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In the last progress report, I noticed that the actual value of the long-term area-wise entropy for N paths of density d at slope factor alpha of length N does indeed show the desired behavior. I'd like to try to produce an approximation valid in the $P \propto N \to \infty$ region now.

Let me return to my Gessel-Viennot matrix form from several reports ago:

$$\begin{bmatrix} R_{11} & R_{12} & \dots & R_{1P} \\ R_{21} & R_{22} & \dots & R_{2P} \\ \dots & \dots & \dots & \dots \\ R_{P1} & R_{P2} & \dots & R_{PP} \end{bmatrix}$$

I will assume that there are an odd number of paths in order to simplify this model.

Now allow the paths to be interleaved as such (let the symbol $\{XD\}$ indicate the starting point X offsets from the center in the first position, and the ending point X offsets from the center in the last position.) In this way, I impose a sort of symmetry:

$$\begin{bmatrix} R_{\{0\},\{0\}} & R_{\{0\},\{-D\}} & R_{\{0\},\{D\}} & R_{\{0\},\{-2D\}} & R_{\{0\},\{2D\}} & \dots \\ R_{\{-D\},\{0\}} & R_{\{-D\},\{-D\}} & R_{\{-D\},\{D\}} & R_{\{-D\},\{-2D\}} & R_{\{-D\},\{2D\}} & \dots \\ R_{\{D\},\{0\}} & R_{\{D\},\{-D\}} & R_{\{D\},\{D\}} & R_{\{D\},\{-2D\}} & R_{\{D\},\{2D\}} & \dots \\ R_{\{-2D\},\{0\}} & R_{\{-2D\},\{-D\}} & R_{\{-2D\},\{D\}} & R_{\{-2D\},\{-2D\}} & R_{\{-2D\},\{2D\}} & \dots \\ R_{\{2D\},\{0\}} & R_{\{2D\},\{-D\}} & R_{\{2D\},\{D\}} & R_{\{2D\},\{-2D\}} & R_{\{2D\},\{2D\}} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

I notice that in the formula set up in previous progress reports that by symmetry the following should hold:

$R_{\{A\},\{B\}} = R(|A-B|,\alpha,N)$. Thus, this matrix should be totally symmetric, but not necessarily in some simple pattern. Let me follow the tactic I used last time; I'll set $\alpha = \frac{1}{2}$ for now and then start writing determinants for the matrix composed of components $R_{\{A\},\{B\}} = R\left(D,\alpha = \frac{1}{2}, N, P = \left\lceil \frac{N}{D} \right\rceil \quad odd\right)$, and then see if I can determine some simple pattern or approximation. I notice, however, that with N, P, and α specified, within the matrix this is really a function of the difference only, as specified earlier. Still keeping P odd and D even, then, I have:

$R_{\{A\},\{B\}} = R\left(D \quad even, \alpha = \frac{1}{2}, N = D(2m+1), P = (2m+1)\right)$. This means that in this case, some truncation will occur so that $R_{\{A\},\{B\}} = 0 \quad if \quad |A-B| > D(2m+1)$.

Looking at GVLongterm11.nb, it looks like so far I'm on the right track. The next step is to try to find a reasonable determinant expression.

Again, there really isn't much freedom in choosing valid expressions: I would like it to have terms like $(X)(X+X)(X+X+X)(X+X+X+X)...$ in order to ensure proper matching. In this case, however, there will be a great deal of truncation that is not accounted for in the model that I previously used.

In this case, I hold N as a fixed proportion of path count P. Now allow me a moment to examine my previous approximation and determine what is systematically going wrong.

For the P = 3 case, the values in my determinant look like (the values inside here are the argument inside the $R(|A-B|)$)

$$(0)(0)(0)+(D)(D)(-2D)+(2D)(-D)(-D)-(-2D)(0)(2D)-(-D)(D)(0)-(0)(-D)(D)$$

My approximation

$$PathCount(\alpha, N, D, P) \approx \binom{N}{\alpha N}^{-P} \prod_{p=1}^{P}\left(\sum_{i=1}^{p}(-1)^{i-1}\binom{N}{\alpha N+(i-1)D}\binom{N}{\alpha N-(i-1)D}\right)$$

would give

$$(0)(0)(0)-(0)(0)\frac{(D)(-D)}{(0)}-(0)(0)\frac{(D)(-D)}{(0)}+(0)(0)\frac{(2D)(-2D)}{(0)}+(0)\frac{(D)(-D)}{(0)}\frac{(D)(-D)}{(0)}-(0)\frac{(2D)(-2D)}{(0)}\frac{(D)(-D)}{(0)}$$

Indeed, when N >> P, this approximation becomes increasingly accurate. The first step towards a proper approximation would be to determine what is beginning to go wrong here. (Look at GVLongterm11.nb)

However, it is clear from GVLongterm10.nb that something does indeed go horribly wrong with my approximation as density grows and as N grows relative to P. The concept here was to let N and P go to infinity together, then examine the remaining behavior on separation D (however, I did previously determine that Gessel-Viennot is fundamentally unfit for the periodic case, since the edge effects have potentially infinite range of propagation).

Let me then discuss the terms that are relevant in a weakly interacting case, and work forward from that.

Most assuredly, the largest contributions will come from the $R(|A-B|=0)$ terms, the second largest from $R(|A-B|=D)$, the third from $R(|A-B|=2D)$, et cetera. Now looking at the contributions in the real determinant, I see that thus the terms considering only nearest neighbor interactions look like this:

$$(0)^P - C_1(P-1)(D)^2(0)^{P-2} + C_2(D)^4(0)^{P-4} - \ldots \pm C_3(D)^{P-1}(0)^1 .$$

I propose this form since the second-nearest neighbor interactions are combinatorially smaller (implying some factorials), where these are merely exponentially smaller interactions.

Let me then try to determine what these coefficients C are. See the Mathematica notebook now to see me attempting to establish a pattern in these nearest-neighbor coefficients, and also attempting to validate whether this appropriately captures the desired behavior in the expansion. Indeed, upon examination, it is clear to see that the nearest-neighbor interactions alone are not sufficient to capture the cubic behavior, so actually determining the coefficients is moot.

Since this does not capture the behavior, let me try instead taking the lowest-order contribution from each of the combinatorial terms. e.g.,

$$(0)^P - D_1(D)^2(0)^{P-2} - D_2(2D)^2(0)^{P-2} - D_3(3D)^2(0)^{P-2} - \ldots$$

Through a rather complex piece of trickery, I've gotten Mathematica to show me which terms fit this description, along with their coefficients. From the notebook (GVLongterm11) , I see that the result is

$$(0)^P - (P-1)(D)^2(0)^{P-2} - (P-2)(2D)^2(0)^{P-2} - (P-3)(3D)^2(0)^{P-2} - \ldots$$

This would be a highly convenient form to work with if it works; almost too good to be true, in fact! Upon testing this in Mathematica, it is clear that this does no better than the earlier concept of only nearest-neighbor contributions. In fact, at the density that misses dramatically every time, 0.5, the paths are in such close proximity on average that one would expect all of the other paths to have a substantial effect, which is exactly the behavior that is occurring. With the number of terms in the determinant growing factorially in the number of paths, these contributions will grow very rapidly in number, apparently somewhat on par with the rate at which their average dies off so as to affect a substantial difference in density, and to add the additional cubic term to the expansion.

So, let me look at the determinant in one final light. In the case I have constructed $\alpha = \frac{1}{2}, D \quad even, P = (2n+1), N = DP$. Then I may write the Gessel-Viennot matrix as such:

$$\begin{bmatrix} \{0\} & \{D\} & \{D\} & \{2D\} & \{2D\} & \{3D\} & \{3D\} & \{4D\} & \{4D\} & \\ \{D\} & \{0\} & \{2D\} & \{D\} & \{3D\} & \{2D\} & \{4D\} & \{3D\} & \{5D\} & \\ \{D\} & \{2D\} & \{0\} & \{3D\} & \{D\} & \{4D\} & \{2D\} & \{5D\} & \{3D\} & \\ \{2D\} & \{D\} & \{3D\} & \{0\} & \{4D\} & \{D\} & \{5D\} & \{2D\} & \{6D\} & \\ \{2D\} & \{3D\} & \{D\} & \{4D\} & \{0\} & \{5D\} & \{D\} & \{6D\} & \{2D\} & \ldots \\ \{3D\} & \{2D\} & \{4D\} & \{D\} & \{5D\} & \{0\} & \{6D\} & \{D\} & \{7D\} & \\ \{3D\} & \{4D\} & \{2D\} & \{5D\} & \{D\} & \{6D\} & \{0\} & \{7D\} & \{D\} & \\ \{4D\} & \{3D\} & \{5D\} & \{2D\} & \{6D\} & \{D\} & \{7D\} & \{0\} & \{8D\} & \\ \{4D\} & \{5D\} & \{3D\} & \{6D\} & \{2D\} & \{7D\} & \{D\} & \{8D\} & \{0\} & \\ & & & & \ldots & & & & & \ldots \end{bmatrix}$$

e.g., if this matrix is considered to be zero-indexed:

$$R_{i,j} = \begin{cases} \left\{\frac{i-j}{2}\right\} & i+j \quad even \\ \left\{\frac{i+j+1}{2}\right\} & i+j \quad odd \end{cases}$$

$$\{X\} = \binom{N}{\alpha N + X} \quad in \quad this \quad case \quad \{X\} = \{|X|\}$$

This matrix is, interestingly, a Toeplitz system and a banded system interleaved, and is totally symmetric. There may well be a valid generating function, since the entries seem to be ultimately a function of only one variable.

Observe the intriguing relationship and similarity between Binomial coefficients and matrix determinants, in a vaguely similar scenario:

http://functions.wolfram.com/GammaBetaErf/Binomial/17/02/0001/

$$\left|\left(\frac{1}{k!}\binom{a\,k+l+x}{a\,k-l+y}\right)_{\substack{0\le k\le n\\ 0\le l\le n}}\right| == a^{\binom{n+1}{2}} \prod_{k=0}^{n} \frac{(a\,k+x)!\,\prod_{l=1}^{k}((k+l-1)\,a+x+y+1)}{(a\,k+y)!\,(2\,k+x-y)!}$$

Perhaps with proper ordering of the matrix I could obtain a great form like this!

Through proper ordering, n in the equation above would come to play the role of the odd P. To make this work, then, I would have to get the form such that:

$ak+l+x=N \quad ak-l+y=\pm D$, and somehow eliminate the $\frac{1}{k!}$ in the denominator.

Indeed, ignoring ordering issues, it is easy to see if you are familiar with the Gessel-Viennot algorithm that I could get my matrix into the aesthetically pleasing form (at least as far as highest-order terms are concerned) above into one like this:

$$\begin{bmatrix} \{0\} & \{D\} & \{2D\} & \{3D\} & \{4D\} & \{5D\} & \{6D\} & \{7D\} & \{8D\} & \\ \{D\} & \{0\} & \{D\} & \{2D\} & \{3D\} & \{4D\} & \{5D\} & \{6D\} & \{7D\} & \\ \{2D\} & \{D\} & \{0\} & \{D\} & \{2D\} & \{3D\} & \{4D\} & \{5D\} & \{6D\} & \\ \{3D\} & \{2D\} & \{D\} & \{0\} & \{D\} & \{2D\} & \{3D\} & \{4D\} & \{5D\} & \\ \{4D\} & \{3D\} & \{2D\} & \{D\} & \{0\} & \{D\} & \{2D\} & \{3D\} & \{4D\} & \ldots \\ \{5D\} & \{4D\} & \{3D\} & \{2D\} & \{D\} & \{0\} & \{D\} & \{2D\} & \{3D\} & \\ \{6D\} & \{5D\} & \{4D\} & \{3D\} & \{2D\} & \{D\} & \{0\} & \{D\} & \{2D\} & \\ \{7D\} & \{6D\} & \{5D\} & \{4D\} & \{3D\} & \{2D\} & \{D\} & \{0\} & \{D\} & \\ \{8D\} & \{7D\} & \{6D\} & \{5D\} & \{4D\} & \{3D\} & \{2D\} & \{D\} & \{0\} & \\ & & & & \ldots & & & & & \ldots \end{bmatrix}$$

This is perfectly Toeplitz and perfectly symmetric. Recall that

$$\{X\} = \binom{N}{\alpha N + X} \quad in \quad this \quad case \quad \{X\} = \{|X|\}.$$

Now let me consider how I might be able to manipulate this general formula a bit. Let me start by writing the form

$$\left| \left(\frac{1}{k!} \binom{ak+l+x}{ak-l+y} \right)_{\substack{0 \le k \le n \\ 0 \le l \le n}} \right|$$

I already determined that n would play the role of P.

$$\left| \left(\frac{1}{k!} \binom{ak+l+x}{ak-l+y} \right)_{\substack{0 \le k \le P-1 \\ 0 \le l \le P-1}} \right|$$

In order to eliminate the stray k-factorial, I may certainly take

$$\left| \binom{ak+l+x}{ak-l+y}_{\substack{0 \le k \le P-1 \\ 0 \le l \le P-1}} \cdot \begin{pmatrix} \frac{1}{k!} & if \quad k=l \\ 0 & otherwise \end{pmatrix}_{\substack{0 \le k \le P-1 \\ 0 \le l \le P-1}} \right| = \left| \binom{ak+l+x}{ak-l+y}_{\substack{0 \le k \le P-1 \\ 0 \le l \le P-1}} \right| \left| \begin{pmatrix} \frac{1}{k!} & if \quad k=l \\ 0 & otherwise \end{pmatrix}_{\substack{0 \le k \le P-1 \\ 0 \le l \le P-1}} \right|$$

$$\left| \begin{pmatrix} \frac{1}{k!} & if \quad k=l \\ 0 & otherwise \end{pmatrix}_{\substack{0 \le k \le P-1 \\ 0 \le l \le P-1}} \right| = \prod_{k=0}^{P-1} \frac{1}{k!}$$

Now all that's left is to determine if I can match the remaining values.

$$\left| \binom{ak+l+x}{ak-l+y}_{\substack{0 \le k \le P-1 \\ 0 \le l \le P-1}} \right|.$$

I note that in this case, I need an integer coefficient like D on the l, at the very least. In fact, it seems that with proper rewriting of the matrix (e.g., "stretch" the matrix by adding rows on the diagonal, effectively only increasing P), this might be possible. In this case, then, let D = 1. I will extend to other cases later. Now I need:

$$ak + l + x = N \quad all \quad k,l \quad \text{int}$$
$$ak - l \propto D \quad all \quad k,l \quad \text{int}$$

Certainly, I see that letting $a = D$ and then properly "stretching" the matrix will give me the desired behavior (here I have let D = 1). In the numerator, this leaves

$$k + l + x = N \quad all \quad k,l \quad \text{int}$$

which again seems like an impossibility. In fact, obtaining any nonzero constant with respect to k and l in the numerator here seems categorically impossible. Further, since k + l is on the order of path count P, I cannot take the k and l contributions in the numerator to be insignificant.

Another identity that might be useful is:

$$A = \{a_{i,j}\} \quad a_{i,j} = \frac{\Gamma(i+j)}{\Gamma(i)\Gamma(j)}$$
$$Det(A) = \Gamma(\dim A)$$

Suppose then that I could get $\begin{matrix} i \to \alpha N + D, j \to (1-\alpha)N - D \\ \alpha = \frac{1}{2} \end{matrix}$.

I have no proof for this so no inkling of where I might fit the values i and j into this.