alpha)}{4\pi\Gamma(D+1+N\alpha)} \right] \cos(\lambda) \times$$

$$\left( e^{-i\lambda} {}_{D+1}F_D \left( \{1\} \cup \bigcup_{j=1}^D \left\{ \frac{D+j-1}{D} - \frac{N}{D} + \frac{N\alpha}{D} \right\}; \bigcup_{j=1}^D \left\{ \frac{D+j}{D} + \frac{N\alpha}{D} \right\}; (-1)^D e^{-i\lambda} \right) + \right.$$

$$\left. e^{i\lambda} {}_{D+1}F_D \left( \{1\} \cup \bigcup_{j=1}^D \left\{ \frac{D+j-1}{D} - \frac{N}{D} + \frac{N\alpha}{D} \right\}; \bigcup_{j=1}^D \left\{ \frac{D+j}{D} + \frac{N\alpha}{D} \right\}; (-1)^D e^{i\lambda} \right) \right)$$

$$t_{N,0} = \binom{N}{\alpha N}$$

I have used Mathematica to determine this.

Next I want to validate that  $f(\lambda) \geq m_f > 0$ . I'll just plot this function for a few specific cases (See GVLlongterm12.nb). It seems to work for the  $D > 1$  cases.

The next step is to find the integral  $\frac{1}{2\pi} \int_0^{2\pi} \ln f_N(\lambda, D) d\lambda$ . In the form above, I have

isolated the parts involving lambda in the last line. Before I spend a lot of effort trying to evaluate the integral, let me try a specific example using numerical integration, and then find out if this line of effort even seems like it might be fruitful (See GVLlongterm12.nb). It does indeed show the desired convergence in the infinite case, but it is important to note that for the actual values of interest, a limit must be taken to get the desired value. Thus, I have interpolated in the Mathematica notebook rather than to try to determine the actual value at the desired points.

Now consider how I might determine this integral: if I could find it, or an approximation thereof I could certainly quickly find the determinant for the large-n case.

From functions.wolfram.com, I have found an expansion of this function valid in the

region in question: namely, that  ${}_{q+1}F_q = 1 + \frac{\prod_{j=1}^{q+1} a_j}{\prod_{j=1}^q b_j} z + \frac{\prod_{j=1}^{q+1} a_j (a_j + 1)}{2 \prod_{j=1}^q b_j (b_j + 1)} z^2 + \dots \quad |z| \leq 1$ .

In this case, my exponential will be exactly on the border of convergence. It's important to note that the numerators will only be somewhat smaller than the denominators: I'll have to justify the first-order expansion in Mathematica. Indeed, in the first order this appears to converge at the large-N limit. Now my approximation reads:

$$f_N(\lambda, D) \approx t_{N,0} -$$

$$4(-1)^{D-1} \left[ \sin(\pi(1+N-N\alpha)) \frac{\Gamma(1+N)\Gamma(D-N+N\alpha)}{4\pi\Gamma(D+1+N\alpha)} \right] \times$$

$$\left[ e^{-i\lambda} \left( 1 + \prod_{j=1}^D \frac{\left( \frac{D+j-1}{D} - \frac{N}{D}(1+\alpha) \right)}{\left( \frac{D+j}{D} + \frac{N\alpha}{D} \right)} (-1)^D e^{-i\lambda} \right) + e^{i\lambda} \left( 1 + \prod_{j=1}^D \frac{\left( \frac{D+j-1}{D} - \frac{N}{D}(1+\alpha) \right)}{\left( \frac{D+j}{D} + \frac{N\alpha}{D} \right)} (-1)^D e^{i\lambda} \right) \right]$$

Unfortunately, comparing this function to the actual one as N grows shows that this is not sufficient to ensure the proper behavior at the border. It looks like I need to perform the integral  $\frac{1}{2\pi} \int_0^{2\pi} f_N(\lambda, D) d\lambda$  before I may do any further expansion, since as I have verified numerically,

$$\frac{1}{N} \ln \text{Det}(N \times N \text{ Actual Gessel-Viennot Matrix Separation } D) \approx \int_0^{2\pi} f_N(\lambda, D) d\lambda$$

where the exact general form is given above. However, this doesn't seem totally

impossible since  $\int {}_pF_q(\{a_p\}; \{b_q\}; z) dz = \frac{\prod_{j=1}^q (b_j - 1)}{\prod_{j=1}^p (a_j - 1)} {}_pF_q(\{a_p - 1\}; \{b_q - 1\}; z)$ . Let me then

try to find the form of the solution of the integral

$$\frac{1}{2\pi} \int \ln(A + B[e^{-i\lambda} F(c, e^{-i\lambda}) + e^{i\lambda} F(c, e^{i\lambda})]) d\lambda$$

$$\int F(c, z) dz = g(c-1)F(c-1, z)$$

Using a sneaky trick in Mathematica, I believe I have found the value of this integral: